STLC

Types $A, B ::= \iota \mid A \to B$

Terms $M, N ::= x \mid \lambda x{:}A.M \mid M\,N$

$\Gamma \vdash c : A$

$\mid c$

β-Reduction $\boxed{M \to_\beta N}$

$$(\lambda x{:}A.M)\,N \to_\beta M[N/x] \qquad (\beta)$$

$$\frac{M \to_\beta M'}{\lambda x{:}A.M \to_\beta \lambda x{:}A.M'} \qquad (\xi)$$

$$\frac{M \to_\beta M'}{M\,N \to_\beta M'\,N} \qquad \frac{N \to_\beta N'}{M\,N \to_\beta M\,N'} \qquad (\text{cong.})$$

Type System $\Gamma ::= \cdot \mid \Gamma, x{:}A$

$\boxed{\Gamma \vdash M : A}$

$$\frac{}{\Gamma \vdash x : A}\,(x{:}A) \in \Gamma \qquad (\text{var})$$

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \to B} \qquad (\text{abs})$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B} \qquad (\text{app})$$

Subject Reduction

If $\Gamma \vdash M : A$ and $M \twoheadrightarrow_\beta M'$
then $\Gamma \vdash M' : A$.

Strong Normalization

- M Does not have any infinite $\to_\beta$ Sequences : $M \to_\beta M_1 \to_\beta M_2 \to_\beta \cdots$
- M is normal if $\neg \exists M'.\ M \not\to_\beta M'$
- Define normal forms like this

$$\begin{aligned} nf &::= \lambda x{:}A.\, nf \mid hnf \\ hnf &::= x \mid hnf\ nf \end{aligned}$$

think of these as predicates

Lemma $nf(M)$ iff $M \not\to_\beta$.

The set of Strongly normalizing terms. $\mathcal{S}$

$$\frac{(\forall N,\ M \to_\beta N \Rightarrow N \in \mathcal{S})}{M \in \mathcal{S}}$$

Lemma nf(M) then $M \in \mathcal{S}$.

Theorem Strong Normalization

If $\Gamma \vdash M : A$ then $M \in \mathcal{S}$.

? Proof: By induction on the derivation:

- $\Gamma \vdash x : A$ then $x \in \mathcal{S}$ ✓

- $$\frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash \lambda x:A.M : A \to B}$$

  by I.H. $M \in \mathcal{S}$ $\lambda x:A.M \in \mathcal{S}$

- $$\frac{\Gamma \vdash M_1 : B \to A \quad \Gamma \vdash M_2 : B}{\Gamma \vdash M_1\,M_2 : A}$$

  - $M_1 \in \mathcal{S}$
  - $M_2 \in \mathcal{S}$ $M_1 M_2 \in S$

??

$M_1 = \lambda x:B.N$

Idea: Logical Relation

Build a family of sets inductively on the types. Those sets will be contained in $\mathcal{S}$.

$\Gamma \nvdash \Omega : A$

$(\lambda x. xx) \in \mathcal{S}$

$\omega\omega \notin \mathcal{S}$

<u>Define</u> $[\![-]\!] : \text{type} \to \mathcal{P}(\text{tm})$

$[\![2]\!] = \mathcal{S} = \{ M \mid \cdot \vdash M : 2 \text{ and } M \in \mathcal{S} \}$

$[\![A \to B]\!] = \{ M \mid \forall N \in [\![A]\!],\ MN \in [\![B]\!] \}$

$\cdot \vdash M : A \to B$

<u>Lemma</u> for all A

- $[\![A]\!] \subseteq \mathcal{S}$

pf by induction A

- case $A = 2$ then $[\![2]\!] = \mathcal{S} \subseteq \mathcal{S}$ ✓
- case $A = B \to C$, then

Let M be given as to show $M \in \mathcal{S}$.

$(\forall N,\ M \to_\beta N \Rightarrow N \in \mathcal{S})$

let N be given assume $M \to_\beta N$

X

- Lemma (take 2)

For all A

- (1) $\forall M \in [\![A]\!]$, $M \in \mathcal{S}$.
- (2) $\forall \underline{M}$, $\left(\forall P,\ M \to_\beta P \Rightarrow P \in [\![A]\!]\right)$

$\underline{Q(M)}$ if M is $\underline{\text{neutral}}$

- $x \in [\![A]\!]$ (by (2)) because $x \in$ neutral.

✓ ✓

M neutral $M \neq \lambda x{:}A.N$ (M is a var or app).

