A Linear/Producer/Consumer Model of Classical Linear Logic

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This paper defines a new proof- and category-theoretic framework for *classical linear logic* that separates reasoning into one linear regime and two persistent regimes corresponding to ! and ?. The resulting linear/producer/consumer (LPC) logic puts the three classes of propositions on the same semantic footing, following Benton's linear/non-linear formulation of intuitionistic linear logic. Semantically, LPC corresponds to a system of three categories connected by adjunctions reflecting the linear/producer/consumer structure. The paper's metatheoretic results include admissibility theorems for the cut and duality rules, and a translation of the LPC logic into category theory. The work also presents several concrete instances of the LPC model.

1 Introduction

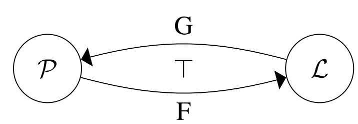

Figure 1: Categorical model of ILL with linear and persistent categories.

Since its introduction by Girard in 1987, linear logic has been found to have a range of applications in logic, proof theory, and programming languages. Its power stems from its ability to carefully manage resource usage: it makes a crucial distinction between *linear* (used exactly once) and *persistent* (unrestricted use) hypotheses, internalizing the latter via the ! connective. From a semantic point of view, the literature has converged (following Benton [3]) on an interpretation of ! as a comonad given by $! = \mathrm{F} \circ \mathrm{G}$ where $\mathrm{F} \dashv \mathrm{G}$ is a symmetric monoidal adjunction between categories $\mathcal{L}$ and $\mathcal{P}$ arranged as shown in Figure 1.

Here, $\mathcal{L}$ (for "linear") is a symmetric monoidal closed category and $\mathcal{P}$ (for "persistent") is a cartesian category. This is, by now, a standard way of interpreting *intuitionistic* linear logic (for details, see Melliès [13]). If, in addition, the category $\mathcal{L}$ is $*$-autonomous, the structure above is sufficient to interpret *classical* linear logic, where the monad ? is determined by $? = (\mathrm{F}^{op}\,(\mathrm{G}^{op}\,(-^{\perp})))^{\perp}$. While sound, this situation unnecessarily commits to a particular implementation of ? in term of $\mathcal{P}^{op}$. The LPC framework absolves us of this commitment by opening up a new range of semantic models, discussed in Section 4.

With that motivation, this paper defines a proof- and category-theoretic framework for full *classical linear logic* that uses *two* persistent categories: one corresponding to ! and one to ?. The resulting categorical structure is shown in Figure 2, where $\mathcal{P}$ takes the place of the "producing" category, in duality with $\mathcal{C}$ as the "consuming" category. This terminology comes from the observations that:

$$
\begin{array}{cc}
!A \vdash 1 & \perp \vdash ?A \\
!A \vdash A & A \vdash ?A \\
!A \vdash !A \otimes !A & ?A \parr ?A \vdash ?A
\end{array}
$$

Intuitively, the left group means that $!A$ is sufficient to *produce* any number of copies of A and, dually, the right group says that $?A$ can *consume* any number of copies of A.

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2 LPC Logic

The syntax of the LPC logic is made up of three syntactic forms for propositions: linear propositions A, producer propositions P, and consumer propositions C.

$$
\begin{array}{lllllllll}
A & ::= & \top & \mid & A_1 \mathbin{\&} A_2 & \mid & 0 & \mid & A_1 \oplus A_2 \\
 & \mid & 1_{\mathsf{L}} & \mid & A_1 \otimes A_2 & \mid & \bot_{\mathsf{L}} & \mid & A_1 \parr A_2 \\
 & \mid & F_! P & \mid & F_? C & & & & \\
P & ::= & 1_{\mathsf{P}} & \mid & P_1 \otimes P_2 & \mid & \lceil A \rceil & & \\
C & ::= & \bot_{\mathsf{C}} & \mid & C_1 \parr C_2 & \mid & \lfloor A \rfloor & &
\end{array}
$$

The syntactic form of a proposition is called its *mode*—linear L, producing P or consuming C. The meta-variable X ranges over propositions of any mode, and the tagged meta-variable X^{m} ranges over propositions of mode m. The term *persistent* refers to either producer or consumer propositions.

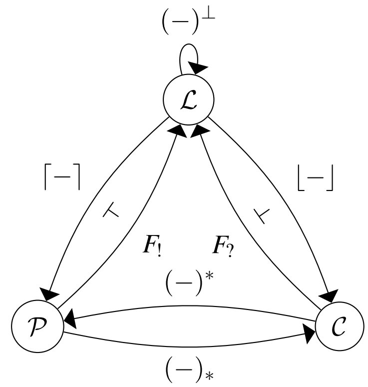

Figure 2: Categorical model of classical linear logic with linear, producing and consuming categories.

LPC replaces the usual constructors ! and ? with two pairs of connectives: $F_!$ and $\lceil - \rceil$ for ! and $F_?$ and $\lfloor - \rfloor$ for ?. If A is a linear proposition, $\lceil A \rceil$ is a producer and $\lfloor A \rfloor$ is a consumer. On the other hand, a producer proposition P may be "frozen" into a linear proposition $F_! P$, effectively discarding its persistent characteristics. Similarly for a consumer C, $F_? C$ is linear. The linear propositions $!A$ and $?A$ are encoded in this system as $F_! (\lceil A \rceil)$ and $F_? (\lfloor A \rfloor)$ respectively.

The inference rules of the logic are shown in Figures 3 and 4. There are two judgments: the linear sequent $\Gamma \vdash \Delta$ and the persistent sequent $\Gamma \Vdash \Delta$. In the linear sequent, the (unordered) contexts Γ and Δ may be made up of propositions of any mode; in the persistent sequent, the contexts may contain only persistent propositions. The meta-variable Γ^{P} refers to contexts made up entirely of producer propositions, and Δ^{C} refers to contexts of consumer propositions.

The linear inference rules in Figures 3 and 4a encompass rules for the units and the linear operators $\oplus$, &, $\otimes$ and $\parr$. It is worth noting that the multiplicative product $\otimes$ is defined only on linear and producer propositions, while the multiplicative sum $\parr$ is defined only on linear and consumer propositions.[1]

Weakening and contraction can be applied for producers on the left-hand-side and consumers on the right-hand-side of both the linear and persistent sequents. For producers, that is:

$$
\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{W}^{\vdash}\text{-L}
\qquad
\frac{\Gamma \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{W}^{\Vdash}\text{-L}
\qquad
\frac{\Gamma, P, P \vdash \Delta}{\Gamma, P \vdash \Delta}\ \mathrm{C}^{\vdash}\text{-L}
\qquad
\frac{\Gamma, P, P \Vdash \Delta}{\Gamma, P \Vdash \Delta}\ \mathrm{C}^{\Vdash}\text{-L}
$$

The rules for the operators $F_!$, $F_?$, $\lceil - \rceil$ and $\lfloor - \rfloor$ are given in Figure 4b. These rules encode dereliction and promotion for ! and ? by passing through the adjunction. For example:

$$
\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}
\quad \Rightarrow \quad
\cfrac{\cfrac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}}{\Gamma, F_! \lceil A \rceil \vdash \Delta}
\qquad\qquad
\frac{\Gamma^! \vdash \Delta^?, A}{\Gamma^! \vdash \Delta^?, !A}
\quad \Rightarrow \quad
\cfrac{\cfrac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! \lceil A \rceil}
$$

Displacement. The commas on the left-hand-side of both the linear and persistent sequents intuitively correspond to the $\otimes$ operator, and the commas on the right correspond to $\parr$. This correspondence motivates the context restriction in the rules that move between the linear and persistent regimes. The

[1] The persistent operators in this paper are necessary for LPC, but in general are not restricted to the sum and product. Other operators, like $\rightarrow$ or $\vee$, could be incorporated so long as every producer operator has a dual for consumers.

