Logical Relations For System F

$$B, A ::= \alpha \mid A \to B \mid \forall \alpha . A$$

$$N, M ::= x \mid \lambda x{:}A.\, M \mid M\, N \mid \Lambda \alpha . M \mid M\,[A]$$

$$\boxed{\Delta ; \Gamma \vdash M : A}$$

implementation types

relation between them

$$[\![\Delta]\!] = (\delta_1, \delta_2, \mathcal{D} : \delta_1 \Leftrightarrow \delta_2)$$

- $\mathrm{dom}(\delta_1) = \mathrm{dom}(\delta_2) = \mathrm{dom}(\mathcal{D}) = \Delta$
- for all $\alpha \in \Delta$, $\mathcal{D}(\alpha) : \delta_1(\alpha) \Leftrightarrow \delta_2(\alpha)$
- for all $\alpha \in \Delta$, $FTV(\delta_1(\alpha)) = FTV(\delta_2(\alpha)) = \emptyset$

<u>Notation</u>: For closed A, B $\quad A \Leftrightarrow B$ is the type of relations $R \subseteq \mathcal{P}(tm) \times \mathcal{P}(tm)$ s.t. $(M, N) \in R$ implies

$\cdot ; \cdot \vdash M : A$

$\cdot ; \cdot \vdash N : B$

$=_{\beta\eta}$ closed relations (may be unnecessary)

[For all $\cdot ; \cdot \vdash M' : A$, $\cdot ; \cdot \vdash N' : B$ if $M =_{\beta\eta} M'$ and $N =_{\beta\eta} N'$ then $(M', N') \in R$]

$\boxed{\Delta; \Gamma \vdash M : A}$

Given $[\![\Delta]\!] = (\delta_1, \delta_2, \mathcal{D})$ implementations

$[\![\Gamma]\!]_{[\![\Delta]\!]} = (\gamma_1, \gamma_2)$

- $\mathrm{dom}(\gamma_1) = \mathrm{dom}(\gamma_2) = \mathrm{dom}(\Gamma)$
- for all $x \in \mathrm{dom}(\Gamma)$

  $\left(\underbrace{\delta_1(\gamma_1(x)), \ \delta_2(\delta_2(x))}_{\text{that are related}}\right) \in [\![\Gamma(x)]\!][\![\Delta]\!]$

$\boxed{[\![A]\!]_{[\![\Delta]\!]}} : \delta_1(A) \Leftrightarrow \delta_2(A)$

- $[\![\alpha]\!]_{[\![\Delta]\!]} = \mathcal{D}(\alpha) : \delta_1(\alpha) \Leftrightarrow \delta_2(\alpha)$
- $[\![A \to B]\!]_{[\![\Delta]\!]} = [\![A]\!]_{[\![\Delta]\!]} \Rightarrow [\![B]\!]_{[\![\Delta]\!]}$

  ↳ see next page

- $[\![\forall\alpha. A]\!]_{[\![\Delta]\!]} =$

  $$\underbrace{\forall(\lambda B, B', R : B \Leftrightarrow B'. \ [\![A]\!]_{[\![\Delta]\!][\alpha \mapsto (B, B', R)]})}_{\text{meta-level function}}$$

  ↳ see next page.

  Extend $[\![\Delta]\!]$ s.t. $\delta_1(\alpha) = B$, $\delta_2(\alpha) = B'$, $\mathcal{D}(\alpha) = R$

Given $R : A \Leftrightarrow A'$, $S : B \Leftrightarrow B'$

$$\boxed{R \Rightarrow S} \triangleq \left\{ (M, M') \;\middle|\; \begin{array}{l} \cdot\,;\cdot \vdash M : A \to B \\ \cdot\,;\cdot \vdash M' : A' \to B' \\ \text{for all } (N, N') \in R, \\ (MN, M'N') \in S \end{array} \right\}$$

Given $f : (B : \text{typ}) \to (B' : \text{typ}) \to (R : B \Leftrightarrow B') \to \left(A[B/\alpha] \Leftrightarrow A'[B'/\alpha]\right)$

$$\boxed{\forall f} \triangleq \left\{ (M, M') \;\middle|\; \begin{array}{l} \cdot\,;\cdot \vdash M : \forall\alpha.\, A \\ \cdot\,;\cdot \vdash M' : \forall\alpha.\, A' \\ \text{for all } B, B', R : B \Leftrightarrow B', \\ (M[B], M'[B']) \in f\, B\, B'\, R \end{array} \right\}$$

Parametricity

Theorem Fundamental theorem of logical Relations.

