Par means Parallel – Multiplicative Linear Logic

Proofs as Concurrent Functional Programs

POPL 2020 – Aschieri + Genco

$$R \triangleq \nu z\,(\,x(y).\,z(w).\,P \mid z[a].\,Q\,)$$

In CP (Wadler's Classical Processes)

$$R \to x(y).\,\nu z\,(\,z(w).\,P \mid z[a].\,Q\,)$$

$\lambda_{⅋}$ $\quad A ::= \alpha \mid \underline{A \multimap B} \mid A ⅋ B \mid \bot$

$$\Rightarrow \left\{ A \otimes B \triangleq ((A \multimap \bot) ⅋ (\underline{B \multimap \bot})) \multimap \bot \right.$$

$$(A \wedge B) = \neg(\neg A \vee \neg B)$$

$$\left. \begin{aligned} A \multimap B &\equiv A^{\perp} ⅋ B \end{aligned} \right]$$

$$A \to B \equiv \neg A \vee B$$

$x, y, z =$ channels

$t, u, v \ ::= \ 0$ terminated

$\bar{x}.u$ output on channel x continue as u

$\bar{x}\bar{y}\ u$ without contination

$u \mid t$

$\underline{close(t)}$ $\qquad close(0) \longrightarrow$

x

$u\ v$

syntactic sugar $\quad \lambda x\ u \triangleq \bar{x}.u \quad (x \in fv(u))$

$\bar{x}v.u \triangleq (\bar{x}.u)\ v$

$D[\ ], C[\ ] \ ::= \ \ldots \mid [\ ]$ ← "hole" that appers once

CBN

$(\lambda x\ u)\ t \longrightarrow u[t/x] \quad (\beta)$

$C[\bar{x}u.s] \mid D[x] \longrightarrow C[s] \mid D[u] \quad (comm)$

(and sym)

$C[\bar{X}\bar{Y}(s \mid t)] \mid D[x][y] \rightarrow C[0] \mid D[s][t]$

(and sym)

$D[x] \mid C[\bar{x}\bar{y}(s \mid t)] \mid E[y] \rightarrow D[s] \mid C[0] \mid E[t]$

(and symm)

$$\frac{t \rightarrow u}{C[t] \rightarrow C[u]}$$

$$
\begin{tikzcd}
(\lambda y\ \bar{x}(y\,w).0)\,v \mid x \arrow[d] \arrow[dr, phantom, "\equiv"] & \\
\bar{x}(v\,w).0 \mid x \arrow[dd] & (\bar{y}.\bar{x}(y\,w).0)\,v \mid x \arrow[d] \\
& (\bar{y}.0)\,v \mid y\,w \arrow[d, phantom, "\equiv"] \\
0 \mid v\,w & (\bar{y}\,v.0) \mid y\,w \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bar{x}\,v.y \mid \bar{y}\,x.0 \arrow[d] \arrow[r] & \bar{x}\,v.x \mid 0 \arrow[d, phantom, "\equiv"] \\
y \mid \bar{y}\,v.0 \arrow[dr] & (\lambda x.x)\,v \mid 0 \arrow[d] \\
& v \mid 0
\end{tikzcd}
$$

$\text{cost prod} \to^{*} \text{price}$

$$
\begin{tikzcd}
\bar{x}\ \text{prod}.y \mid \bar{y}(\text{cost}\ x).0 \arrow[d] \arrow[r] & \bar{x}\ \text{prod}.\text{cost}\,x \mid 0 \arrow[d, phantom, "\equiv"] \\
y \mid \bar{y}(\text{cost prod}).0 \arrow[d, "*"] & (\lambda x.\text{cost}\,x)\,\text{prod} \mid 0 \arrow[d] \\
y \mid \bar{y}(\text{price}).0 \arrow[dr] & \text{cost prod} \mid 0 \arrow[d, "*"] \\
& \text{price} \mid 0
\end{tikzcd}
$$

$$\bar{x}(\lambda a \lambda b\ a\ (b\ prod)).y \mid x(\bar{y}.0)\ cost$$

$$: \mathbb{N} \parr \bot$$

$$y \mid (\lambda a \lambda b\ a\ (b\ prod))\ (\bar{y}.0)\ cost$$

$$\to^* y \mid \bar{y}(cost\ prod).0$$

$$\to^* y \mid \bar{y}(price).0$$

$$\to price \mid 0$$

$$(cost\ prod) \mid 0 \to^* price \mid 0$$

Types

$$A ::= \alpha \mid A \multimap B \mid A \parr B \mid \bot$$

$$\Gamma ::= \cdot \mid \Gamma, x : A$$

$$\Delta ::= \cdot \mid \Delta, t : A \mid \Delta, u$$

$$\boxed{\Gamma \Rightarrow \Delta}$$

$$x:A \Rightarrow x:A$$

$$\frac{\Gamma \Rightarrow s:A,\ t:B,\ \Delta}{\Gamma \Rightarrow (s|t) : A \parr B,\ \Delta}\ \parr I$$

$$\frac{\Gamma \Rightarrow s : A \parr B, \Delta \qquad \Sigma_1, x:A \Rightarrow \Theta_1 \qquad \Sigma_2, y:B \Rightarrow \Theta_2}{\Gamma, \Sigma_1, \Sigma_2 \Rightarrow \overline{x}\,\overline{y}\,s : \bot, \Delta, \Theta_1, \Theta_2}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, o:\bot} \qquad\qquad \frac{\Gamma \Rightarrow t:\bot, \Delta}{\Gamma \Rightarrow \mathrm{close}(t), \Delta}$$

$$= \frac{\Gamma, x:A \Rightarrow t:B, \Delta}{\Gamma \Rightarrow \lambda x\, t : A \multimap B, \Delta} \quad (\text{if } x \text{ occurs in } t)$$

$$\overline{x}.t \stackrel{\Delta}{=} \lambda x.t$$

$$x \in fv(t)$$

$$\frac{\Gamma, x:A \Rightarrow t:B, \Delta}{\Gamma \Rightarrow \overline{x}.t : A \multimap B, \Delta} \quad (x \text{ occurs in } \Delta)$$

$$\frac{\Gamma \Rightarrow s : A \multimap B, \Delta \qquad \Sigma \Rightarrow t:A, \Theta}{\Gamma, \Sigma \Rightarrow s\,t : B, \Delta, \Theta}$$

Substitution $\Sigma \Rightarrow t : A, \Theta$

$\Gamma, x : A \Rightarrow \Delta$

then $\Sigma, \Gamma \Rightarrow \underbrace{\Delta[t/x]}, \Theta$

Subject Reduction $\Gamma \Rightarrow \Delta$

and $\Delta \to \Delta'$

$\Delta = \quad t_1 : A_1 \mid \ldots \mid t_n : A_n \mid close(t_{n+1}) \ldots$

Δ'

$\Gamma \Rightarrow \Delta'$

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

$!x(\underline{y}).P \qquad !P \equiv P \mid !P$