Intuitionistic Linear Logic

Linear Contexts $\Delta ::= \underbrace{\cdot \mid \Delta, x:A}_{\text{Sets (implicit exchange)}}$

Δ_1, Δ_2 is disjoint union (undefined if $dom(\Delta_1) \cap dom(\Delta_2) \neq \emptyset$)

$$\boxed{\Delta \vdash M : A}$$

"under linear assumptions Δ M has type A"

Multiplicative L.L.

$$\Gamma; x:A \vdash x:A$$

$$\frac{\Gamma; \Delta, x:A \vdash M:B}{\Gamma; \Delta \vdash \lambda x{:}A.M : A \multimap B} \qquad \frac{\Gamma; \Delta_1 \vdash M : A \multimap B \quad \Gamma; \Delta_2 \vdash N : A}{\Gamma; \Delta_1, \Delta_2 \vdash (M\,N) : B}$$

$$\frac{\Delta_1 \vdash M : A \quad \Delta_2 \vdash N : B}{\Delta_1, \Delta_2 \vdash \langle M, N\rangle : A \otimes B} \qquad \frac{\Delta_1 \vdash M : A \otimes B \quad \Delta_2, x:A, y:B \vdash N : C}{\Delta_1, \Delta_2 \vdash \text{let } \langle x, y\rangle = M \text{ in } N : C}$$

$$\Gamma; \cdot \vdash \langle\rangle : 1 \qquad \frac{\Delta_1 \vdash M : 1 \quad \Delta_2 \vdash N : C}{\Delta_1, \Delta_2 \vdash \text{let } \langle\rangle = M \text{ in } N : C}$$

$M ; N$

Additive

$$\frac{\Delta \vdash M : A}{\Delta \vdash \text{inl } M : A \oplus B}$$

$$\frac{\Delta \vdash M : B}{\Delta \vdash \text{inr } M : A \oplus B}$$

$$\frac{\Delta_1 \vdash M : A \oplus B \qquad \Delta_2, x : A \vdash N_1 : C \qquad \Delta_2, y : B \vdash N_2 : C}{\Delta_1, \Delta_2 \vdash \text{match } M \text{ with } \mid \text{inl } x \to N_1 \mid \text{inr } y \to N_2 \text{ end} : C}$$

$$\frac{\Delta_1 \vdash M : \emptyset}{\Delta_1, \Delta_2 \vdash \text{match } M \text{ with end} : C}$$

$$\frac{\Delta \vdash M : A \qquad \Delta \vdash N : B}{\Delta \vdash \{\text{fst} = M; \text{snd} = N\} : A \,\&\, B}$$

$$\frac{\Delta \vdash M : A \,\&\, B}{\Delta \vdash \text{fst } M : A}$$

$$\frac{\Delta \vdash M : A \,\&\, B}{\Delta \vdash \text{snd } M : B}$$

$$\Delta \vdash \{\} : \top$$

Exponentials

$$\Gamma ::= \cdot \mid u:A, \Gamma \quad \leftarrow \text{"unrestricted contexts"}$$

$$\boxed{\Gamma ; \Delta \vdash M : A}$$

$$\frac{}{\Gamma ; \cdot \vdash u : A}\ (u:A) \in \Gamma$$

$$\frac{\Gamma ; \cdot \vdash M : A}{\Gamma ; \cdot \vdash !M : !A}$$

$$\frac{\Gamma ; \Delta_1 \vdash M : !A \qquad \Gamma, u:A ; \Delta_2 \vdash N : B}{\Gamma ; \Delta_1, \Delta_2 \vdash \text{let } !u = M \text{ in } N : B}$$

$$!A \approx 0 + A + A \otimes A + A \otimes A \otimes A + \cdots$$

$$[\![b]\!] = b$$
$$[\![\sigma \to \tau]\!] = ![\![\sigma]\!] \multimap [\![\tau]\!]$$
$$[\![\sigma \vee \tau]\!] = ![\![\sigma]\!] \oplus ![\![\tau]\!]$$
$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \& [\![\tau]\!]$$
$$[\![\text{True}]\!] = \top$$
$$[\![\text{False}]\!] = 0$$

Evaluation

$$w, V ::= x \mid \mathsf{inl}\ v \mid \mathsf{inr}\ v \mid \langle v, w \rangle \mid \langle\rangle$$
$$\mid \{\mathsf{fst} = M, \mathsf{snd} = N\} \mid \lambda x{:}A.\,M \mid \{\}$$
$$\mid\ !M$$

$$E := [\,] \mid \mathsf{inl}\ E \mid \mathsf{inr}\ E \mid \langle E, M \rangle \mid \langle v, E \rangle$$
$$\mid \mathsf{let}\ \langle x, y \rangle = E\ \mathsf{in}\ M \mid \mathsf{let}\ \langle\rangle = E\ \mathsf{in}\ N \mid$$
$$\mid \mathsf{match}\ E\ \mathsf{with}\ \mathsf{inl}\ x \to N_1\ \mathsf{inr}\ y \to N_2\ \mathsf{end} \mid \mathsf{match}\ E\ \mathsf{with}\ \mathsf{end} \mid$$
$$\mid \mathsf{fst}\ E \mid \mathsf{snd}\ E \mid E\ M \mid V\ E$$
$$\mid \mathsf{let}\ !u = E\ \mathsf{in}\ M$$

