Linear Logic & Linear Types

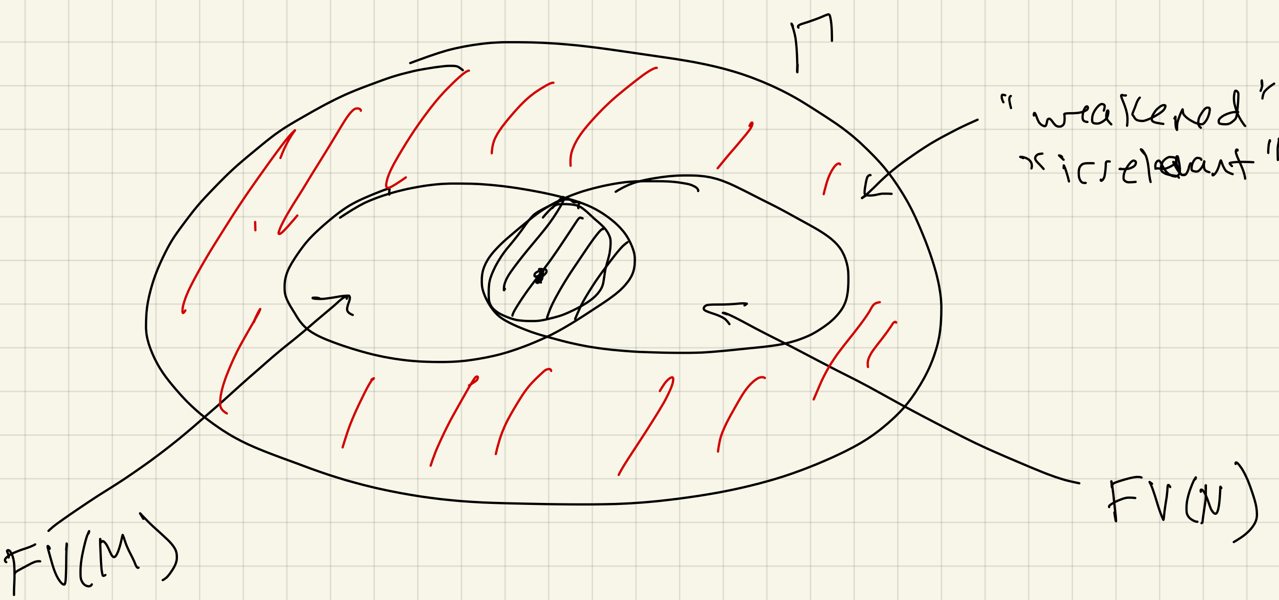
- <u>Weakening</u>: If $\Gamma \vdash M : A$ and $\Gamma' \supseteq \Gamma$ then $\Gamma' \vdash M : A$.

$$\Gamma ::= \cdot \mid \Gamma, x : A \qquad \Gamma \uplus \{(x : A)\}$$

$$\frac{}{\Gamma \vdash x : A}\ (x : A) \in \Gamma \qquad \Gamma' \supseteq \Gamma$$

- <u>Contraction</u>: If $\Gamma, x : A, y : A \vdash M : B$ then $\Gamma, x : A \vdash M\{x/y\} : B$.

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M N) : B}$$



list not set

- Exchange: If $\Gamma, x:A, y:B, \Gamma' \vdash M : C$

  then $\Gamma, y:B, x:A, \Gamma' \vdash M : C$.

Substructural Logics

W = weakening

C = contraction

usual "unrestricted"

$$\begin{tikzcd}
 & \text{W and C} & \\
\neg W, C \arrow[ur] & & W, \neg C \arrow[ul] \\
 & \neg W, \neg C \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

"relevance" logic

$\geq \underline{1}$ uses of a hypothesis

"Affine logic" (0 or 1 uses of a hyp.)

"linear logic"

"exactly one use of a hyp."

