Session Types as Intuitionistic Linear Logic propositions — Caires and Pfenning 2010

$$P ::= 0 \mid (\nu x)P \mid x[y].P \mid x(y).P \mid P|Q$$

$$!x(y).P$$

$$x.\mathrm{inl}; P \mid x.\mathrm{inr}; P \mid x.\mathrm{case}(P, Q)$$

$$x.\mathrm{inl}; P \mid x.\mathrm{case}(Q, R) \rightarrow P \mid Q$$

$$x.\mathrm{inr}; P \mid x.\mathrm{case}(Q, R) \rightarrow P \mid R$$

Server that allows a client to either
- make a purchase sends product id, payment
- query the price

$$\mathrm{ServerBody}_s \triangleq s.\mathrm{case}(\, s(pn).s(cn).(\nu rc)\, s[rc].0 \,,\; s(pn).(\nu p).s[p].0 \,)$$

$$\mathrm{Server}_c \triangleq\ !c(s).\, \mathrm{ServerBody}_s$$

$$\mathrm{ClientBody}_s \triangleq s.\mathrm{inl}; (\nu\, item)\, s[item].(\nu\, pin)\, s[pin].\, s(receipt).0$$

$$\mathrm{ServerBody}_s \mid \mathrm{ClientBody}_s \mid \mathrm{ClientBody}_s$$

$$\mathrm{Client}_c \triangleq (\nu s)\, c[s].\, \mathrm{clientBody}_s$$

$$\mathit{SessionProt} \triangleq \left(\begin{array}{l} N \multimap I \multimap (N \otimes 1) \\ \quad \& \\ N \multimap (I \otimes 1) \end{array} \right)$$

$$!\,\mathit{SessionProt}$$

$$\mathit{ClientProt} \triangleq \begin{array}{l} N \otimes (I \otimes (N \multimap 1)) \\ \quad \oplus \\ N \otimes (I \multimap 1) \end{array}$$

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send A, then continue as B

$$A, B ::= 1 \mid\ !A \mid A \otimes B \mid A \multimap B$$
$$\mid A \oplus B \mid A \,\&\, B$$

receive A and continue as B

$$\boxed{\Gamma ; \Delta \vdash A}$$

$$\Gamma ::= \cdot \mid u : A, \Gamma$$
$$\Delta ::= \cdot \mid x : A, \Delta$$

$$\Gamma ; \Delta \vdash P :: x : A$$

" A service that offers to follow protocol A along channel x, implemented by P "

$$\boxed{\cdot\ ; s : \mathit{SessionProt} \vdash \mathit{ClientBody}_s :: \_ : 1}$$

$$\boxed{\Gamma ; \Delta \vdash P :: x : A} \qquad T ::= x : A$$

$$\Gamma ; \cdot \vdash 0 :: x : \mathbf{1} \qquad \frac{\Gamma ; \Delta \vdash P :: T}{\Gamma ; \Delta , x : \mathbf{1} \vdash P :: T}$$

$$\frac{\Gamma ; \Delta , y : A \vdash P :: x : B}{\Gamma ; \Delta \vdash x(y).P :: x : A \multimap B}$$

$$\frac{\Gamma ; \Delta \vdash P :: y : A \qquad \Gamma ; \Delta' , x : B \vdash Q :: T}{\Gamma ; \Delta , \Delta' , x : A \multimap B \vdash (\nu y)\, x[y].(P \mid Q) :: T}$$

$$\frac{\Gamma ; \Delta \vdash P :: y : A \qquad \Gamma ; \Delta' \vdash Q :: x : B}{\Gamma ; \Delta , \Delta' \vdash (\nu y)\, x[y].(P \mid Q) :: x : A \otimes B}$$

$$\frac{\Gamma ; \Delta , y : A , x : B \vdash P :: T}{\Gamma ; \Delta , x : A \otimes B \vdash x(y).P :: T}$$

$$\frac{\Gamma ; \Delta \vdash P :: x : A}{\Gamma ; \Delta \vdash x.\mathrm{inl} ; P :: x : A \oplus B}$$

(and sim. for inr)

$$\frac{\Gamma ; \Delta , x : A \vdash P :: T \qquad \Gamma ; \Delta , x : B \vdash Q :: T}{\Gamma ; \Delta , x : A \oplus B \vdash x.\mathrm{case}(P, Q) :: T}$$

$$\frac{\Gamma ; \Delta \vdash P :: x : A \qquad \Gamma ; \Delta \vdash Q :: x : B}{\Gamma ; \Delta \vdash x.\mathrm{case}(P, Q) :: x : A \,\&\, B}$$

$$\frac{\Gamma ; \Delta , x : A \vdash P :: T}{\Gamma ; \Delta , x : A \,\&\, B \vdash x.\mathrm{inl} ; P :: T}$$

(and sim. for inr)

$$\frac{\Gamma ; \cdot \vdash Q :: y : A}{\Gamma ; \cdot \vdash !x(y).Q :: x : !A}$$

$$\frac{\Gamma , u : A ; \Delta \vdash P\{u/x\} :: T}{\Gamma ; \Delta , x : !A \vdash P :: T}$$

$$\frac{\Gamma , u : A ; \Delta , y : A \vdash P :: T}{\Gamma , u : A ; \Delta \vdash (\nu y) u[y].P :: T}$$

$$\frac{\Gamma ; \Delta \vdash P :: x : A \qquad \Gamma ; \Delta', x : A \vdash Q :: T}{\Gamma ; \Delta, \Delta' \vdash (\nu x)(P \mid Q) :: T} \text{(cut)}$$

$$\frac{\Gamma ; \cdot \vdash P :: y : A \qquad \Gamma, u : A ; \Delta \vdash Q :: T}{\Gamma ; \Delta \vdash (\nu u)(!u(y).P \mid Q) :: T} \text{(cut)}$$

$$\Gamma ; \Delta \vdash P :: z : A \qquad P \to Q \qquad \Gamma ; \Delta \vdash Q :: z : A$$

$$\frac{\Gamma ; \Delta \vdash P :: - : 1 \qquad \Gamma ; \Delta' \vdash Q :: T}{\Gamma ; \Delta, \Delta' \vdash P \mid Q :: T}$$

<u>Lemma</u> for any type A and distinct channel names x, y there is a cut-free process $id_A(x,y)$

$$\cdot ; \; x : A \vdash id_A(x;y) :: y : A$$

x

$$x : A \otimes B \vdash x(z).\ (\nu n)\ y[n].\ (\mathrm{id}_B(x, n) \mid \mathrm{id}_A(z, y)) :: \underline{y : B \otimes A}$$