System F - Girard-Reynolds

Polymorphic λ-calculus

$K ::= * \mid K \Rightarrow K$

Types: $A, B ::= \alpha \mid A \to B \mid \forall \alpha . A \mid \mathbb{N}$

$\mid k \mid \lambda \alpha : K . A \mid A[B]$

"pre terms" $M, N ::= x \mid M N \mid \lambda x : A . M \mid \dots$

$(\lambda x : \alpha . x)\ (\lambda y : \beta . \beta)$ $\qquad M A \mid \Lambda \alpha . M$

$$\frac{}{\Gamma \vdash x : A}\ x : A \in \Gamma$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma : M\ N : B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A . M : A \to B}\ (x \notin FV(\Gamma))$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \Lambda \alpha . M : \forall \alpha . A}\ (\alpha \notin FTV(\Gamma))$$

$\boxed{\Delta ; \Gamma \vdash M : A}$

(alternative)

$$\frac{\Gamma \vdash M : \forall \alpha . A}{\Gamma \vdash M\ B : A[B/\alpha]}$$

$$\frac{M \to M'}{\Lambda \alpha . M \to \Lambda \alpha . M}$$

$$\frac{M \to M'}{M A \to M' A}$$

$(\Lambda \alpha . M)\ A \to M[A/\alpha]$

$\lambda g : q \to q \to q .$ $\underbrace{\lambda f : q \to q . \lambda x . q .}_{\text{interface}}$ $\underbrace{f\ x}_{\text{"client"}}$ $\Big\}\ M$

$(\Lambda q . M)\ [p \to p]\ g_1 f_1 x_1$ "implement $q := \overbrace{p \Rightarrow p}^{\text{little-endian}}$"

$ss \quad ss \quad ss \quad ss$

$(\Lambda q . M)\ [r]\ g_2\ f_2\ x_2$ "implement $\underbrace{q := r}_{\text{big endian}}$"

STLC Logical Relations

$[\![-]\!] : (A : \text{type}) \to Set$

$[\![-]\!]_\delta :$

$$[\![\forall \alpha . A]\!] \delta := \bigcap_{S} [\![A]\!] \delta[\alpha \mapsto S]$$

$R_{Q\,\omega} = \{ M \mid Q \vdash M : \omega \}$

$\Gamma \qquad A$

Let V be for any $Q : V \to \text{type}$ and $x \in V$

$$\frac{}{Q \vdash x : Q(x)}$$

$\boxed{x \in R_{Q\,Q(x)}}$

- Compared to STLC

  - Nat $\triangleq \forall \alpha.\ \underbrace{(\alpha \to \alpha)}_{S} \to \underbrace{(\alpha \to \alpha)}_{z}$ $\lambda x : \mathrm{Nat}.$
  - Bool $\triangleq \forall \alpha.\ \alpha \to \alpha \to \alpha$

    SUM(A,B) $\forall \alpha. (A \to \alpha) \to (B \to \alpha) \to \alpha$

    $\mathrm{inl}_{A+B}\ \underset{:A}{M} : \Lambda \alpha. (\lambda k_1 : A \to \alpha). (\lambda k_2 : B \to \alpha).$
    $k_1 M.$

  - $\mathrm{PAIR}(A,B) \triangleq \forall \alpha.\ (A \to B \to \alpha) \to \alpha$

    $\pi_1 \underset{\mathrm{PAIR}(A,B)}{M} = (M, A)(\lambda x : A. \lambda y. B. x)$

- $\mathrm{List}\ A \triangleq \forall \alpha.\ (A \to \alpha \to \alpha) \to \alpha \to \alpha$

- Type inference
  - STLC $\cdot\ \underline{\lambda x.\ x}$
  - $\underline{\Lambda \alpha. \lambda x : \alpha.\ x}$
  -