$$\frac{}{X \vdash X}\ \mathrm{Ax}^{\vdash} \qquad \frac{}{\Gamma \vdash \Delta, \top}\ \top^{\vdash}_{\mathsf{L}}\text{-R} \qquad \frac{}{\Gamma, 0 \vdash \Delta}\ 0^{\vdash}_{\mathsf{L}}\text{-L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L1} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta}\ \&^{\vdash}_{\mathsf{L}}\text{-L2} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R1} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \oplus B}\ \oplus^{\vdash}_{\mathsf{L}}\text{-R2}$$

$$\frac{\Gamma \vdash \Delta, A_1 \quad \Gamma \vdash \Delta, A_2}{\Gamma \vdash \Delta, A_1 \,\&\, A_2}\ \&^{\vdash}_{\mathsf{L}}\text{-R} \qquad \frac{\Gamma, A_1 \vdash \Delta \quad \Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \oplus A_2 \vdash \Delta}\ \oplus^{\vdash}_{\mathsf{L}}\text{-L}$$

$$\frac{\Gamma, X_1{}^{\mathsf{m}}, X_2{}^{\mathsf{m}} \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma, (X_1 \otimes X_2)^{\mathsf{m}} \vdash \Delta}\ \otimes^{\vdash}\text{-L} \qquad \frac{\Gamma_1 \vdash \Delta_1, X_1{}^{\mathsf{m}} \quad \Gamma_2 \vdash \Delta_2, X_2{}^{\mathsf{m}} \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, (X_1 \otimes X_2)^{\mathsf{m}}}\ \otimes^{\vdash}\text{-R}$$

$$\frac{\Gamma \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\Gamma, 1_{\mathsf{m}} \vdash \Delta}\ 1^{\vdash}\text{-L} \qquad \frac{\mathsf{m} \in \{\mathsf{L}, \mathsf{P}\}}{\cdot \vdash 1_{\mathsf{m}}}\ 1^{\vdash}\text{-R}$$

$$\frac{\Gamma_1, X_1{}^{\mathsf{m}} \vdash \Delta_1 \quad \Gamma_2, X_2{}^{\mathsf{m}} \vdash \Delta_2 \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma_1, \Gamma_2, (X_1 \parr X_2)^{\mathsf{m}} \vdash \Delta_1, \Delta_2}\ \parr^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta, X_1{}^{\mathsf{m}}, X_2{}^{\mathsf{m}} \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma \vdash \Delta, (X_1 \parr X_2)^{\mathsf{m}}}\ \parr^{\vdash}\text{-R}$$

$$\frac{\mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\bot_{\mathsf{m}} \vdash \cdot}\ \bot^{\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta \quad \mathsf{m} \in \{\mathsf{L}, \mathsf{C}\}}{\Gamma \vdash \Delta, \bot_{\mathsf{m}}}\ \bot^{\vdash}\text{-R}$$

Figure 3: Inference Rules for Linear Sequent

restriction ensures that almost all of the propositions have the "natural" mode—producers on the left and consumers on the right. We say "almost" because the principal formula in each of these rules defies this classification. We call such propositions *displaced*.

Definition 1. *In a derivation of* $\Gamma \Vdash \Delta$*, a producer P is displaced if it appears in* Δ*. A consumer C is displaced if it appears in* Γ*.*

Proposition 2 (Displacement)**.** *Every derivation of* $\Gamma \Vdash \Delta$ *contains exactly one displaced proposition.*

Cut. The cut rules are presented in Figure 4c. The rules with persistent cut terms have the property that whenever the cut term is displaced in a subderivation, that derivation must be persistent and satisfy the restrictions of Proposition 2. Lacking this restriction, simpler formulations of the cut rules disallow cut admissibility.

To show admissibility of the CUT rules, it is sufficient to show admissibility of an equivalent set of rules called CUT+. The versions differ in their treatment of cut terms. The CUT+ formulation uses the observation that when a persistent proposition is *not* displaced in a sequent, it can be replicated any number of times. Let $(X)_n$ be n copies of a proposition X. It is easy to see that the following propositions are admissible in the linear sequent (and similarly for the persistent sequent):

$$\frac{\Gamma, (P)_n \vdash \Delta}{\Gamma, P \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, (C)_n}{\Gamma \vdash \Delta, C}$$

$$\frac{}{P \Vdash P}\ \textsc{Ax}^{\Vdash}_{\mathsf{P}} \qquad \frac{}{C \Vdash C}\ \textsc{Ax}^{\Vdash}_{\mathsf{C}}$$

$$\frac{\Gamma, P_1, P_2 \Vdash \Delta}{\Gamma, P_1 \otimes P_2 \Vdash \Delta}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{\Gamma_1 \Vdash \Delta_1, P_1 \quad \Gamma_2 \Vdash \Delta_2, P_2}{\Gamma_1, \Gamma_2 \Vdash \Delta_1, \Delta_2, P_1 \otimes P_2}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-R}$$

$$\frac{\Gamma \Vdash \Delta}{\Gamma, 1_{\mathsf{P}} \Vdash \Delta}\ 1^{\Vdash}_{\mathsf{P}}\text{-L} \qquad \frac{}{\cdot \Vdash 1_{\mathsf{P}}}\ 1^{\Vdash}_{\mathsf{P}}\text{-R}$$

$$\frac{\Gamma_1, C_1 \Vdash \Delta_1 \quad \Gamma_2, C_2 \Vdash \Delta_2}{\Gamma_1, \Gamma_2, C_1 \parr C_2 \Vdash \Delta_1, \Delta_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C_1, C_2}{\Gamma \Vdash \Delta, C_1 \parr C_2}\ \parr^{\Vdash}_{\mathsf{C}}\text{-R}$$

$$\frac{}{\bot_{\mathsf{C}} \Vdash \cdot}\ \bot^{\Vdash}_{\mathsf{C}}\text{-L} \qquad \frac{\Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, \bot_{\mathsf{C}}}\ \bot^{\Vdash}_{\mathsf{C}}\text{-R}$$

(a) Inference Rules for Persistent Sequent

$$\frac{\Gamma, P \vdash \Delta}{\Gamma, F_! P \vdash \Delta}\ F_!\text{-L} \qquad \frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P}{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, F_! P}\ F_!\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}}, C \Vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, F_? C \vdash \Delta^{\mathsf{C}}}\ F_?\text{-L} \qquad \frac{\Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, F_? C}\ F_?\text{-R}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}\ \lceil - \rceil\text{-L} \qquad \frac{\Gamma^{\mathsf{P}} \vdash \Delta^{\mathsf{C}}, A}{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, \lceil A \rceil}\ \lceil - \rceil\text{-R}$$

$$\frac{\Gamma^{\mathsf{P}}, A \vdash \Delta^{\mathsf{C}}}{\Gamma^{\mathsf{P}}, \lfloor A \rfloor \Vdash \Delta^{\mathsf{C}}}\ \lfloor - \rfloor\text{-L} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, \lfloor A \rfloor}\ \lfloor - \rfloor\text{-R}$$

(b) Adjunction Inference Rules

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textsc{Cut}^{\vdash}_{\mathsf{L}}$$

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad P, \Gamma_2 \vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}^{\vdash}_{\mathsf{P}} \qquad \frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad P, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut}^{\Vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1 \vdash \Delta_1, C \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut}^{\vdash}_{\mathsf{C}} \qquad \frac{\Gamma_1 \Vdash \Delta_1, C \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut}^{\Vdash}_{\mathsf{C}}$$

(c) Cut Inference Rules

Figure 4: Persistent and Auxiliary Inference Rules

Thus the Cut+ rules, given below, are equivalent to the Cut rules.

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \textsc{Cut+}^{\vdash}_{\mathsf{L}}$$

$$\frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut+}^{\vdash}_{\mathsf{P}} \qquad \frac{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P \quad (P)_n, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^{\mathsf{P}}, \Gamma_2 \Vdash \Delta_1^{\mathsf{C}}, \Delta_2}\ \textsc{Cut+}^{\Vdash}_{\mathsf{P}}$$

$$\frac{\Gamma_1 \vdash \Delta_1, (C)_n \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut+}^{\vdash}_{\mathsf{C}} \qquad \frac{\Gamma_1 \Vdash \Delta_1, (C)_n \quad C, \Gamma_2^{\mathsf{P}} \Vdash \Delta_2^{\mathsf{C}}}{\Gamma_1, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1, \Delta_2^{\mathsf{C}}}\ \textsc{Cut+}^{\Vdash}_{\mathsf{C}}$$

Lemma 3 (Cut+ Admissibility)**.** *The* Cut+ *rules are admissible in* LPC.

Proof. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the hypotheses of one of the cut rules. The proof is by induction on the cut term primarily and the sum of the depths of $\mathcal{D}_1$ and $\mathcal{D}_2$ secondly.

1. Suppose $\mathcal{D}_1$ or $\mathcal{D}_2$ ends in a weakening or contraction rule on the cut term. In particular, consider the weakening case where the cut term is a producer and $\mathcal{D}_2$ is a linear judgment. In this case $\mathcal{D}_1$ is a derivation of $\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P$ and $\mathcal{D}_2$ is the derivation shown to the right. By the inductive hypothesis on P, $\mathcal{D}_1$ and $\mathcal{D}_2'$, there exists a cut-free derivation of $\Gamma_1^{\mathsf{P}}, \Gamma_2 \vdash \Delta_1^{\mathsf{C}}, \Delta_2$.

$$\mathcal{D}_2 = \frac{\dfrac{\mathcal{D}_2'}{\Gamma_2, (P)_n \vdash \Delta_2}}{\Gamma_2, (P)_{n+1} \vdash \Delta_2}\ \text{W-L}$$

2. If $\mathcal{D}_1$ or $\mathcal{D}_2$ is an axiom, the case is trivial.

3. Suppose the cut term is the principle formula in both $\mathcal{D}_1$ and $\mathcal{D}_2$ (excluding weakening and contraction rules). We consider a few of the subcases here:

($\otimes_{\mathsf{L}}$)
$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_{11}}{\Gamma_{11} \vdash \Delta_{11}, A_1} \qquad \cfrac{\mathcal{D}_{12}}{\Gamma_{12} \vdash \Delta_{12}, A_2}}{\Gamma_{11}, \Gamma_{12} \vdash \Delta_{11}, \Delta_{12}, A_1 \otimes A_2}\ \otimes^{\vdash}_{\mathsf{L}}\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \cfrac{\cfrac{\mathcal{D}_2'}{\Gamma_2, A_1, A_2 \vdash \Delta_2}}{\Gamma_2, A_1 \otimes A_2 \vdash \Delta_2}\ \otimes^{\vdash}_{\mathsf{L}}\text{-L}$$

By the inductive hypothesis on A_2, $\mathcal{D}_{12}$ and $\mathcal{D}_2'$, there exists a derivation $\mathcal{E}$ of $\Gamma_{12}, \Gamma_2, A_1 \vdash \Delta_{12}, \Delta_2$. Then the desired derivation of $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \vdash \Delta_{11}, \Delta_{12}, \Delta_2$ exists by the inductive hypothesis on A_1, $\mathcal{D}_{11}$ and $\mathcal{E}$.

