$$A ::= \alpha \mid \forall\alpha . A \mid A \to B$$

[1 Bool]
Pair
List

$$\Delta ; \Gamma \vdash M : A$$

$\delta : \Delta \to \text{type}$ maps $\alpha \mapsto \underbrace{\delta(\alpha)}_{\text{closed}}$

$\delta(\Gamma)$

$\gamma \in [\![\Gamma]\!]$ a substitutions maps $x \mapsto M$

$\gamma(x) : \Gamma(x)$ (closed M)

$\underline{\gamma(\delta(M))} : \underline{\delta(A)}$ where $\gamma \in [\![\delta\Gamma]\!]$

$[\![A]\!] = \{ M \mid \cdot \vdash M : A \}$ ⟸

for closed A

$$\gamma(\delta(M)) \in [\![\delta(A)]\!]$$

- $[\![A]\!]$: type → term → $\mathbb{P}$
- $[\![A]\!]$: type → term → term → $\mathbb{P}$

$\delta_1(\alpha) = \text{int}$

$\delta_2(\alpha) = \text{bool}$

$[\![\alpha]\!]_{\delta : \delta_1 \leftrightarrow \delta_2}$: int → bool → $\mathbb{P}$

$R_{\text{int bool}} = \{(x, b) \mid x = 0 \Leftrightarrow b = \text{false}\}$

isFalse : $\alpha \to$ bool

isFalse false $\to_\beta^*$ true

isFalse 0 $\to_\beta^*$ true

isFalse k $\to_\beta^*$ false   k>0

$$\mathrm{isFalse} \in [\![\alpha \to \mathrm{bool}]\!]_{\mathcal{D}}$$

$$\mathcal{D}(\alpha) = R_{\mathrm{int}\,\mathrm{Bool}}$$

$$\forall (x,b) \in [\![\alpha]\!]_{\mathcal{D}(\alpha)}.$$

$$(\mathrm{isfalse}\ x,\ \mathrm{isfalse}\ b) \in [\![\mathrm{bool}]\!]_{\mathcal{D}}$$

$$\iff \forall (x,b) \in \underbrace{R_{\mathrm{int},\mathrm{Bool}}}_{},$$

$$(\mathrm{isFalse}\ x,\ \mathrm{isFalse}\ b) \in [\![\mathrm{bool}]\!]_{\mathcal{D}}$$

$\iff$ $\forall x, b$, If

$x = 0 \iff b = \mathrm{false}$,

then $(\mathrm{isFalse}\ x,\ \mathrm{isFalse}\ b) \in \underbrace{[\![\mathrm{bool}]\!]_{\mathcal{D}}}_{}$

$$\mathrm{Id}_{\mathrm{bool}} = \{ (b_1, b_2) \mid b_1 = b_2 \}$$

$$\Delta, \Gamma \vdash M : A$$

~~If~~ given $\delta_1, \delta_2 : \Delta \to \text{type}$

$$R_{AB} : A \to B \to \mathbb{P} \qquad A \Leftrightarrow B$$

$$\mathcal{D}_{\delta_1\delta_2} : \Delta \Rightarrow \text{Rel} \qquad \mathcal{D} : \delta_1 \Leftrightarrow \delta_2$$

$$\underline{\mathcal{D}_{\delta_1\delta_2}(\alpha)} : \underline{\delta_1(\alpha) \to \delta_2(\alpha)} \to \mathbb{P}$$

$$\gamma_1, \gamma_2 : \Gamma \to \text{term}$$

iff $\forall \underline{x \in \text{dom}(\Gamma)}.$ $\qquad \underbrace{\delta_1(\underbrace{\Gamma(x)})}\qquad \underbrace{\delta_2(\underbrace{\Gamma(x)})}$

$$(\gamma_1(x), \gamma_2(x)) \in [\![\Gamma(x)]\!]_{\mathcal{D} : \delta_1 \Leftrightarrow \delta_2}$$