Proof Sketch

- Ind on A

(1) $A = 2$ $\quad \underline{[\![2]\!] = \mathcal{S}} \subseteq \mathcal{S}$

(2) $A = \dot{2}$ Let M be given. (and neutral M)

for all P $\underbrace{M \to_\beta P}$, $\underbrace{P \in [\![2]\!]}_{= \mathcal{S}}$

✓.

$A = B \to C$ $\quad M : B \to C$

idea: pick free variable $y \in [\![B]\!]$

by I.H. 2 and neutral y.

$y \in [\![B]\!]$ $\quad \underline{My} \in [\![C]\!]$ by I.H.

My

Lemma $\Gamma \vdash M : A$ then $M \in [\![A]\!]$.

By ind. on typing

- $\Gamma \vdash x : A$ $\quad x \in [\![A]\!]$ ✓

- $\Gamma \vdash M\,N : B$ $\quad M\,N \in [\![B]\!]$

  by. I.H. $\Gamma \vdash M : C \to B$

  an $M \in [\![C \to B]\!]$

  and $N \in [\![C]\!]$

  by $[\![C \to B]\!]$ $\quad (M\,N) \in [\![B]\!]$

- $$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A . M : A \to B}$$

to show :

$$(\lambda x : A . M) \in [\![A \to B]\!]$$

so. Let $N \in [\![A]\!]$

to show $\underline{((\underbrace{\lambda x : A . M) N}_{M[N/x]}) \in [\![B]\!]}$??

Know: $M \in [\![B]\!]$ $\quad \underline{\underline{M[N/x]}}$

Lemma $M \in [\![A]\!]$ and $M \to_\beta M'$ then

$M' \in [\![A]\!]$

Define a substitution $\sigma : \underline{Var \to Tm}$.

$\underline{\text{Defn}}$. $\underline{\Gamma} \vDash \sigma$ iff $dom(\Gamma) = dom(\sigma)$

$\Gamma \vdash \gamma$

$\gamma \in \Gamma$

and $\forall x \in dom(\Gamma).\ \sigma(x) \in \underline{[\![\Gamma(x)]\!]}$

$\boxed{\begin{array}{l} \underline{\text{Theorem}}\ \text{The fundamental theorem of logical relations.} \\ \text{If } \Gamma \vdash M : A \text{ and } \Gamma \vDash \sigma \text{ then} \\ \underline{\sigma(M)} \in [\![A]\!]. \end{array}}$

$\underline{\text{Proof}}$ By ind. on M.

- $\underline{\text{case}}$ $M = x$ $\left\{ \sigma(x) = \begin{cases} x & \text{if } x \notin dom(\sigma) \\ N & \text{if } x \in dom(\sigma) \end{cases} \right.$

$\sigma(x) \in [\![A]\!]$ ✓

- $\underline{\text{case}}$ $M_1 M_2$ $\sigma(M_1 M_2) = (\sigma(M_1)\ \sigma(M_2))$

✓

- $\underline{\text{case}}$ $\underbrace{\lambda x{:}A.M}$ $: A \to B$ $\sigma(\lambda x{:}A.M) = \lambda x{:}A.\underbrace{\sigma(M)}$

to show: for $N \in [\![A]\!]$ by I.H.

$(\lambda x{:}A.M)\ N \in [\![B]\!]$

pick $\sigma' = \underline{\sigma[x \mapsto N]}$ claim $\underline{\Gamma, x{:}A \vDash \sigma'}$

by. I.H. $\sigma'(M) \in [\![B]\!]$

$= \underline{\sigma(M)[N/x]}$ want: $(\lambda x{:}A.M)N \in [\![B]\!]$ $\square$

but that follows by

<u>Lemma</u>: $[\![-]\!]$ is closed under β-expansion.

<u>Lemma</u> if $M[N/x] \in [\![B]\!]$ and $N \in [\![A]\!]$

then $(\lambda x{:}A.M)N \in [\![B]\!]$.

$(\lambda x.\lambda y.y)\,\Omega \notin \mathcal{S}$

but $(\lambda y.y)[\Omega/x] \in \mathcal{S}$

<u>Closed</u> $\square$

<u>Note</u> $x \in [\![A]\!]$ the identity substitution

$\Gamma \models id$

Instantiate the fundamental thm. with $\sigma = id$.

Conclude: If $\Gamma \vdash M : A$ then $M \in \mathcal{S}$. $\square$

- Tait's method
- Girard's candidates of reducibility.