($\otimes_{\mathsf{P}}$)
$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_{11}}{\Gamma_{11}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, P_1} \qquad \cfrac{\mathcal{D}_{12}}{\Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{12}^{\mathsf{C}}, P_2}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, P_1 \otimes P_2}\ \otimes^{\Vdash}_{\mathsf{P}}\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \cfrac{\cfrac{\mathcal{D}_2'}{\Gamma_2, (P_1 \otimes P_2)_n, P_1, P_2 \vdash \Delta_2}}{\Gamma_2, (P_1 \otimes P_2)_{n+1} \vdash \Delta_2}\ \otimes^{\vdash}_{\mathsf{P}}\text{-L}$$

The inductive hypothesis on $P_1 \otimes P_2$, $\mathcal{D}_1$ itself and $\mathcal{D}_2'$ gives us a derivation $\mathcal{E}$ of

$$\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1, P_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2.$$

Multiple applications of the inductive hypothesis give the following derivation:

$$\cfrac{\cfrac{\mathcal{D}_{11}}{\Gamma_{11}^{\mathsf{P}} \Vdash \Delta_{11}^{\mathsf{C}}, P_1} \qquad \cfrac{\cfrac{\mathcal{D}_{12}}{\Gamma_{12}^{\mathsf{P}} \Vdash \Delta_{12}^{\mathsf{C}}, P_2} \qquad \cfrac{\mathcal{E}}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1, P_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}}{\Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2, P_1 \vdash \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}\ \text{IH}(P_2)}{\Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_{11}^{\mathsf{P}}, \Gamma_{12}^{\mathsf{P}}, \Gamma_2 \vdash \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_{11}^{\mathsf{C}}, \Delta_{12}^{\mathsf{C}}, \Delta_2}\ \text{IH}(P_1)$$

Because the replicated contexts are made up exclusively of non-displaced propositions, it is possible to apply contraction multiple times to obtain the desired sequent.

($F_!$)
$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, P}}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, F_! P}\ F_!\text{-R} \qquad \text{and} \qquad \mathcal{D}_2 = \cfrac{\cfrac{\mathcal{D}_2'}{\Gamma_2, P \vdash \Delta}}{\Gamma_2, F_! P \vdash \Delta}\ F_!\text{-L}$$

Because $\mathcal{D}_1'$ is a persistent derivation, we can apply the inductive hypothesis for P with $n = 1$ to obtain the desired derivation.

4. Suppose the cut term is *not* the principle formula in $\mathcal{D}_1$ or $\mathcal{D}_2$. Most of the subcases are straightforward in that the last rule in the derivation commutes with the inductive hypotheses.

If the cut term is a producer, then $\mathcal{D}_1$ is a persistent judgment so it cannot be the case that the last rule of $\mathcal{D}_1$ is a $F_!$ rule or a $\lceil - \rceil$-L or $\lfloor - \rfloor$-R rule. But it also cannot be the case that the last rule in $\mathcal{D}_1$ is a in a $\lceil - \rceil$-R or $\lfloor - \rfloor$-L rule because there is a non-principle formula—namely, the cut formula—which is in a displaced position.

Suppose on the other hand that the cut term is a consumer and $\mathcal{D}_1$ is the derivation to the right. Then $\mathcal{D}_2$ is a derivation of $\Gamma_2^{\mathsf{P}}, C \Vdash \Delta_2^{\mathsf{C}}$. By the inductive hypothesis on C, $\mathcal{D}_1'$ and $\mathcal{D}_2$, there is a derivation $\mathcal{E}$ of $\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A, \Delta_2^{\mathsf{C}}$. Because the contexts in $\mathcal{D}_2$ were undisplaced, it is possible to apply the $\lceil - \rceil$-R rule to $\mathcal{E}$ to obtain a derivation of $\Gamma_1^{\mathsf{P}}, \Gamma_2^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, \lceil A \rceil, \Delta_2^{\mathsf{C}}$.

$$\mathcal{D}_1 = \cfrac{\cfrac{\mathcal{D}_1'}{\Gamma_1^{\mathsf{P}} \vdash \Delta_1^{\mathsf{C}}, A, (C)_n}}{\Gamma_1^{\mathsf{P}} \Vdash \Delta_1^{\mathsf{C}}, \lceil A \rceil, (C)_n}\ \lceil - \rceil\text{-R}$$

For the full proof of Lemma 3, see the accompanying technical report [14]. □

Theorem 4 (CUT Admissibility)**.** *The* CUT *rules in Figure 4c are admissible in* LPC.

Duality. Every rule in the LPC inference rules has a clear dual, but unlike standard presentations of classical linear logic, LPC does not contain an explicit duality operator $(-)^{\perp}$, nor a linear implication $\multimap$ with which to encode duality. Instead, we define $(-)^{\perp}$ to be a meta-operation on propositions and prove that the following duality rules are admissible in LPC:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}\text{-L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\ (-)^{\perp}\text{-R}$$

In fact, there are three versions of this duality operation: $(-)^{\perp}$ for linear propositions, $(-)^{*}$ for producers and $(-)_{*}$ for consumers. For a linear proposition A, $A^{\perp}$ is linear, but for a producer P, P^{*} is a consumer, and for a consumer C, C_{*} is a producer. We define these (invertible) duality operations as follows:

$$\top^{\perp} := 0 \qquad 1_{\mathsf{L}}^{\perp} := \bot_{\mathsf{L}} \qquad 1_{\mathsf{P}}^{*} := \bot_{\mathsf{C}} \qquad (F_{!}\,P)^{\perp} := F_{?}\,P^{*}$$
$$(A \,\&\, B)^{\perp} := A^{\perp} \oplus B^{\perp} \qquad (A \otimes B)^{\perp} := A^{\perp} \parr B^{\perp} \qquad (P \otimes Q)^{*} := P^{*} \parr Q^{*} \qquad \lceil A \rceil^{*} := \lfloor A^{\perp} \rfloor$$

We will show that the inference rules given in Figure 5 (as well as the respective right rules) are admissible in LPC.

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}\text{-L}$$

$$\frac{\Gamma \vdash \Delta, P}{\Gamma, P^{*} \vdash \Delta}\ (-)^{*\vdash}\text{-L} \qquad \frac{\Gamma \vdash \Delta, C}{\Gamma, C_{*} \vdash \Delta}\ (-)_{*}{}^{\vdash}\text{-L}$$

$$\frac{\Gamma \Vdash \Delta, P}{\Gamma, P^{*} \Vdash \Delta}\ (-)^{*\Vdash}\text{-L} \qquad \frac{\Gamma \Vdash \Delta, C}{\Gamma, C_{*} \Vdash \Delta}\ (-)_{*}{}^{\Vdash}\text{-L}$$

Figure 5: Left Duality Inference Rules

Lemma 5. *The following axioms hold in* LPC*:*

$$\frac{}{A, A^{\perp} \vdash \cdot} \qquad \frac{}{\cdot \vdash A, A^{\perp}} \qquad \frac{}{P, P^{*} \Vdash \cdot} \qquad \frac{}{\cdot \Vdash P, P^{*}}$$

Proof. By mutual induction on the proposition. □

The variations $P, P^{*} \vdash \cdot$ and $\cdot \vdash P, P^{*}$ on the other hand cannot be proved by induction because of the subcase $P = \lceil A \rceil$; there is no way to apply the inductive hypothesis to the goal $\lceil A \rceil, \lfloor A^{\perp} \rfloor \vdash \cdot$. However we can construct the desired derivations using cut rules:

$$\frac{\dfrac{}{P^{*} \vdash P^{*}} \qquad \dfrac{}{P, P^{*} \Vdash \cdot}}{P, P^{*} \vdash \cdot} \qquad \frac{\dfrac{}{\cdot \Vdash P, P^{*}} \qquad \dfrac{}{P \vdash P}}{\cdot \vdash P, P^{*}}$$

Theorem 6. *The duality rules in Figure 5 are admissible in* LPC.