If $\Delta ; \Gamma \vdash M : A$ then for any $[\![\Delta]\!]$ and $[\![\Gamma]\!]_{[\![\Delta]\!]}$

$$(\gamma_1 M, \gamma_2 M) \in [\![A]\!]_{[\![\Delta]\!]}$$

Corollary : If $\cdot\,;\cdot \vdash M : A$ then

$$(M, M) \in [\![A]\!]_{[\![\cdot]\!]}$$

Note

$$[\![\text{bool}]\!]_{[\![\Delta]\!]} = \{(\text{true}, \text{true}), (\text{false}, \text{false})\}^{=_{\beta\eta}}$$

$[\![A \text{ list}]\!]_{[\![\Delta]\!]} : \delta_1(A) \text{ list} \Longleftrightarrow \delta_2(A) \text{ list}$

- nil $\quad \{([\,], [\,])\}^{=_{\beta\eta}}$

$$\left\{ (M, N) \;\middle|\; \begin{array}{l} \cdot;\cdot \vdash M : \delta_1(A) \text{ list} \\ \cdot;\cdot \vdash N : \delta_2(A) \text{ list} \\ \text{s.t. } M =_{\beta\eta} [\,] \text{ and } N =_{\beta\eta} [\,] \end{array} \right\}$$

- cons

$\forall (M', N') \in [\![A \text{ list}]\!]_{[\![\Delta]\!]}$

for all $(X, Y) \in [\![A]\!]_{[\![\Delta]\!]}$

if $M =_{\beta\eta} X :: M' \quad N =_{\beta\eta} Y :: N'$

then $(M, N) \in [\![A \text{ list}]\!]_{[\![\Delta]\!]}$

Examples of Parametricity

- Claim: there is no closed M s.t.

  $\cdot ; \cdot \vdash M : \forall \alpha . \alpha$

Proof: By the fundamental

$$(M, M) \in [\![\forall \alpha . \alpha]\!]_{[\![\cdot]\!]} = \underbrace{\forall (\lambda B B'. R)}$$

$$= \left.\begin{array}{l} \{ (N, N') \mid \dots \\ \quad \text{for all } B, B', R : B \Leftrightarrow B' \\ \quad (N[B], N'[B']) \in R \ \} \end{array}\right\} X$$

$= \emptyset$ (suppose $(N, N') \in X$, pick $R = \emptyset$

$(N, N') \in \emptyset$ ✗. $\square$

Type : Type

F + fix   diverge : A

A $A_\perp$

Example Claim: If $\cdot;\cdot \vdash M : \forall\alpha.\alpha\to\alpha$

then for any $F : A \to B$ (closed A, B, F)

$$F \circ (M[A]) = (M[B]) \circ F$$

Proof $(M,M) \in [\![\forall\alpha.\alpha\to\alpha]\!]_{[\cdot]}$

$$= \forall\,(\lambda A,B,R.\ [\![\alpha\to\alpha]\!]_{(\alpha\mapsto(A,B,R))})$$

$$= \left\{ (N,N') \;\middle|\; \begin{array}{l} \cdot;\cdot \vdash N : \forall\alpha.\alpha\to\alpha \\ \cdot;\cdot \vdash N' : \forall\alpha.\alpha\to\alpha \\ \text{for all } A,B,R \\ \quad (N[A], N'[B]) \in [\![\alpha\to\alpha]\!]_{(\alpha\mapsto A,B,R)} \end{array} \right\}$$

Pick $R : A \Leftrightarrow B = \{ (a, F\,a) \}^{=_{\beta\eta}}$ for $\cdot;\cdot \vdash a : A$, $\vdash F\,a : B$

$(M[A], M[B]) \in [\![\alpha\to\alpha]\!]_{(\alpha\mapsto A,B,R)}$

$\Leftrightarrow$ for all $(a,b)\in R$, $(M[A]\,a, M[B]\,b) \in R$

$\Leftrightarrow$ for all a $(\underbrace{M[A]\,a}, M[B](F\,a)) \in \underline{R}$

$$F(M[A]\,a) =_{\beta\eta} M[B](F\,a)$$

$$\lambda a.\,F(M[A]\,a) =_{\beta\eta} \lambda a.\,M[B](F\,a)$$

$$\begin{array}{ccc} \| & & \| \\ F\circ(M[A]) & & M[B]\circ F \end{array}$$

$$\text{map} : \forall \alpha \beta.\ (\alpha \to \beta) \to (\alpha\ \text{list}) \to (\beta\ \text{list})$$

<u>Lemma</u> : For any $\ell : A$ list $F : A \to B$

$$(\ell,\ \text{map}\,[A][B]\,F\,\ell) \in [\![\alpha\ \text{list}]\!]_{(\alpha \mapsto A, B, R_F)}$$

$$R_F = \{(a, F\,a)\}^{=_{\beta\eta}}$$

<u>Pf</u> by ind. on ℓ

- nil $\ell = []$, $\text{map}\,[A][B]\,F\,[] =_{\beta\eta} []$

  $([], []) \in [\![\alpha\ \text{list}]\!]\,(\alpha \mapsto A, B, R_F)$ ✓

- cons $\ell = a :: \ell'$ by

  I.H. $(\ell', \text{map}\,[A][B]\,F\,\ell') \in [\![\alpha\ \text{list}]\!]_{([\Delta])}$

