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Lecture Notes 4
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11 A Second Proof for the 0-1 Law for FOL

Relying on the Ehrenfeucht-Fraisse Theorem we gave in the previous lectures a first proof of the 0-1 Law (for the case of the random graph of constant probability). Here we will present a second proof of the 0-1 Law which allows us to derive a very nice additional fact: given an FO sentence, it is *decidable* whether it holds almost surely or almost never!

The strategy of the second proof starts from an observation about the set of sentences that hold almost surely:

$$\mathcal{S}_{p(n)} = \{\varphi \text{ in FO} \mid \mu_{p(n)}(\varphi) = 1\}$$

Again we omit the subscript $p(n)$ when clear from the context. The main observation is that the 0-1 Law would follow if we can show that for any FO sentence φ, either $\varphi \in \mathcal{S}$, or $\neg\varphi \in \mathcal{S}$. This is a common issue in logic and it is usually asked of *theories*.

Definition 11.1 (Theory, consistent, complete) *We say that a set of sentences T is a* theory *if it is closed under proof, i.e., for any φ, if $T \vdash \varphi$ then $\varphi \in T$. We say that a theory T is* consistent *when there is no φ such that both φ and $\neg\varphi$ are in T. This is equivalent to $false \notin T$, and further equivalent to $T \neq$ FO (an inconsistent theory contains all sentences!). We say that a theory is* complete *if for any sentence φ, either $\varphi \in T$, or $\neg\varphi \in T'$.*

For example, if Γ is a set of sentences then $Th(\Gamma) = \{\varphi \mid \Gamma \vdash \varphi\}$ is a theory ("generated" by Γ). By the way, we say that Γ is consistent whenever $Th(\Gamma)$ is consistent. For another example, if CC is a class of models then $Th(CC) = \{\varphi \mid \forall \mathcal{A} \in CC\ \mathcal{A} \models \varphi\}$ is a theory ("of" CC). An important particular case is when CC consists of a single model because such a theory is consistent and complete!

A simple but fundamental property of theories is the following:

Proposition 11.1 *A consistent and complete theory that is recursively axiomatizable is decidable.*

Proof Proofs from a recursive (decidable) set of axioms are recursively enumerable. To decide if φ is in the theory we enumerate all such proofs. Eventually, a proof of each sentence in the theory shows up. We stop when either a proof of φ or a proof of $\neg\varphi$ shows up. By completeness, one of these must happen and by consistency if a proof of $\neg\varphi$ shows up then φ is not in the theory. $\square$

Now look at $\mathcal{S}$ defined above.

Proposition 11.2 *$\mathcal{S}$ is a consistent theory (called the* almost sure *theory).*

Proof We need to show that $\mathcal{S}$ is closed under proof. If $\mathcal{S} \vdash \varphi$ then there is a finite subset $\Gamma_0 \subset \mathcal{S}$ such that $\Gamma_0 \vdash \varphi$. The conjunction of the sentences in Γ_0 must then also hold almost surely and since this implies φ the latter must too, hence it belongs to $\mathcal{S}$. For consistency notice that $\mu(false) = 0$ hence $false \notin \mathcal{S}$. $\square$

We see now that if we show that $\mathcal{S}$ is complete and recursively axiomatizable then the 0-1 Law follows and we also have the desired decidability property.

To prove that $\mathcal{S}$ is complete it would suffice to show that it is the theory of a single model. Getting to this is more complicated and it involves the following general result.

Proposition 11.3 *Assume that the vocabulary (signature) is at most countable. Let T be a theory that is consistent, has no finite models and is $\omega-$categorical (i.e., any two countable models of the theory are isomophic). Then T is complete.*

The proof of this proposition will require a sharpened form of Gödel's Completeness, which, you will recall, we stated as follows:

Theorem 11.4 (Godel's Completeness Theorem) $\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$.

