Linear Logic

Intuitionistic Linear Logic

\[ \begin{array}{c}
\text{A, B ::= } \alpha \mid b \mid A \rightarrow B \mid \top \mid \bot \mid \exists \alpha. A \\
\mid A \otimes B \mid \bot \mid A \oplus B \mid \top \mid \forall \alpha. A \mid \! \! A
\end{array} \]

Classical Logic

Excluded Middle : \( A \lor \neg A \)

LK - Gentzen's Classical Sequent Calculus

\( \Gamma \vdash \Delta \)

"and" \( \Gamma, A \vdash \Delta \)

"or" \( \Gamma, A \lor B \vdash \Delta \)

"multiple conclusion" \( \Gamma, A \rightarrow \Delta \)

(1) restrict to single conclusion \( \Rightarrow \) intuitionistic

(2) restrict weakening & contraction \( \Rightarrow \) linear logic
\[ \Gamma, A \vdash A \]  
\[ \text{(Axiom)} \]

\[ A \vdash A \]  
\[ \text{(Axiom)} \]

\[ \vdash \neg A, A \]  
\[ \text{(neg)} \]

\[ \vdash A \lor A, A \]  
\[ \text{(lin \ "or")} \]

\[ \vdash A, \neg A \lor A \]  
\[ \text{(exchange)} \]

\[ \vdash \neg A \lor A, \neg A \lor A \]  
\[ \text{(lin \ "or")} \]

\[ \vdash \neg A \lor A \]  
\[ \text{(contraction)} \]

\[ A \& B \vdash \text{unit for } \& \]

\[ A, B ::= \alpha \mid b \mid A \rightarrow B \mid \perp \mid A \& B \mid \top \mid \exists \alpha. A \mid ?A \]

\[ b^\perp \mid A \& B \mid \perp \mid A \oplus B \mid 0 \mid \forall \alpha. A \mid !A \]

\[ A \nrightarrow B \overset{\text{def}}{=} A^\perp \& B \]

\[ \vdash A \nrightarrow \perp, A^\perp \& \perp \]

\[ \Gamma, A, B \vdash \Delta \]

\[ \Gamma, A \& B \vdash \Delta \]

\[ \Gamma, A \& B \vdash \Delta \]

\[ \Gamma, A \& B \vdash \Delta \]

\[ (b^\perp) = b^\perp \]

\[ (A \& B)^\perp = A^\perp \& B^\perp \]

\[ (A \nrightarrow B)^\perp = A^\perp \oplus B^\perp \]

\[ (\exists \alpha. A)^\perp = \forall \alpha. A^\perp \]

\[ (\perp)^\perp = \top \]

\[ (\top)^\perp = \perp \]

\[ (?A)^\perp = ! (A^\perp) \]
• Add $\mathcal{B}$, 1 to recover multiconclusion
• Work with $A^\top$ to recover negation

$A \to 1 = A^\top \quad A \to 0$

In linear logic $\vdash A \otimes A^\top \leq$

but $\not\vdash A \oplus A \otimes 0$

$\begin{array}{c}
\begin{array}{c}
A \otimes A^\top \\
\hline
A \to B \overset{\text{def}}{=} A^\top \otimes B \\
\hline
A^\top \oplus B
\end{array}
\end{array}$
Full Classical Linear Logic

$\Gamma \vdash \Delta$

$\Gamma \vdash \Delta$ is a multiset of L.L. propositions

$\Gamma, A \vdash A^\perp, \Delta$

$\Gamma, A \vdash \Delta$

$\Gamma, A \vdash \Delta, B$

$\Gamma, A \otimes B, \Delta$

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\[ \vdash \Gamma, A \& B \]
\[ \vdash \Gamma, A \quad \vdash \Gamma, A \oplus B \]
\[ \vdash \Gamma, A \& B \]
\[ \vdash \Gamma, A [B/\alpha] \]
\[ \vdash \Gamma, \exists \alpha. A \]
\[ \vdash ? \Gamma, A \quad \vdash ? \Gamma, ! A \]
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\[(A^\perp)^\perp = A\]

\[\Gamma, A, A^\perp \vdash \Gamma, A, A^\perp, \Delta\]

**Inductive Type**: 

- **Positive base**: \((s : \text{String})\)
- **Negative base**: \((s : \text{String})\)
- **One**, **Zero**, **Tensr**

**Fixpoint dual** 

\[(A : \text{Type}) : \text{Type} :=\]

**Inductive judgments**: 

- **List**: \([\text{Type} \Rightarrow \text{Prop}]\)
- **Identity**: \(\forall (A : \text{Type}), \Gamma A, \text{dual } A\]
- \(\vdash\)
\[ \Gamma + \Delta \]

\[ x : A, y : A' \ldots \vdash (s : A), (t : B), (u : C) \]

\[ A \rightarrow B = A^t \rightarrow B \]

\[ \Gamma, x : A \vdash s : B, \Delta \]

\[ \Gamma, \vdash (\lambda x : S) : A \rightarrow B, \Delta \]