  $(a, F\,a) \in [\![\alpha]\!]_{[\Delta]} = R_F$

  $$(a :: \ell,\ \underbrace{(F\,a) :: \text{map}\,[A][B]\,F\,\ell'}_{=_{\beta\eta}\ \text{map}\,[A][B]\,F\,(a :: \ell')}) \in [\![\alpha\ \text{list}]\!]_{[\Delta]}$$ ✓

$$\left[\begin{array}{l} \text{map}\,[A][B]\,F\ \text{nil} \to_\beta \text{nil} \\ \text{map}\,[A][B]\,F\,(a :: \ell) \to_\beta (F\,a) :: \text{map}\,[A][B]\,F\,\ell \end{array}\right.$$

<u>Example</u> For all $r : \forall\alpha.\,\alpha\ \text{list} \to \alpha\ \text{list}$

$$F : A \to B \qquad \ell : A\ \text{list}$$

$$\text{map}[A][B]F\,(r[A]\,\ell) = r[B]\,(\text{map}[A][B]F\ell)$$

<u>Proof</u> $(r, r) \in [\![\forall\alpha.\,\alpha\ \text{list} \to \alpha\ \text{list}]\!]$

$$= \mathbb{V}\left(\lambda ABR.\ [\![\alpha\ \text{list} \to \alpha\ \text{list}]\!]\underbrace{(\alpha \mapsto A, B, R)}_{\Delta}\right)$$

pick $R = \{(a, F\,a)\}^{=_{\beta\eta}}$

$$(r[A], r[B]) \in [\![\alpha\ \text{list} \to \alpha\ \text{list}]\!]_{[\![\Delta]\!]}$$

$\iff$ for all $(\ell, \ell') \in [\![\alpha\ \text{list}]\!]_{[\![\Delta]\!]}$

$$(r[A]\,\ell,\ r[B]\,\ell') \in [\![\alpha\ \text{list}]\!]_{[\![\Delta]\!]}$$

- $(\ell,\ \text{map}[A][B]F\,\ell) \in [\![\alpha\ \text{list}]\!]_{[\![\Delta]\!]}$

$$(\underbrace{r[A]\ell}_{},\ r[B]\,(\text{map}[A][B]\,F\ell)) \in [\![\alpha\ \text{list}]\!]_{[\![\Delta]\!]}$$

$$\|$$

$$(r[A]\,\ell,\ \text{map}[A][B]F\,(r[A]\,\ell)) \in [\![\alpha\ \text{list}]\!]_{[\![\Delta]\!]}$$

"by Functionality" of $[\![\alpha\ \text{list}]\!]_{[\![\Delta]\!]}$

$$\left[\begin{array}{l} (x, M) \in R \\ (x, N) \in R \quad \text{then} \quad M =_{\beta\eta} N \end{array}\right.$$

- Consider closed A

  $\hat{A} = \forall \alpha.\ (A \to \alpha) \to \alpha$

<u>claim</u> $A \simeq \hat{A}$

- $i : A \to \hat{A}$

  $i \triangleq \lambda a{:}A.\ \Lambda\alpha.\ \lambda k{:}(A \to \alpha).\ k\ a$

  $j \triangleq \lambda x{:}\hat{A}.\ x[A](\lambda z{:}A.\ z)$

<u>claim</u> $i \circ j \cong id_{\hat{A}}$ $\quad \underbrace{j \circ i \cong id_A}$ $\beta\eta$ this

- $j \circ i = id_A$ $\quad \lambda a{:}A.\ (j(i\ a))$ $\rightsquigarrow$ a

- $i \circ j = id_{\hat{A}}$

  $\lambda x{:}\hat{A}.\ i(j(x))$

  $=_{\beta\eta} \lambda x{:}\hat{A}.\ i\ \underbrace{(x[A](\lambda z{:}A.\ z))}$

  $=_{\beta\eta} \lambda x{:}\hat{A}.\ \Lambda\alpha.\ \lambda k{:}A \to \alpha.\ k\ (x[A](\lambda z{:}A.z))$

  $= k(x[A](\lambda z{:}A.z))$

  $x[\alpha](\lambda y{:}A.\ k((\lambda z{:}A.z)\ y))$

  $\cong \lambda x{:}\hat{A}.\ x$

  $\lambda y.\ k((\lambda z{:}A.z)(x))$

  $\lambda x{:}\hat{A}.\ \underline{\Lambda\alpha.}\ \lambda k{:}A \to \alpha.\ x[\alpha](k \circ (\lambda z{:}A.z))$

  $x[\alpha](\lambda x.(k\ x)\ y)$

  $=_{\beta\eta}$

  $x[\alpha]\ k$

  $\underbrace{\lambda k.\ \quad x[\alpha]\ k}_{x[\alpha]}$

  $\lambda x{:}\hat{A}.\ x$