The "if" direction is completeness. The "only if" direction is soundness. The theorem is also often stated equivalently as follows: for any set Ω of sentences, Ω is consistent iff it is satisfiable. Assuming soundness, the completeness part is often stated equivalently in terms of properties of theories: any consistent theory is satisfiable. (Might as well state an important corollary known as the Compactness Theorem: if any finite subset of Γ is satisfiable then Γ is satisfiable. This leads to an easy proof that connectivity is not definable in FOL over *all* models, try to prove this! This proof does not show that connectivity is not definable in FOL over finite models, which can be shown as a consequence of the Ehrenfeucht-Fraisse Theorem.)

We will need the following more precise form of the completeness implication:

Theorem 11.5 (Sharpened Completeness Theorem) *Assume that the vocabulary (signature) is at most countable. If a theory T is consistent, then T is satisfiable in a model that is at most countable.*

This is shown in all logic textbooks but a particularly clear proof appears in Enderton. Now:

Proof (of Proposition 11.3). We prove this by contradiction. Suppose that T is not complete, then there exists φ such that both φ and $\neg\varphi$ are not in T. Consider $T_1 = Th(T \cup \{\varphi\})$ and $T_2 = Th(T \cup \{\neg\varphi\})$. We argue that T_1 is consistent. Suppose not, then we have

$$\begin{aligned} & false \in T_1 \\ \Rightarrow\ & T, \varphi \vdash false \\ \Rightarrow\ & T \vdash (\varphi \rightarrow false) \text{ (deduction property)} \\ \Rightarrow\ & T \vdash \neg\varphi \\ \Rightarrow\ & \neg\varphi \in T, \end{aligned}$$

which is a contradiction. Similarly, T_2 is also consistent. By the Sharpened Completeness Theorem they have models, say M_i for T_i, that are at most countable. Hence, M_1 and M_2 are both at most countable models for T. Since T has no finite models, the two models are countable and by $\omega-$categoricity they are isomorphic. They must therefore satisfy the same sentences, hence they satisfy both φ and $\neg\varphi$ which is impossible. □

With this result, our strategy now leaves us to prove that $\mathcal{S}$ does not have finite models at that it is $\omega-$categorical. Enter the extension axioms. By Lemma8.1 the extension axioms hold almost always when $p(n)$ is constant. When $p(n)$ is not constant this property does hold in general so we continue under the following assumption:

Assumption $p(n)$ is such that every extension axiom is almost surely true. That is, $EA = \{EA_{r,s} \mid r, s \geq 0\} \subset \mathcal{S}$

We have immediately

Proposition 11.6 *$\mathcal{S}$ has no finite models (because EA doesn't).*

But more importantly

Proposition 11.7 *$Th(EA)$ (and therefore $\mathcal{S}$) is $\omega-$categorical.*

Proof Let G_1 and G_2 bet two countable graphs that satisfy EA. Without loss of generality we assume that the nodes of each graph are actually the natural numbers. We play an Ehrenfeuch-Fraisse game with countably many rounds in which the Spoiler uses the following special strategy. He will play the smallest unplayed node of G_1 in each even-numbered round and the smallest unplayed node of G_2 in each odd-numbered round. Like in the proof of Lemma 10.2 the validity of the extension axioms give the Duplicator a winning strategy, one in which, we note, she always chooses an unplayed node. When the game is played with these two strategies for Spoiler, respectively Duplicator, each graph node, for both G_1 and G_2, is eventually played, and is played exactly once. After round k the game establishes a partial isomorphism between k nodes of G_1 and k nodes of G_2. This partial isomorphism is a subset of the partial isomorphism established after round $k+1$. Take the (countable) union of these partial isomorphisms and this must be an isomorphism between G_1 and G_2. □

Therefore $\mathcal{S}$ is complete and we have the second proof of the 0-1 Law. Note that it also uses the Ehrenfeucht-Fraisse games, but *not* the Ehrenfeucht-Fraisse theorem. For decidablity we need an extra feature: a recursive axiomatization. This is now obvious:

Proposition 11.8 $\mathcal{S} = Th(EA)$

Proof From $EA \subset \mathcal{S}$ and since $\mathcal{S}$ is a theory it follows that $Th(EA) \subseteq \mathcal{S}$. The same propositions that we used to prove that $\mathcal{S}$ is complete also imply that $Th(EA)$ is complete. Now $\mathcal{S} = Th(EA)$ follows from the following simple general observation: if $T_1 \subseteq T_2$ are theories such that T_2 is consistent and T_1 is complete then $T_1 = T_2$. □

In conclusion, given an FO sentence φ it is decidable whether $\varphi \in \mathcal{S}$ and therefore $\mu(\varphi) = 1$ or $\varphi \notin \mathcal{S}$ and therefore $\mu(\varphi) = 0$. The decision procedure given by this proof consists of enumerating all FO proofs with axioms from EA until either φ or $\neg\varphi$ results. A careful analysis of the problem has shown that it is PSPACE-complete (Grandjean, 1983).

Remark Since it is consistent $\mathcal{S}$ has a countable model which, by $\omega-$categoricity, is unique up to isomorphism. Call this model $\mathcal{R}$. (It is known under the name Rado Graph, sometimes as the Random Graph or the Erdös-Rényi Graph.) Since $\mathcal{S} \subseteq Th(\mathcal{R})$, $\mathcal{S}$ is complete and $Th((\mathcal{R})$ is consistent we have $\mathcal{S} = Th(\mathcal{R})$. Thus, the sentences true in $\mathcal{R}$ aer exactly the sentences which hold almost always. This is therefore a very interesting model and it is remarkable that it has an (apparently) simple alternative definition (discovered by Rado): The nodes are the natural numbers and we have an edge between i and j iff the j'th bit in the binary expansion of i is 1. By the results above, to check that this is the unique model of $\mathcal{S}$ it suffices to show that it satisfies the extension axioms. Given $r + s$ distinct natural numbers the existential in $EA_{r,s}$ will be satisfied by any natural number whose binary expansion has 1's in the positions designated by the r numbers, 0's in the positions designated by the s numbers and arbitrary bits elsewhere.

12 0-1 or not? Threshold Functions

Both proofs of the 0-1 Law that we have given rely on just one feature of the probability distribution: that it will make the extension axioms hold almost surely. In fact, for certain $p(n)$ this fails and the 0-1 Law for FOL also fails. However, a fascinating discovery of Erdös and Rényi was that the functions for which the 0-1 Law fails in such a way that probabilities "move" from 0 to 1 are "discrete occurrences" and that different graph properties (FO and beyond) have different such functions, which they were able to identify precisely. Developing this material is beyond the scope of this course. However, we can illustrate the basic ideas for structures with a very simple FO vocabulary.

Let U be a *unary* predicate symbol and for each n consider the random structure $SU[n, p(n)]$ with the universe $\{1, \ldots, n\}$ and such that U holds of each of its elements independently with probability $p(n)$. As in the case of random graphs we denote

$$\mu(\varphi) = \lim_{n \to \infty} \Pr(SU[n, p(n)] \models \varphi)$$

whenever it exists. Now consider the following simple sentence $YES = \exists x\ U(x)$. It is easy to see that

$$\Pr(SU[n, p(n)] \models YES) = 1 - (1 - p(n))^n$$

Observe that for $p(n) = 1/2$ (or any constant probability) $\mu(YES) = 1$, for $p(n) = \frac{1}{n^2}$ we have $\mu(YES) = 0$, but for $p(n) = \frac{1}{n}$ we have $\mu(YES) = 1 - \frac{1}{e}$ and therefore there exist probability functions for which the 0-1 Law fails! This leads us to Erdös-rényi's seminal concept of a "threshold" probability function:

Definition 12.1 (Threshold Function) *$p(n)$ is a threshold function for a property P if*