Proof. Three of the rules can be generated by a straightforward application of cut:

$$\frac{\Gamma \vdash \Delta, A \qquad \dfrac{}{A, A^{\perp} \vdash \cdot}}{\Gamma, A^{\perp} \vdash \Delta}\ \textsc{Cut}_{\mathsf{L}}^{\vdash} \qquad \frac{\Gamma \vdash \Delta, C \qquad \dfrac{}{C, C_{*} \Vdash \cdot}}{\Gamma, C_{*} \vdash \Delta}\ \textsc{Cut}_{\mathsf{C}}^{\vdash} \qquad \frac{\Gamma \Vdash \Delta, C \qquad \dfrac{}{C, C_{*} \Vdash \cdot}}{\Gamma, C_{*} \Vdash \Delta}\ \textsc{Cut}_{\mathsf{C}}^{\Vdash}$$

When we try to do the same for the left producer rules, the context restriction around the displaced cut term leads to the following derivations:

$$\frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P \qquad \dfrac{}{P, P^{*} \Vdash \cdot}}{\Gamma^{\mathsf{P}}, P^{*} \Vdash \Delta^{\mathsf{C}}}\ \textsc{Cut}_{\mathsf{P}}^{\Vdash} \qquad \frac{\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P \qquad \dfrac{}{P, P^{*} \vdash \cdot}}{\Gamma^{\mathsf{P}}, P^{*} \vdash \Delta^{\mathsf{C}}}\ \textsc{Cut}_{\mathsf{P}}^{\vdash}$$

For the first of these, recall that due to displacement, every derivation of $\Gamma, P^{*} \Vdash \Delta$ in fact has the restriction that $\Gamma = \Gamma^{\mathsf{P}}$ and $\Delta = \Delta^{\mathsf{C}}$. So this derivation is actually equivalent to the one in Figure 5. The second derivation, on the other hand, is not equivalent to the one in Figure 5, nor an acceptable variant. The hypothesis and conclusion of the derivation are different kinds of sequents, and linear propositions are completely excluded from the contexts.

Instead we can prove the more general form of the rule directly: For any derivation $\mathcal{D}$ of $\Gamma \vdash \Delta, P$, there is a derivation of $\Gamma, P^* \vdash \Delta$. We prove this by induction on $\mathcal{D}$. Most of the cases commute directly with the inductive hypothesis, which the following exception: If $\mathcal{D}$ is the axiom $P \vdash P$ then there is a derivation of $P, P^* \vdash \cdot$, as expected. □

Consistency. Define the negation of a linear proposition to be $\neg A := A^\perp \parr 0$.

Theorem 7 (Consistency)**.** *There is no proposition A such that A and ¬A are both provable in* LPC.

Proof. Suppose there were such an A, along with derivations $\mathcal{D}_1$ of $\cdot \vdash A$ and $\mathcal{D}_2$ of $\cdot \vdash A^\perp \parr 0$. Then there exists a derivation of $\cdot \vdash 0$ as seen to the right. However, there is no cut-free proof of $\cdot \vdash 0$ in LPC, which contradicts cut admissibility. □

$$\dfrac{\dfrac{\mathcal{D}_2}{\cdot \vdash A^\perp, 0} \qquad \dfrac{\dfrac{\mathcal{D}_1}{\cdot \vdash A}}{A^\perp \vdash \cdot}\,(-)^\perp\text{-L}}{\cdot \vdash 0}\,\textsc{Cut}^{\vdash}_{\mathsf{L}}$$

3 Categorical Model

In this section we describe a categorical axiomatization of LPC based on the three-category Figure 2. Certain definitions have been omitted for brevity; these can be found in the companion paper [14].

Preliminaries. We start with some basic definitions about symmetric monoidal structures.

Definition 8. *A* symmetric monoidal category *is a category $\mathcal{C}$ equipped with a bifunctor $\otimes$, an object* 1*, and the following natural isomorphisms:*

$$\alpha_{A_1,A_2,A_3} : A_1 \otimes (A_2 \otimes A_3) \to (A_1 \otimes A_2) \otimes A_3 \qquad \lambda_A : 1 \otimes A \to A$$
$$\sigma_{A,B} : A \otimes B \to B \otimes A \qquad \rho_A : A \otimes 1 \to A$$

These must satisfy the following coherence conditions:

$$\begin{aligned} & A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) \xrightarrow{\alpha_{A_1 \otimes A_2, A_3, A_4} \circ \alpha_{A_1,A_2,A_3 \otimes A_4}} ((A_1 \otimes A_2) \otimes A_3) \otimes A_4 \xrightarrow{{\alpha_{A_1,A_2,A_3}}^{-1} \otimes id_{A_4}} (A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \\ = \; & A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) \xrightarrow{id_{A_1} \otimes \alpha_{A_2,A_3,A_4}} A_1 \otimes ((A_2 \otimes A_3) \otimes A_4) \xrightarrow{\alpha_{A_1,A_2 \otimes A_3, A_4}} (A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \end{aligned} \tag{1}$$

$$id_A \otimes \lambda_B = A \otimes (1 \otimes B) \xrightarrow{\alpha_{A,1,B}} (A \otimes 1) \otimes B \xrightarrow{\rho_A \otimes id_B} A \otimes B \tag{2}$$

$$\begin{aligned} & A_1 \otimes (A_2 \otimes A_3) \xrightarrow{id_{A_1} \otimes \sigma_{A_2,A_3}} A_1 \otimes (A_3 \otimes A_2) \xrightarrow{\alpha_{A_1,A_3,A_2}} (A_1 \otimes A_3) \otimes A_2 \xrightarrow{\sigma_{A_1,A_3} \otimes id_{A_2}} (A_3 \otimes A_1) \otimes A_2 \\ = \; & A_1 \otimes (A_2 \otimes A_3) \xrightarrow{\alpha_{A_1,A_2,A_3}} (A_1 \otimes A_2) \otimes A_3 \xrightarrow{\sigma_{A_1 \otimes A_2, A_3}} A_3 \otimes (A_1 \otimes A_2) \xrightarrow{\alpha_{A_3,A_1,A_2}} (A_3 \otimes A_1) \otimes A_2 \end{aligned} \tag{3}$$

$$id_{A \otimes B} = A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \xrightarrow{\sigma_{B,A}} A \otimes B \tag{4}$$

$$\lambda_A = 1_{\mathsf{L}} \otimes A \xrightarrow{\sigma_{1_{\mathsf{L}},A}} A \otimes 1_{\mathsf{L}} \xrightarrow{\rho_A} A \tag{5}$$

Definition 9. *Let* $(\mathcal{C},\otimes,1,\alpha,\lambda,\rho,\sigma)$ *and* $(\mathcal{C}',\otimes',1',\alpha',\lambda',\rho',\sigma')$ *be symmetric monoidal categories. A* symmetric monoidal functor $\mathrm{F}:\mathcal{C}\Rightarrow\mathcal{C}'$ *is a functor along with a map* $m_1^{\mathrm{F}}:1'\to\mathrm{F}\,1$ *and a natural transformation* $m_{A,B}^{\mathrm{F}}:\mathrm{F}(A)\otimes'\mathrm{F}(B)\to\mathrm{F}(A\otimes B)$ *that satisfy the following coherence conditions:*

$$\begin{array}{ccc}
(\mathrm{F}(A_1)\otimes'\mathrm{F}(A_2))\otimes'\mathrm{F}(A_3) & \xrightarrow{\alpha'} & \mathrm{F}(A_1)\otimes'(\mathrm{F}(A_2)\otimes'\mathrm{F}(A_3)) \\
\downarrow{\scriptstyle m^{\mathrm{F}}_{A_1,A_2}\otimes' id} & & \downarrow{\scriptstyle id\otimes' m^{\mathrm{F}}_{A_2,A_3}} \\
\mathrm{F}(A_1\otimes A_2)\otimes'\mathrm{F}(A_3) & & \mathrm{F}(A_1)\otimes'\mathrm{F}(A_2\otimes A_3) \\
\downarrow{\scriptstyle m^{\mathrm{F}}_{A_1\otimes A_2,A_3}} & & \downarrow{\scriptstyle m^{\mathrm{F}}_{A_1,A_2\otimes A_3}} \\
\mathrm{F}((A_1\otimes A_2)\otimes A_3) & \xrightarrow{\mathrm{F}(\alpha)} & \mathrm{F}(A_1\otimes(A_2\otimes A_3))
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{F}(A)\otimes'\mathrm{F}(B) & \xrightarrow{\sigma'} & \mathrm{F}(B)\otimes'\mathrm{F}(A) \\
\downarrow{\scriptstyle m^{\mathrm{F}}_{A,B}} & & \downarrow{\scriptstyle m^{\mathrm{F}}_{B,A}} \\
\mathrm{F}(A\otimes B) & \xrightarrow{\mathrm{F}(\sigma)} & \mathrm{F}(B\otimes A)
\end{array}$$

$$\begin{array}{ccc}
1'\otimes'\mathrm{F}(A) & \xrightarrow{\lambda'} & \mathrm{F}(A) \\
\downarrow{\scriptstyle m^{\mathrm{F}}_1\otimes' id} & & \uparrow{\scriptstyle \mathrm{F}(\lambda_A)} \\
\mathrm{F}(1)\otimes'\mathrm{F}(A) & \xrightarrow[m^{\mathrm{F}}_{1,A}]{} & \mathrm{F}(1\otimes A)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{F}(A)\otimes'1' & \xrightarrow{\rho'} & \mathrm{F}(A) \\
\downarrow{\scriptstyle id\otimes' m^{\mathrm{F}}_1} & & \uparrow{\scriptstyle \mathrm{F}(\rho)} \\
\mathrm{F}(A)\otimes'\mathrm{F}(1) & \xrightarrow[m^{\mathrm{F}}_{A,1}]{} & \mathrm{F}(A\otimes 1)
\end{array}$$

