Linear Logic

Intuitionistic Linear Logic

$$A, B ::= \alpha \mid b \mid A \multimap B \mid \quad \mid A \,\&\, B \mid \top \mid \exists \alpha . A$$
$$\mid A \otimes B \mid 1 \mid A \oplus B \mid 0 \mid \forall \alpha . A \mid\ !A$$

Classical Logic

Excluded Middle : $A \vee \neg A$

LK - Gentzen's Classical Sequent Calculus

$\Gamma \vdash \underline{\Delta}$

"multiple conclusion"

"and" "or"

$p, q \vdash r, s$

"p and q imply r or s"

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma \vdash \bot}{\Gamma, \bot \vdash \Delta}$$

$\bot \vdash$

(1) restrict to single conclusion $\Longrightarrow$ intuitionistic

(2) restrict weakening & contraction $\Longrightarrow$ linear logic

$$\frac{}{\Gamma, A \vdash A}\ (\text{Axiom})$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\ (\text{Axiom})}{\vdash \neg A, A}\ (\text{neg})}{\vdash \neg A \vee A, A}\ (\text{inl "or"})}{\vdash A, \neg A \vee A}\ (\text{exchange})}{\vdash \neg A \vee A, \neg A \vee A}\ (\text{inr "or"})}{\vdash \neg A \vee A}\ (\text{contraction})$$

$A \parr B$ unit for $\parr$

$$\begin{aligned} A, B ::= {} & \alpha \mid b \mid A \multimap B \mid \bot \mid A \,\&\, B \mid \top \mid \exists \alpha. A \mid ?A \\ & b^\perp \mid A \otimes B \mid 1 \mid A \oplus B \mid 0 \mid \forall \alpha. A \mid !A \end{aligned}$$

$$A \multimap B \overset{\text{def}}{=} A^\perp \parr B \qquad \underbrace{A \multimap \bot}, \ \underbrace{A^\perp} \parr \bot$$

$$A \parr \bot \cong A$$

nonlinear

linear context

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}$$

$$(b)^\perp = b^\perp \qquad (b^\perp)^\perp = b$$

$$(A \otimes B)^\perp = A^\perp \parr B^\perp \qquad \bot^\perp = 1$$

$$(A \,\&\, B)^\perp = A^\perp \oplus B^\perp \qquad \top^\perp = 0$$

$$(\exists \alpha. A)^\perp = \forall \alpha. A^\perp \qquad (?A)^\perp = !(A^\perp)$$

- Add $⅋$, $\bot$ to recover multiconclusion
- Work with $A^\perp$ to recover negation

$$A \multimap \bot \simeq A^\perp \qquad A \multimap 0$$

In linear logic $\vdash A ⅋ A^\perp$ $\Longleftarrow$

but $\nvdash A \oplus A \multimap 0$

$A \oplus A^\perp$

$A \multimap B \overset{def}{=} A^\perp ⅋ B$

$A^\perp \oplus B$

Full Classical Linear Logic

$\boxed{\vdash \Delta}$ Δ is a multiset of L.L. propositio[ns]

$$\frac{}{\vdash A, A^\perp}\ (\text{identity}) \qquad [A \vdash A]$$

$\vdash 1, \perp$

$\vdash 0, \top$

$\vdash b, \bar{b}$

$$\frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ (\text{cut}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma'}\ (\Gamma' \text{ is a permutation of } \Gamma)$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, A \otimes B, \Delta} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

[Compare

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$$\frac{\Gamma_1 \vdash^{M} A \otimes B \qquad \Gamma_2, \underset{x}{A}, \underset{y}{B} \vdash C}{\Gamma_1, \Gamma_2 \vdash_{n} C} \qquad \vdash \Gamma_2^\perp, A^\perp, B^\perp, C$$

let $\langle x, y \rangle = M$ in N

$\vdash \Gamma_1^\perp, \Gamma_2^\perp, C$

$$\frac{\vdash \Gamma_1, A \otimes B \qquad \cfrac{\cfrac{\cfrac{\vdots}{\vdash \Gamma_2, A^\perp, B^\perp, C}}{\vdash \Gamma_2, A^\perp \parr B^\perp, C}}{\vdash A^\perp \parr B^\perp, \Gamma_2, C}}{\vdash \Gamma_1, \Gamma_2, C}\ (\text{cut})$$

]

$[A, (\Gamma \otimes \Delta), B]$

$$\vdash \Gamma, \top \qquad \text{(no rule for } 0\text{)}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall \alpha. A}\ (\alpha \notin fv(\Gamma)) \qquad \frac{\vdash \Gamma, A[B/\alpha]}{\vdash \Gamma, \exists \alpha. A}$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$$

$?\Gamma$ means that every prop. in Γ starts with $?$

$$\left[\ \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ (\forall B \in \Gamma,\ B = ?C \text{ for some } C)\right.$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \text{(weakening)} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \text{(contraction)}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \text{(dereliction)}$$

Compared to JILL $\Leftarrow$

$\Gamma ; \Delta \vdash A$ $\qquad \vdash A, \Delta^\perp, \Gamma^?$ $\qquad \Gamma, u{:}A ; \cdot \vdash u{:}A$

$\Gamma^? \qquad \cdot^? =$ $\qquad \Gamma, A ; \cdot \vdash A$

$(\Gamma, A)^? = \Gamma^?, ?A^\perp \quad ; \quad (!A)^\perp$

$$\frac{\vdash A, ?A^\perp, \Gamma^\perp}{\vdash A, A^\perp, \Gamma^\perp}$$

$$(A^\perp)^\perp = A$$

$$\vdash A, A^\perp \qquad \frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

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Inductive ltype :=
 | positive_base : (s: String)
 | negative_base : (s: String)
 | One | Zero | Tensor .

Fixpoint dual (A : ltype) : ltype :=

Inductive judgments := : list ltype -> Prop
 | Identity : ∀ (A : ltype), [A, dual A]
 | :
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$$\Gamma \vdash \Delta$$

$$x : A, y : A' \ldots \vdash (s : A), (t : B), (u : C)$$

$$\boxed{A \multimap B} = A^{\perp} \parr B$$

$x \in S$

$$\frac{\Gamma, x : A \vdash s : B, \Delta \ldots}{\Gamma, \vdash (\lambda x : S) : A \multimap B, \Delta}$$