1. *Whenever $p'(n) \ll p(n)$, in other words, $p'(n) = O(p(n))$ but $p(n) \neq O(p'(n))$, we have $\mu_{p'(n)}(P) = 0$, and*
2. *Whenever $p(n) \ll p'(n)$, we have $\mu_{p'(n)}(P) = 1$.*

A extra bit of calculus in the example above with the random unary structure shows that $p(n) = \frac{1}{n}$ is a threshold function for the property YES. The need for ordering the probability functions asymptotically is easy to see: if $p(n) = \frac{1}{2n}$ we have $\mu(YES) = 1 - \frac{1}{\sqrt{e}}$ and $\frac{1}{2n}$ is also a threshold function.

It can be shown that in the of the unary random structure $p(n) = \frac{1}{n}$ is a threshold function for *all* properties expressible in FOL not just YES, and it is the (asymptotically) *only* threshold function. For example, the following can be shown:

Theorem 12.1 *For the random unary structure and for $p(n)$ such that $p(n) \gg \frac{1}{n}$, $1 - p(n) \gg \frac{1}{n}$ the 0-1 Law holds for FO sentences.*

The proof of this theorem follows exactly the strategy of the second proof we gave for random graphs while using the following sentences in the role previously played by the extension axioms:

For each $r \geq 0$ define

$$A_r = \exists x_1 \ldots x_r\ Distinct(x_1, \ldots, x_r) \wedge U(x_1) \wedge \cdots \wedge U(x_r)$$

and

$$B_r = \exists x_1 \ldots x_r\ Distinct(x_1, \ldots, x_r) \wedge \neg U(x_1) \wedge \cdots \wedge \neg U(x_r).$$

Let

$$UEA = \{A_r \mid r \geq 0\} \cup \{B_r \mid r \geq 0\}$$

Proving that $Th(UEA)$ is $\omega-$categorical and that with the assumptions we made about $p(n)$ the sentences in UEA hold almost surely will be assigned as homework.

Now back to the random graph. A useful intuition is to think of $G[n, p(n)]$ as "evolving from empty to full" as the probability $p(n)$ varies from from 0 to 1. When $p(n)$ is a threshold function for a property P we say that P "appears" at $p(n)$ as the random graph fills up. Dually, we can define threshold functions for those properties that "disappear"as the random graph fills up.

We list without proof some of the fascinating results that were shown about the random graph.

Already in their seminal paper that started the subject Erdös and Rényi found threshold functions for some graph properties. Most of these properties are formulated in terms of containing certain kinds of subgraphs. Some of these properties are FO definable, some not.

- At $p(n) = \frac{1}{n^2}$, edges appear (FO definable).
- At $p(n) = \frac{1}{n\sqrt{n}}$, hinges (pairs of edges with one common vertex) appear (FO definable).
- At $p(n) = \frac{1}{n\sqrt[k]{n}}$, where k is fixed, trees on $k+1$ vertices appear (FO definable).
- At $p(n) = \frac{1}{n}$, triangles (in fact, cycles of any fixed size) appear (FO definable) and planarity disappears (not FO definable).
- At $p(n) = \frac{\ln n}{n}$, connectivity appears (not FO definable)

Moreover, it was shown that (asymptotically) below $\frac{1}{n^2}$ and (asymptotically) in-between the threshold functions listed above the 0-1 law for FO definable properties holds.

Finally, we mention that Shelah and Spencer have shown that for any $p(n) = \frac{1}{n^\alpha}$ where $\alpha \in (0,1)$ is *irrational* the 0-1 law holds for all FOL sentences.

But if α is rational, then $\frac{1}{n^\alpha}$ is a threshold function for some FO definable property. An example is $\alpha = \frac{2}{3}, p(n) = \frac{1}{\sqrt[3]{n^2}}$ when cliques of size 4 appear (Cliques of fixed size $k \geq 3$ appear at $\frac{1}{n^{2/k-1}}$.)