A functor $\mathrm{F}:\mathcal{C}\Rightarrow\mathcal{C}'$ *is* symmetric comonoidal *if it is equipped with a map* $n_1^{\mathrm{F}}:\mathrm{F}\,1\to 1'$ *and natural transformation* $n_{A,B}^{\mathrm{F}}:\mathrm{F}(A\otimes B)\to\mathrm{F}(A)\otimes'\mathrm{F}(B)$ *such that the appropriate (dual) diagrams commute.*

Definition 10. *Let* F *and* G *be symmetric monoidal functors* $\mathrm{F},\mathrm{G}:\mathcal{C}\Rightarrow\mathcal{C}'$. *A monoidal natural transformation* $\tau:\mathrm{F}\to\mathrm{G}$ *is a natural transformation satisfying*

$$\tau_{A\otimes B}\circ m^{\mathrm{F}}_{A,B}=m^{\mathrm{G}}_{A,B}\circ(\tau_A\otimes'\tau_B) \qquad \textit{and} \qquad \tau_{1_{\mathsf{L}}}\circ m^{\mathrm{F}}_1=m^{\mathrm{G}}_1.$$

For F *and* G *symmetric comonoidal functors, a natural transformation* $\tau:\mathrm{F}\to\mathrm{G}$ *is comonoidal if it satisfies the appropriate dual diagrams.*

Definition 11. *A* symmetric (co-)monoidal adjunction *is an adjunction* $\mathrm{F}\dashv\mathrm{G}$ *between symmetric (co-)monoidal functors* F *and* G *where the unit and counit of the adjunction are symmetric (co-)monoidal natural transformations.*

The LPC model. Traditionally the multiplicative fragment of linear logic is modeled by a *-autonomous category. For LPC, we use an equivalent notion that puts the tensor $\otimes$ and co-tensor $\invamp$ on equal footing, by modeling the category $\mathcal{L}$ as a symmetric linearly distributive category with negation [5].

Definition 12. *Let* $\mathcal{L}$ *be a category with two symmetric monoidal structures* $\otimes$ *and* $\invamp$, *and a natural transformation*

$$\delta_{A_1,A_2,A_3}:A_1\otimes(A_2\invamp A_3)\to(A_1\otimes A_2)\invamp A_3$$

Then $\mathcal{L}$ *is a* symmetric linearly distributive category *if* δ *satisfies a number of coherence conditions described by Cockett and Seely [5].*

$\mathcal{L}$ *is said to* have negation *if there exists a map* $(-)^{\perp}$ *on objects of* $\mathcal{L}$ *and families of maps*

$$\gamma^{\perp}_A:A^{\perp}\otimes A\to\perp_{\mathsf{L}} \qquad \textit{and} \qquad \gamma^{1}_A:1_{\mathsf{L}}\to A\invamp A^{\perp}$$

commuting with δ *in certain ways.*

Theorem 13 (Cockett and Seely)**.** *Symmetric linearly distributive categories with negation correspond to *-autonomous categories.*

Definition 14. *A linear/producing/consuming (*LPC*) model consists of the following components:*

1. *A symmetric linearly distributive category $(\mathcal{L}, \otimes, ⅋)$ with negation $(-)^\perp$, finite products $\&$ and finite coproducts $\oplus$.*
2. *Symmetric monoidal categories $(\mathcal{P}, \otimes)$ and $(\mathcal{C}, ⅋)$ in duality by means of contravariant functors $(-)^* : \mathcal{P} \Rightarrow \mathcal{C}$ and $(-)_* : \mathcal{C} \Rightarrow \mathcal{P}$, where $(-)^*$ is monoidal and $(-)_*$ is comonoidal, with natural isomorphisms*
$$\varepsilon^*_{*C} : (C_*)^* \to C \qquad \text{and} \qquad \eta^*_{*P} : P \to (P^*)_*.$$
3. *Monoidal natural transformations $e^{\otimes}_P : P \to 1_{\mathsf{P}}$ and $d^{\otimes}_P : P \to P \otimes P$ in $\mathcal{P}$ and comonoidal natural transformations $e^{⅋}_C : \bot_{\mathsf{C}} \to C$ and $d^{⅋}_C : C ⅋ C \to C$ in $\mathcal{C}$, interchanged under duality, such that:*
   (a) for every P, $(P, d^{\otimes}_P, e^{\otimes}_P)$ forms a commutative comonoid in $\mathcal{P}$; and
   (b) for every C, $(C, d^{⅋}_C, e^{⅋}_C)$ forms a commutative monoid in $\mathcal{C}$.
4. *Symmetric monoidal functors $\lceil - \rceil : \mathcal{L} \Rightarrow \mathcal{P}$ and $F_! : \mathcal{P} \Rightarrow \mathcal{L}$ and symmetric comonoidal functors $\lfloor - \rfloor : \mathcal{L} \Rightarrow \mathcal{C}$ and $F_? : \mathcal{C} \Rightarrow \mathcal{L}$, which respect the dualities in that $(F_! P)^\perp \simeq F_?(P^*)$ and $\lceil A \rceil \simeq \lfloor A^\perp \rfloor$, and that form monoidal/comonoidal adjunctions $\lceil - \rceil \dashv F_!$ and $F_? \dashv \lfloor - \rfloor$.*

To unpack condition (3), consider the definition of a commutative comonoid:

Definition 15. *Let $(\mathcal{P}, \otimes, 1_{\mathsf{P}})$ be a symmetric monoidal category. A commutative comonoid in $\mathcal{P}$ is an object P in $\mathcal{P}$ along with two morphisms $e^{\otimes} : P \to 1_{\mathsf{P}}$ and $d^{\otimes} : P \to P \otimes P$ that commute with the symmetric monoidal structure of $\mathcal{P}$. Dually, a commutative monoid in a symmetric monoidal category $(\mathcal{C}, ⅋, \bot_{\mathsf{C}})$ is an object C along with morphisms $e^{⅋} : \bot_{\mathsf{C}} \to C$ and $d^{⅋} : C ⅋ C \to C$.*

The commutative comonoids in $\mathcal{P}$ ensure that all propositions are duplicable in the producer category. This property is then preserved by the exponential decomposition $F_!$, leading to the property that linear propositions of the form $!A = F_! \lceil A \rceil$ are similarly duplicable.

Because $\lceil - \rceil \dashv F_!$ forms a monoidal adjunction, $F_!$ is necessarily a strong monoidal functor[**?**], which implies that $F_!$ is both monoidal and comonoidal. A similar result can be stated for $F_?$.

LPC and other linear logic models. As LPC is inspired by Benton's linear/non-linear paradigm, this section formalizes the relationship between LPC, LNL, and single-category models of linear logic.

Definition 16 (Melliès [12])**.** *A linear/non-linear (*LNL*) model consists of: (1) a symmetric monoidal closed category $\mathcal{L}$; (2) a cartesian category $\mathcal{P}$; and (3) functors $\mathrm{G} : \mathcal{L} \Rightarrow \mathcal{P}$ and $\mathrm{F} : \mathcal{P} \Rightarrow \mathcal{L}$ that form a symmetric monoidal adjunction $\mathrm{F} \dashv \mathrm{G}$.*[2]

In LPC, because every object in $\mathcal{P}$ forms a commutative comonoid, $\mathcal{P}$ is cartesian [8]. Therefore:

Proposition 17. *Every* LPC *model is an* LNL *model.*

In addition, a *-autonomous category in a linear/non-linear model induces an LPC triple:

Proposition 18. *If the category $\mathcal{L}$ in an* LNL *model is *-autonomous, then $(\mathcal{L}, \mathcal{P}, \mathcal{P}^{op})$ is an* LPC *model.*

Next we prove that every LPC model contains a classical linear category as defined by Schalk [16]. This definition is just the extension of Benton et al's linear category [2] to classical linear logic.

Definition 19 (Schalk [16])**.** *A category $\mathcal{L}$ is a model for classical linear logic if and only if it: (1) is *-autonomous; (2) has finite products $\&$ and thus finite coproducts $\oplus$; and (3) has a linear exponential comonad $!$ and thus a linear exponential monad $?$.*

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[2]The LNL model given by Benton [3] has the added stipulation that the cartesian category be cartesian closed, but other works have since disregarded this condition.