(Im)Predicativity   System F

$$\underbrace{\underbrace{(\Lambda\alpha.\,\lambda x{:}\alpha.\,x)}_{ID}\;\underbrace{[\forall\beta.\beta\to\beta]}}\;ID \quad : \forall\beta.\beta\to\beta$$

$$\underbrace{(\forall\beta.\beta\to\beta)}\to(\forall\beta.\beta\to\beta)$$

$$\to (\lambda x{:}(\forall\beta.\beta\to\beta).\,x)\;ID$$

$$\to_\beta ID$$

$\forall\alpha_1\ldots\alpha_n.\,\sigma$

$[\;* \Rightarrow *\;]$

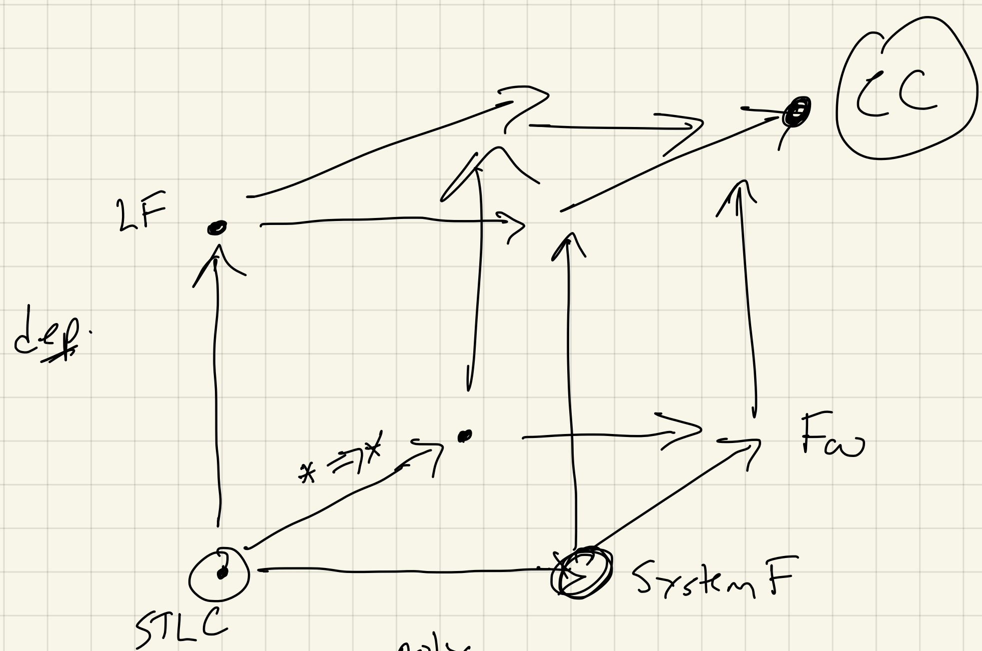
$A : *$ $* : ?$

$Type_0$ $Type_0 : Type_1$ . . .

$$\left[\;(\forall \cancel{X}{:}*.\;\ldots\;---)\quad(\forall X{:}*.\;\_\_)\right.$$

$\underline{F}$ + fix

CIC



poly

$\Lambda\alpha. M$

$\underline{List} : * \Rightarrow *$

- Inductive ✓
- Strong sums • Σ
- Universes $*_0 : *_1 \quad *_1 : *_2 \quad \ldots$
- Divergence
  Fix
  $\mu X. X \Rightarrow X$

$M : \boxed{\forall K : * \Rightarrow * . \_\_\_\_}$

M

Definition f {M:Type⇒Type}(Monad M)

type: $A, B ::= \underline{\alpha} \mid \underbrace{A \to B} \mid \underbrace{\forall \alpha . A}$

$\underline{\text{fTV}(M)} = \alpha \quad \text{fv} \quad \underbrace{\{\underline{f} : \alpha \to \alpha,\ \underline{g} : \alpha \to \alpha \to ?,\ \underline{x} : \alpha\}}_{\Gamma}$

M

$\Gamma \vdash M : A$

$\Delta \qquad \delta : \Delta \to \text{type}$ closed "representation"

$\delta_1(M) \qquad \delta_1(\alpha) = \mathbb{N}$

$\delta_2(M) \qquad \delta_2(\alpha) = \underbrace{(\forall \beta . (\beta \to \beta) \to (\beta \to \beta))}_{CE}$

$\gamma_1 : (x\,\text{var}) \to \delta(\Gamma(x))$

$\gamma_2 : (x : \text{var}) \to \delta_2(\Gamma(x))$ closed

$\gamma_1(\delta_1(M)) \underbrace{\simeq_R} \gamma_2 \delta_2(M))$

$R : \Delta \to \text{Rel} \qquad R(\alpha) : \mathbb{N} \longleftrightarrow CE$

$0 \longleftrightarrow \Lambda \beta . \lambda f . \lambda z . z$

$\vdots$