$$\boxed{M \to_\beta M'}$$

$$(\lambda x{:}A.\,M)\ V \to_\beta M\{V/x\}$$
$$\mathsf{let}\ \langle x, y \rangle = \langle V_1, V_2 \rangle\ \mathsf{in}\ M \to_\beta M\{V_1/x\}\{V_2/y\}$$
$$\mathsf{let}\ \langle\rangle = \langle\rangle\ \mathsf{in}\ M \to_\beta M$$
$$\mathsf{match}\ \mathsf{inl}\ v\ \mathsf{with}\ \mathsf{inl}\ x \to N_1 \mid \mathsf{inr}\ y \to N_2\ \mathsf{end} \to N_1\{v/x\}$$
$$\mathsf{fst}\ \{\mathsf{fst} = M, \mathsf{snd} = N\} \to M$$
$$\mathsf{let}\ !u = !M\ \mathsf{in}\ N \to N\{M/u\}$$

Substitution lemmas

(1) If $\Gamma ; \Delta_1 \vdash M : A$ and $\Gamma ; \Delta_2, x : A \vdash N : B$

then $\Gamma ; \Delta_1, \Delta_2 \vdash N\{M/x\} : B$

(2) If $\Gamma ; \cdot \vdash M : A$ and $\Gamma, u : A ; \Delta \vdash N : B$

then $\Gamma ; \Delta \vdash N\{M/u\} : B$

$$[\![\lambda u{:}\tau.\, e]\!] \;:\; ![\![\tau]\!] \multimap [\![\sigma]\!]$$

$$\overset{\text{def}}{=} \lambda x{:}![\![\tau]\!].\ \text{let } !u = x \text{ in } [\![e]\!]$$

$$[\![e_1\ e_2]\!] \overset{\text{def}}{=} [\![e_1]\!]\ !([\![e_2]\!])$$

$$[\![\Gamma \vdash e : \tau]\!] \approx [\![\Gamma]\!] ; \cdot \vdash [\![e]\!] : [\![\tau]\!]$$

$$!(A \,\&\, B) \cong\ !A \otimes !B$$

$$\frac{\Gamma ; \Delta \vdash M : A}{\Gamma ; \Delta \vdash \Lambda\alpha.M : \forall\alpha.A}\ (\alpha \notin fv(\Gamma) \cup fv(\Delta))$$

$$\frac{\Gamma ; \Delta \vdash M : \forall\alpha.A}{\Gamma ; \Delta \vdash M[B] : A\{B/\alpha\}}$$

$$(\Lambda\alpha.M)[B] \to M\{B/\alpha\}$$

$$\frac{\Gamma ; \Delta \vdash M : A\{B/\alpha\}}{\Gamma ; \Delta \vdash [B, M] : \exists \alpha . A} \quad (\alpha \notin fv(\Gamma, \Delta))$$

$$\frac{\Gamma ; \Delta \vdash M : \exists \alpha . A \qquad \Gamma ; \Delta , x : A \vdash N : B}{\Gamma ; \Delta \vdash \text{let } [\alpha, x] = M \text{ in } N : B}$$

$$\text{let } [\alpha, x] = [B, M] \text{ in } N \Rightarrow N[B/\alpha]\{M/x\}$$

$$\forall \alpha . \; !\alpha \multimap !\alpha \multimap !\alpha$$

$$!A \qquad !_r A \qquad !_r \mathbb{N} \multimap \mathbb{N}$$

File handle Protocol

open : string $\multimap$ fh

$$\mathrm{Ops}(fh) \stackrel{def}{=} !\{\, read : fh \multimap fh \otimes !string \;;\; write : fh \multimap string \multimap fh \otimes 1 \;;\; close : fh \multimap 1 \,\}$$

$$open : !string \multimap \exists fh.\ fh \otimes \mathrm{Ops}(fh)$$

let doit : string $\multimap$ 1 = λ filename:string in
  let $[\alpha, \langle h, !o \rangle]$ = open filename in
    let $\langle h, !s \rangle$ = o.read h in
    let $\langle h, \langle\rangle \rangle$ = o.write h s in
    let $\langle h, \langle\rangle \rangle$ = o.write h s in
      close h ;
      close h ← ill typed

ill typed

.... $\langle$ write ~~h "x"~~ ,
write ~~h "y"~~ $\rangle$ X

let $[\alpha \langle h_1, !o_1 \rangle = open\ f_1]$ in
let $[\beta, \langle h_2, !o_2 \rangle = open\ f_2]$ in