Proposition 20. *The category $\mathcal{L}$ from the* LPC *model is a model for classical linear logic.*

Proof. From Theorem 13 we know that $\mathcal{L}$ is *-autonomous, and by construction it has finite products and coproducts. Because the LPC model is also an LNL model, we may apply Benton's proof that every LNL model has a linear exponential comonad [3]. □

Proposition 21. *Every model for classical linear logic forms an* LPC *category.*

Proof. Benton proved that every SMCC with a linear exponential comonad has an LNL model. Because the linear category is *-autonomous the LNL model induces an LPC model. □

Interpretation of the Logic. We define an interpretation of the LPC logic that maps propositions to objects in the categories, and derivations to morphisms. For objects, the $[\![-]\!]_{\mathsf{L}}$ interpretation function maps any mode of proposition into the linear category. The interpretation of linear propositions is straightforward, and for persistent propositions we define

$$[\![P]\!]_{\mathsf{L}} = F_! [\![P]\!]_{\mathsf{P}} \qquad [\![C]\!]_{\mathsf{L}} = F_? [\![C]\!]_{\mathsf{C}}.$$

The functions $[\![-]\!]_{\mathsf{P}}$ and $[\![-]\!]_{\mathsf{C}}$ map propositions into the producer and the consumer categories $\mathcal{P}$ and $\mathcal{C}$ respectively, but they are defined only on the persistent propositions. To map producer propositions into the consumer category and vice versa, we define

$$[\![C]\!]_{\mathsf{P}} = ([\![C]\!]_{\mathsf{C}})_* \qquad [\![P]\!]_{\mathsf{C}} = ([\![P]\!]_{\mathsf{P}})^*.$$

Linear contexts are interpreted as a single proposition in the linear category. The comma is represented by the tensor connector $\otimes$ if the context is meant to appear on the left-hand-side of a sequent, and by the cotensor $⅋$ if the context is meant to appear on the right. These interpretations of linear contexts are represented as $[\![\Gamma]\!]_{\mathsf{L}}^{\otimes}$ and $[\![\Delta]\!]_{\mathsf{L}}^{⅋}$ respectively. In the producer category there is no cotensor and vice versa for the consumer category, so $[\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}}$ interprets the comma as the tensor in the producer category, and $[\![\Gamma^{\mathsf{C}}]\!]_{\mathsf{C}}$ interprets the comma as the cotensor in the consumer category.

In this way a linear derivation $\mathcal{D}$ of the form $\Gamma \vdash \Delta$ will be interpreted as a morphism $[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \to [\![\Delta]\!]_{\mathsf{L}}^{⅋}$. However, it is not clear in which category we should interpret a persistent sequent of the form $\Gamma \Vdash \Delta$, since Γ and Δ may contain both producer and consumer propositions. Recall Proposition 2, which states that every such derivation $\mathcal{D}$ contains exactly one displaced propositions. This means that $\mathcal{D}$ is either of the form $\Gamma^{\mathsf{P}} \Vdash \Delta^{\mathsf{C}}, P$ or $\Gamma^{\mathsf{P}}, C \Vdash \Delta^{\mathsf{C}}$. In the category $\mathcal{P}$, this derivation will be interpreted as a morphism

$$[\![\mathcal{D}]\!]_{\mathsf{P}} : [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \to [\![P]\!]_{\mathsf{P}} \quad \text{or} \quad [\![\mathcal{D}]\!]_{\mathsf{P}} : [\![\Gamma^{\mathsf{P}}]\!]_{\mathsf{P}} \otimes [\![\Delta^{\mathsf{C}}]\!]_{\mathsf{P}} \to [\![C]\!]_{\mathsf{P}},$$

respectively. In the same way every derivation can be interpreted as a morphism in $\mathcal{C}$.

The interpretation is defined by mutual induction on the derivations.

1. The interpretation of the linear inference rules given in Figure 3 as well as the persistent rules in Figure 4a are straightforward from the categorical structures.

2. The interpretation of weakening and contraction rules is defined using the monoid in $\mathcal{C}$ and comonoid in $\mathcal{P}$. For weakening in the linear sequent, suppose $\mathcal{D}$ is the derivation to the right. The interpretation of $\mathcal{D}$ inserts the comonoidal component $e^{\otimes}$ in $\mathcal{P}$ into the linear category:

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma \vdash \Delta}}{\Gamma, P \vdash \Delta}\,\mathrm{W}^{\vdash}\text{-L}$$

$$[\![\mathcal{D}]\!]_{\mathsf{L}} : [\![\Gamma]\!]_{\mathsf{L}}^{\otimes} \otimes F_! [\![P]\!]_{\mathsf{P}} \xrightarrow{[\![\mathcal{D}]\!]_{\mathsf{L}} \otimes F_! e^{\otimes}} [\![\Delta]\!]_{\mathsf{L}}^{⅋} \otimes F_! 1_{\mathsf{P}} \xrightarrow{\mathrm{id} \otimes (m^{F_!})^{-1}} [\![\Delta]\!]_{\mathsf{L}}^{⅋} \otimes 1_{\mathsf{L}} \xrightarrow{\rho^{\otimes}} [\![Q]\!]_{\mathsf{P}}$$

3. If the last rule in the derivation is an $F_!$-L or $F_?$-R rule, its interpretation is just the interpretation of its subderivation. On the other hand, if the last rule is the right $F_!$ rule, the The inductive hypothesis states that there exists a morphism $[\![\mathcal{D}']\!]_\mathsf{P} : [\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes [\![\Delta^\mathsf{C}]\!]_\mathsf{P} \to [\![P]\!]_\mathsf{P}$. It is necessary to undo this duality transformation for interpretation in the linear category.

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma^\mathsf{P} \Vdash \Delta^\mathsf{C}, P}}{\Gamma^\mathsf{P} \vdash \Delta^\mathsf{C}, F_!\, P}\ F_!\text{-R}$$

Notice that for any persistent context Γ, there is an isomorphism $\pi : [\![\Gamma]\!]_\mathsf{L}^\otimes \cong F_!\,[\![\Gamma]\!]_\mathsf{P}$ given by the monoidal components of $F_!$. Furthermore, there is an isomorphism τ between $([\![\Gamma]\!]_\mathsf{L}^\parr)^\perp$ and $F_!\,[\![\Gamma]\!]_\mathsf{P}$ given by the isomorphism $(F_?\,C)^\perp \cong F_!\,C_*$. Using π and τ we define the interpretation of $\mathcal{D}$:

$$\begin{array}{llllll}
[\![\mathcal{D}]\!]_\mathsf{L} : [\![\Gamma^\mathsf{P}]\!]_\mathsf{L}^\otimes & \xrightarrow{\rho^\otimes;(\mathrm{id}\otimes\gamma^1)} & [\![\Gamma^\mathsf{P}]\!]_\mathsf{L}^\otimes \otimes (([\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr)^\perp \parr [\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr) & \xrightarrow{\pi\otimes(\tau\parr\mathrm{id})} & F_!\,[\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes (F_!\,[\![\Delta^\mathsf{C}]\!]_\mathsf{P} \parr [\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr) \\
& \xrightarrow{\delta^{\mathrm{L,L}}} & (F_!\,[\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes F_!\,[\![\Delta^\mathsf{C}]\!]_\mathsf{P}) \parr [\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr & \xrightarrow{m^{F_!}\parr\mathrm{id}} & F_!\,([\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes [\![\Delta^\mathsf{C}]\!]_\mathsf{P}) \parr [\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr \\
& \xrightarrow{F_!\,[\![\mathcal{D}']\!]_\mathsf{P}\parr\mathrm{id}} & F_!\,[\![P]\!]_\mathsf{P} \parr [\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr & \xrightarrow{\sigma^\parr} & [\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr \parr [\![F_!\,P]\!]_\mathsf{L}
\end{array}$$

4. Suppose the last rule in $\mathcal{D}$ is the left $\lceil - \rceil$ rule. The interpretation of $\mathcal{D}$ should be a morphism from $[\![\Gamma]\!]_\mathsf{L}^\otimes \otimes F_!\,(\lceil [\![A]\!]_\mathsf{L} \rceil)$ to $[\![\Delta]\!]_\mathsf{L}^\parr$; we use the unit of the adjunction, $\varepsilon : F_!\,\lceil A \rceil \to A$ to cancel out the exponentials.

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma, A \vdash \Delta}}{\Gamma, \lceil A \rceil \vdash \Delta}\ \lceil - \rceil\text{-L}$$

$$[\![\mathcal{D}]\!]_\mathsf{L} : [\![\Gamma]\!]_\mathsf{L}^\otimes \otimes F_!\,(\lceil [\![A]\!]_\mathsf{L} \rceil) \xrightarrow{\mathrm{id}\otimes\varepsilon} [\![\Gamma]\!]_\mathsf{L}^\otimes \otimes [\![A]\!]_\mathsf{L} \xrightarrow{[\![\mathcal{D}']\!]_\mathsf{L}} [\![\Delta]\!]_\mathsf{L}^\parr$$

Similarly, the $\lceil - \rceil$-R rule uses the counit of the adjunction, along with the isomorphisms π and τ defined previously. If the last rule in $\mathcal{D}$ is the $\lceil - \rceil$-R rule, its interpretation is defined as follows:

$$\mathcal{D} = \dfrac{\dfrac{\mathcal{D}'}{\Gamma^\mathsf{P} \vdash \Delta^\mathsf{C}, A}}{\Gamma^\mathsf{P} \Vdash \Delta^\mathsf{C}, \lceil A \rceil}\ \lceil - \rceil\text{-R}$$

$$\begin{array}{llllll}
[\![\mathcal{D}]\!]_\mathsf{P} : [\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes [\![\Delta^\mathsf{C}]\!]_\mathsf{P} & \xrightarrow{\eta\otimes\eta} & \lceil F_!\,[\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \rceil \otimes \lceil F_!\,[\![\Delta^\mathsf{C}]\!]_\mathsf{P} \rceil & \xrightarrow{m^{\lceil - \rceil}} & \lceil F_!\,[\![\Gamma^\mathsf{P}]\!]_\mathsf{P} \otimes F_!\,[\![\Delta^\mathsf{C}]\!]_\mathsf{P} \rceil \\
& \xrightarrow{\lceil \pi^{-1}\otimes\tau^{-1} \rceil} & \lceil [\![\Gamma^\mathsf{P}]\!]_\mathsf{L}^\otimes \otimes ([\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr)^\perp \rceil & \xrightarrow{\lceil [\![\mathcal{D}']\!]_\mathsf{L}\otimes\mathrm{id} \rceil} & \lceil ([\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr \parr [\![A]\!]_\mathsf{L}) \otimes ([\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr)^\perp \rceil \\
& \xrightarrow{\lceil \delta^{\mathrm{R,L}} \rceil} & \lceil ([\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr \otimes ([\![\Delta^\mathsf{C}]\!]_\mathsf{L}^\parr)^\perp) \parr [\![A]\!]_\mathsf{L} \rceil & \xrightarrow{\gamma^\perp\parr\mathrm{id};\lceil \lambda^\parr \rceil} & \lceil [\![A]\!]_\mathsf{L} \rceil = [\![\lceil A \rceil]\!]_\mathsf{P}
\end{array}$$

4 Examples

This section provides some concrete instances of the LPC model. The following chart summarizes the three examples and their LPC categories.