$\vdots$

Then $\left(\gamma_1(\delta_1(M)), \gamma_2(\delta_2(M))\right) \in [\![A]\!]_{\mathcal{D} : \delta_1 \Leftrightarrow \delta_2}$

$$[\![A]\!]_{\mathcal{D}} \qquad [\![\alpha]\!]_{\mathcal{D}} = \mathcal{D}(\alpha)$$

$$[\![A \to B]\!]_{\mathcal{D}} = [\![A]\!]_{\mathcal{D}} \Rightarrow [\![B]\!]_{\mathcal{D}}$$

$$\underbrace{R_{AA'} \Rightarrow R_{BB'}} = \{ (f, f') \mid \forall (a, a') \in R_{AA'}, (fa, f'a') \in R_{BB'} \}$$

$$\{ (M, M') \mid \cdot \vdash M : A \to B, \cdot \vdash M' : A' \to B', \forall (a, a') \in R_{AA'}, (Ma, M'a') \in R_{BB'} \}$$

- $[\![\forall \alpha . A]\!]_{\mathcal{D}} =$

$\mathcal{D} : \Delta \to rel$

$$\Big\{ (M, M') \;\Big|\; \forall_{R_{BB'}}^{BB'} \big(M[B/\alpha], M'[B'/\alpha] \big) \in [\![A]\!]_{\mathcal{D}'}$$

$$\mathcal{D}'(\beta) = \begin{cases} \mathcal{D}(\beta) & \text{if } \beta \neq \alpha \\ R_{BB'} & \text{if } \beta = \alpha \end{cases} \Big\}$$

$$\Big\{ (M, M') \;\Big|\; \cdot \vdash M : (\forall \alpha . A)\delta_1 \quad \cdot \vdash M' : (\forall \alpha A)\delta_2$$

$$\big(M[B], M'[B'] \big) \in [\![A]\!]_{\mathcal{D}'}$$

$$\ldots \Big\}$$

$\mathcal{V}[\![A]\!]$

$$\mathcal{C}[\![A]\!] = \{ (M, M') \mid M \Downarrow V \wedge M' \Downarrow V' \wedge (V, V') \in \mathcal{V}[\![A]\!] \}$$

$[\![\underline{\text{List}}\ A]\!]$

$\delta(\alpha) = A_{impl}$

- $[\![\overset{m}{\underline{map}} : \underbrace{\forall \alpha\beta.\ (\alpha \to \beta) \to \underbrace{\text{list}\ \alpha \to \text{list}\ \beta}_{TM}}$

- $\underline{r} : \forall \alpha.\ \text{list}\ \alpha \to \text{list}\ \alpha.$

$f : \text{int} \to \text{bool}$

$r_{bool} (map_{int,bool}\ f_{int \to bool}\ l_{int\ list})$ $\qquad l : \text{int list}$

$=$

$map_{int,bool}\ f_{int \to bool}\ \underbrace{(r_{int}\ l)}\ _{int\ list}$ $\qquad r : \text{int}$

$(\overset{m}{map}, \overset{m}{map}) \in \underbrace{[\![TM]\!]_{\emptyset}}$

- pick $\alpha \mapsto A, A'$
  pick $\beta \mapsto B, B'$

$\underbrace{(f, f') \in (A \to B) \Leftrightarrow (A \to B)}$

$\underbrace{m_{AB}} f = map\ f \qquad m_{AA}(Id_A) = \underbrace{m_{BB}} Id \circ (map\ f)$

if

$R_f : A \to B \to Prop =$

$\lambda a. \lambda b.\ b = fa)$

$= \underbrace{\forall\alpha.\ \alpha.}$

$$\underline{(m, m)} \in [\![\forall \alpha . \alpha]\!]_{\emptyset}$$

$$\alpha = (\forall \beta . \beta \to \beta)$$

$$R = \emptyset$$