Stateful Programming

State Monad

$$\underbrace{S \longrightarrow S \times A}$$

initial state   final state

map (loc ⇒ val)

$\lambda s_0 : S.$
  let s_1 = update s_0 $x := 3$ in
  let s_2 = update s_1 $y := 4$ in
  let s_3 z = read s_2 y in
  $\langle s_3, z \rangle$

Intuitionistic Linear Logic

Linear Contexts $\Delta ::= \underbrace{\cdot \mid \Delta, x : A}_{\text{Sets (implicit exchange)}}$

Δ_1, Δ_2 is disjoint union (undefined if $dom(\Delta_1) \cap dom(\Delta_2) \neq \emptyset$)

$\boxed{\Delta \vdash M : A}$ "under linear assumptions Δ M has type A"

Multiplicative L.L.

$$\overline{x : A \vdash x : A}$$

$$\frac{\Delta, x{:}A \vdash M : B}{\Delta \vdash \lambda x{:}A. M : A \multimap B} \qquad \frac{\Delta_1 \vdash M : A \multimap B \quad \Delta_2 \vdash N : A}{\Delta_1, \Delta_2 \vdash (M\,N) : B}$$

$$\frac{\Delta_1 \vdash M : A \quad \Delta_2 \vdash N : B}{\Delta_1, \Delta_2 \vdash \langle M, N \rangle : A \otimes B} \qquad \frac{\Delta_1 \vdash M : A \otimes B \quad \Delta_2, x : A, y : B \vdash N : C}{\Delta_1, \Delta_2 \vdash \text{let } \langle x, y \rangle = M \text{ in } N : C}$$

$$\overline{\cdot \vdash \langle\rangle : 1} \qquad \frac{\Delta_1 \vdash M : 1 \quad \Delta_2 \vdash N : C}{\Delta_1, \Delta_2 \vdash \text{let } \langle\rangle = M \text{ in } N : C}$$

$M ; N$

$$(A \otimes B) \multimap C \cong (A \multimap (B \multimap C))$$

$$A \otimes B \not\multimap A$$

$$A \wedge B \Rightarrow A$$

$$A \times B \Rightarrow A$$

$$A \multimap A \multimap A \qquad \forall \alpha.\ \alpha \multimap \alpha \multimap \alpha$$

void

$$A \to A \to A \qquad \forall \alpha.\ \alpha \to \alpha \to \alpha$$

bool

$$A \multimap B \multimap B$$

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let x = [] in f(x)
               g(x)
               h(x)
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Additive

$$\frac{\Delta \vdash M : A}{\Delta \vdash \mathrm{inl}\ M : A \oplus B}$$

$$\frac{\Delta \vdash M : B}{\Delta \vdash \mathrm{inr}\ M : A \oplus B}$$

$$\frac{\Delta_1 \vdash M : A \oplus B \qquad \Delta_2, x : A \vdash N_1 : C \qquad \Delta_2, y : B \vdash N_2 : C}{\Delta_1, \Delta_2 \vdash \mathrm{match}\ M\ \mathrm{with}\ |\mathrm{inl}\ x \to N_1\ |\mathrm{inr}\ y \to N_2\ \mathrm{end} : C}$$

$$\frac{\Delta_1 \vdash M : \emptyset}{\Delta_1, \Delta_2 \vdash \mathrm{match}\ M\ \mathrm{with}\ \mathrm{end} : C}$$

$$\frac{\Delta \vdash M : A \qquad \Delta \vdash N : B}{\Delta \vdash \{\mathrm{fst} = M;\ \mathrm{snd} = N\} : A \mathbin{\&} B}$$

$$\frac{\Delta \vdash M : A \mathbin{\&} B}{\Delta \vdash \mathrm{fst}\ M : A}$$

$$\frac{\Delta \vdash M : A \mathbin{\&} B}{\Delta \vdash \mathrm{snd}\ M : B}$$

$$\Delta \vdash \{\} : \top$$