	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{C}$
Vectors	FINVECT	FINSET	FINSET^{op}
Relations	REL	SET	SET^{op}
Bool. Alg.	FINBOOLALG	FINPOSET	FINLAT

Vector Spaces. Linear logic shares many features with linear algebra, based on the natural interpretations of the tensor product and duality of vector spaces. To construct an LPC model, let $\mathcal{L}$ be the category of finite-dimensional vector spaces over a finite field $\mathbb{F}$, $\mathcal{P}$ be the category of finite sets and functions, and $\mathcal{C}$ be the opposite category of $\mathcal{P}$.

The $\otimes$ operator of linear logic is easily interpreted as the tensor product in $\mathcal{L}$. The $\parr$ operator has no natural interpretation in terms of vector spaces, but we may define $U \parr V := U \otimes V$. The units 1_L and $\perp_\mathsf{L}$ may be any one-dimensional vector space; for concreteness let them be generated by the basis $\{\mathbb{1}\}$.

The free vector space $\mathbf{Free}(X)$ of a finite set X over $\mathbb{F}$ is the vector space with vectors the formal sums $\alpha_1 x_1 + \cdots + \alpha_n x_n$, addition defined pointwise, and scalar multiplication defined by distribution over the x_i's. A basis for $\mathbf{Free}(X)$ is the set $\{\delta_x \mid x \in X\}$ where δ_x is the free sum x.

The dual of a vector space V (with basis B over $\mathbb{F}$ is the set $V^\perp$ of linear maps from V to $\mathbb{F}$. For any vector $v \in V$, we can define $\overline{v} \in V^\perp$ to be the linear map acting on basis elements x of V by

$$\overline{v}[x] = \begin{cases} 1 & x = v \\ 0 & x \neq v \end{cases}$$

Addition and scalar multiplication are defined pointwise. Then $\{\overline{x} \mid x \in B\}$ is a basis for $V^\perp$.

The additives & and $\oplus$ are embodied by the notions of the direct product and direct sum, which in the case of finite-dimensional vector spaces, coincide.

Lemma 22. *The category* FINVECT *is a symmetric linearly distributive category with negation, products and coproducts.*

Proof. Since $\otimes$ and $\parr$ overlap, the distributivity transformation δ is simply associativity. The coherence diagrams for linear distribution then depend on the commutativity of tensor, associativity, and swap morphisms. To show the category has negation, we define $\gamma_A^\perp : A^\perp \otimes A \to \perp$ and $\gamma_A^1 : 1 \to A \parr A^\perp$ as follows, where B is a basis for A:

$$\gamma_A^\perp(\delta_u \otimes v) = \delta_u[v] \cdot \mathbb{1} \qquad\qquad \gamma_A^1(\mathbb{1}) = \sum_{v \in B} v \otimes \overline{v}$$

It then suffices to check that $\lambda \circ (\gamma^\perp \otimes \mathrm{id}) \circ \alpha \circ (\mathrm{id} \otimes \lambda) = \rho$. □

We will present only the adjunction between FINVECT and the producing category FINSET; the other can be inferred from the opposite category. Define $\lceil - \rceil$: FINVECT $\Rightarrow$ FINSET to be the forgetful functor, which takes a vector space to its underlying set of vectors. It is a monoidal functor with components $m_1^{\lceil - \rceil} : 1_{\mathsf{P}} \to \lceil 1 \rceil$ and $m_{A,B}^{\lceil - \rceil} : \lceil A \rceil \times \lceil B \rceil \to \lceil A \otimes B \rceil$, where

$$m_{1_{\mathsf{L}}}^{\lceil - \rceil}(\emptyset) = \mathbb{1} \qquad \text{and} \qquad m_{A,B}^{\lceil - \rceil}(u, v) = u \otimes v$$

On objects, the functor $F_!$: FINSET $\Rightarrow$ FINVECT takes a set X to the free vector space generated by X. For a morphism $f : X_1 \to X_2$ in FINSET, we define $F_! f : \mathbf{Free}(X_1) \to \mathbf{Free}(X_2)$ to be $F_! f(\delta_x) = \delta_{f(x)}$. Then $F_!$ is monoidal with components $m_1^{F_!} : 1 \to F_!\, 1_{\mathsf{P}}$ and $m_{X_1,X_2}^{F_!} : F_! X_1 \otimes F_! X_2 \to F_!(X_1 \times X_2)$ defined as

$$m_{1_{\mathsf{L}}}^{F_!}(\mathbb{1}) = \delta_\emptyset \qquad\qquad m_{X_1,X_2}^{F_!}(\delta_{x_1} \otimes \delta_{x_2}) = \delta_{(x_1,x_2)}$$

Lemma 23. *The functors* $\lceil - \rceil$ *and* $F_!$ *form a symmetric monoidal adjunction* $\lceil - \rceil \dashv F_!$.

Proof. We define the unit $\varepsilon_A : F_! \lceil A \rceil \to A$ and counit $\eta_P : P \to \lceil F_! P \rceil$ of the adjunction as follows:

$$\varepsilon_A(\delta_{\lceil v \rceil}) = v \qquad\qquad \eta_P(x) = \lceil \delta_x \rceil$$

It is easy to check that ε and η form an adjunction, and are both monoidal natural transformations. □

Corollary 24. FINVECT, FINSET, *and* FINSET^{op} *together form an* LPC *model.*

Linear algebra has been considered as a model for linear logic multiple times in the literature. Ehrhard [7] presents finiteness spaces, where the objects are spaces of vectors with finite support. In his model, the ! operator sends a space A to the space supported by finite multisets over A; it takes some effort to show that this comonad respects the finiteness conditions. Pratt [15] proves that finite dimensional vector spaces over a field of characteristic 2 is a Chu space and thus a model of linear logic. Valiron and Zdancewic [17] show that the LPC model of FINVECT is a sound and complete semantic model for an algebraic λ-calculus.

Relations. Let REL be the category of sets and relations, and let SET be the category of sets and functions. (Notice that the sets in either category here may be infinite, unlike in the FINVECT case.) It is easy to see that REL is linearly distributive where the tensor and the cotensor are both cartesian product, and distributivity is just associativity. The unit is a singleton set, and negation on REL is the identity operation.

SET is cartesian and its opposite category SET^{op}, cocartesian. The $F_!$ and $F_?$ functors are the forgetful functors which interpret a function as a relation. The $\lceil - \rceil$ functor takes a set to its powerset. Suppose R is a relation between A and B. Then $\lceil R \rceil : \lceil A \rceil \to \lceil B \rceil$ is defined as

$$\lceil R \rceil(X) = \{y \in B \mid \exists x \in X, (x,y) \in R\}.$$

Then $\lceil - \rceil$ has monoidal components $m_1^{\lceil - \rceil} : 1_{\mathsf{P}} \to \lceil 1_{\mathsf{L}} \rceil$ and $m_{A,B}^{\lceil - \rceil} : \lceil A \rceil \times \lceil B \rceil \to \lceil A \times B \rceil$ defined by

$$m_{1_{\mathsf{L}}}^{\lceil - \rceil}(\emptyset) = \emptyset \qquad m_{A,B}^{\lceil - \rceil}(X_1, X_2) = X_1 \times X_2$$

The dual notion $\lfloor - \rfloor$ is just the inverse.

Melliés [12] discusses a non-model of linear logic based on REL, where the exponential takes a set X to the finite subsets of X. That "model" fails because the comonad unit $\varepsilon_A : !A \to A$ is not natural. In the LPC formulation, ε is derived from the adjunction, ensuring naturality.

Boolean Algebras. Next we examine an example of the LPC categories where $\mathcal{P}$ and $\mathcal{C}$ are related by a non-trivial duality. The relationship is based on Birkhoff's representation theorem [4], which can be interpreted as a duality between the categories of finite partial orders and order-preserving maps ($\mathcal{P}$) on the one hand, and finite distributive lattices with bounded lattice homomorphisms ($\mathcal{C}$) on the other hand.

The linear category is the category $\mathcal{L}$ of finite boolean algebras with bounded lattice homomorphisms. For the monoidal structure, the units are both the singleton lattice $\emptyset$, and the tensors $A \otimes B$ and $A \parr B$ are the boolean algebra with base set $A \times B$ and lattice structure as follows:

$$\begin{array}{ccc} \bot = (\bot, \bot) & \top = (\top, \top) & \\ (x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2) & (x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2) & \neg(x,y) = (\neg x, \neg y) \end{array}$$

Given a partially ordered set $(P, \leq)$, a subset $X \subseteq P$ is called *lower* if it is downwards closed with respect to $\leq$. The set of all lower sets of P forms a lattice with $\top = P$, $\bot_{\mathsf{L}} = \emptyset$, meet as union and join and intersection. Let P^* refer to this lattice.

Meanwhile, given a lattice C, an element x is join-irreducible if x is neither $\bot_{\mathsf{L}}$ nor the join of any two elements less than x. That is, $x \neq y \vee z$ for $y, z \neq x$. Let C_* be the partially ordered set with base set the join-irreducible elements of C, with the ordering $x \leq y$ iff $x = y \wedge x$.

The operators $(-)^*$ and $(-)_*$ extend to functors that form a duality between $\mathcal{P}$ and $\mathcal{C}$ [**?**].

The monoidal structure on $\mathcal{P}$ is given by the cartesian product with the ordering $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$. The unit is the singleton order $\{\emptyset\}$. It is easy to check that every poset has a communitive comonoid.

Finite distributive lattices have a monoidal structure with the unit the singleton lattice $\{\emptyset\}$ and the tensor $C_1 \parr C_2$ the lattice where the base set is $C_1 \times C_2$. For every lattice C in $\mathcal{C}$ there exists a commutative monoid with components $e_C^{\parr} : \bot_{\mathsf{C}} \to C$ and $d_C^{\parr} : C \parr C \to C$ as follows:

$$e_C^{\parr}(\emptyset) = \bot \qquad\qquad d_C^{\parr}(x,y) = x \wedge y$$

The components of the monoid in $\mathcal{C}$ and the comonoid in $\mathcal{P}$ are interchanged under the Birkhoff duality.

Next we define the symmetric monoidal functors. Define $\lceil - \rceil : \mathcal{L} \Rightarrow \mathcal{P}$ and $\lfloor - \rfloor : \mathcal{L} \Rightarrow \mathcal{C}$ to be forgetful functors. For $\lceil - \rceil$ in particular, the order induced by the boolean algebra is $x \leq y$ iff $x = y \wedge x$.

Define $F_!$ and $F_?$ to be the powerset algebra, which takes a structure with base set X to the boolean algebra with base set X, with top, bottom, join, meet and negation corresponding to X, $\emptyset$, union, intersection and complementation respectively. On morphisms, define

$$F_! f(X) = F_? f(X) = \{f(x) \mid x \in X\}.$$

It is easy to check that these functors respect the dualities in that $(F_! P)^{\perp} \simeq F_? P^*$ and $\lceil A \rceil^* \simeq \lfloor A^{\perp} \rfloor$. To prove $F_! \dashv \lceil - \rceil$, it suffices to show a bijection of homomorphism sets $\mathrm{Hom}(F_! P, A) \cong \mathrm{Hom}(P, \lceil A \rceil)$. Suppose $f : F_! P \to A$ in $\mathcal{L}$. Then define $f^{\sharp} : P \to \lceil A \rceil$ by

$$f^{\sharp}(x) = f(\{z \in X \mid z \leq x\})$$

This morphism is in fact order-preserving. Next, for $g : P \to \lceil A \rceil$ define $g^{\flat} : F_! P \to A$ as follows:

$$g^{\flat}(X) = \bigvee_{x \in X} g(x)$$

Again $g^{\flat}$ is a lattice homomorphism. It remains to check that $(f^{\sharp})^{\flat} = f$ and $(g^{\flat})^{\sharp} = g$.

From these definitions, the unit $\varepsilon_A : F_! \lceil A \rceil \to A$ and counit $\eta_P : P \to \lceil F_! P \rceil$ of the adjunction are

$$\varepsilon_A(X) = \mathrm{id}^{\flat}_{\lceil A \rceil}(X) = \bigvee_{x \in X} \mathrm{id}_{\lceil A \rceil}(x) = \bigvee X \qquad\qquad \eta_P(x) = (\mathrm{id}_{F_! P})^{\sharp}(x) = \{z \mid z \leq x\}$$

To show the adjunction is monoidal, it suffices to prove ε and η are monoidal natural transformations.

The proof of the monoidal adjunction $\lfloor - \rfloor \dashv F_?$ is similar.

5 Related work

Girard [9] first introduced linear logic to mix the constructivity of intuitionistic propositional logic with the duality of classical logic. Partly because of this constructivity, there has been great interest in the semantics of linear logic in both the classical and intuitionistic fragments. Consequently, there exist several categorical frameworks for its semantic models.

One influential framework is Benton *et al.*'s *linear category* [2], consisting of a symmetric monoidal closed category with products and a linear exponential comonad !. Schalk [16] adapted linear categories to the classical case by requiring that the symmetric monoidal closed category be *-autonomous. The coproduct $\parr$ and coexponential ? are then induced from the duality.

Cockett and Seely [5], seeking to study $\otimes$ and $\parr$ as independent structures unobscured by duality, introduced linearly distributive categories, which make up the linear category in the LPC model. The authors extended this motivation to the exponentials by modeling ! and ? as linear functors [6], meaning that ? is *not* derived from ! and $(-)^{\perp}$. The LPC model reflects that work by allowing ! and ? to have different adjoint decompositions.

Other variations of classical linear logic, notably Girard's Logic of Unity [10], distinguish linear propositions from persistent ones. In the intuitionistic case, Benton [3] developed the linear/non-linear logic and categorical model described in Section 3. Barber used this model as the semantics for a term calculus called DILL [1]. A Lafont category [11] is a canonical instance of an LNL model where $!A$ is the free commutative comonoid generated by A. This construction automatically admits an adjunction between between a linear category $\mathcal{L}$ and the category of commutative comonoids over $\mathcal{L}$. However, the LNL and LPC models have an advantage over Lafont categories by allowing a much greater range of interpretations for the exponential. Lafont's construction excludes traditional models of linear logic like coherence spaces and the category REL.

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References

[1] Andrew Barber (1996): *Dual intuitionistic linear logic*. Technical Report.

[2] Nick Benton, G. M. Bierman, J. Martin E. Hyland & Valeria de Paiva (1993): *A Term Calculus for Intuitionistic Linear Logic*. In: Proceedings of the International Conference on Typed Lambda Calculi and Applications, Springer-Verlag LNCS 664, pp. 75–90.

[3] P. N. Benton (1995): *A mixed linear and non-linear logic: proofs, terms and models*. In: Proceedings of Computer Science Logic (CSL '94), Kazimierz, Poland., Springer-Verlag, pp. 121–135.

[4] Garrett Birkhoff (1937): *Rings of sets*. Duke Mathematical Journal 3(3), pp. 443–454.

[5] J.R.B. Cockett & R.A.G. Seely (1997): *Weakly distributive categories*. Journal of Pure and Applied Algebra 114(2), pp. 133 – 173.

[6] J.R.B. Cockett & R.A.G. Seely (1999): *Linearly distributive functors*. Journal of Pure and Applied Algebra 143(13), pp. 155 – 203.

[7] Thomas Ehrhard (2005): *Finiteness spaces*. Mathematical Structures in Computer Science 15(4), pp. 615–646.

[8] Thomas Fox (1976): *Coalgebras and cartesian categories*. Communications in Algebra 4(7), pp. 665–667.

[9] Jean-Yves Girard (1987): *Linear logic*. Theoretical computer science 50(1), pp. 1–101.

[10] Jean-Yves Girard (1993): *On the unity of logic*. Annals of pure and applied logic 59(3), pp. 201–217.

[11] G Max Kelly (1974): *Doctrinal adjunction*. In: Category Seminar, Springer, pp. 257–280.

[12] Yves Lafont (1988): *The Linear Abstract Machine*. Theoretical Computer Science 59, pp. 157–180. Corrections in vol. 62, pp. 327–328.

[13] Paul André Melliés (2003): *Categorical models of linear logic revisited*.

[14] Paul-André Melliès (2009): *Categorical semantics of linear logic*. Panoramas et Syntheses 27.

[15] Jennifer Paykin & Steve Zdancewic (2014): *A Linear/Producer/Consumer model of Classical Linear Logic*. Technical Report MS-CIS-14-03, University of Pennsylvania.

[16] V.R. Pratt (1994): *Chu spaces: complementarity and uncertainty in rational mechanics*. Course notes, TEMPUS summer school, Budapest.

[17] Andrea Schalk (2004): *Whats is a categorical model of linear logic*. Technical Report, University of Manchester.

[18] Richard P Stanley (2011): *Enumerative combinatorics*. 49, Cambridge university press.

[19] Benoît Valiron & Steve Zdancewic (2014): *Finite Vector Spaces as a Model of Simply-Typed Lambda-Calculi*.