

Untyped λ -Calculus - Functions

terms $N, M := x \mid (\lambda x. M) \mid (MN)$

α -equivalence $M =_{\alpha} M'$ (rename bound variables)

Free Variables:

$$\begin{aligned} \text{fv}(x) &= \{x\} \\ \text{fv}(\lambda x. M) &= \text{fv}(M) \setminus \{x\} \\ \text{fv}(MN) &= \text{fv}(M) \cup \text{fv}(N) \end{aligned}$$

Capture avoiding Substitution: $M[N/x]$

- replaces free occurrences of x in M by N
- renames bound variables in M to avoid capturing the free variables of N

e.g. $(\lambda x. y x) [\lambda z. x / y] =_{\alpha} (\lambda w. (\lambda z. x) w)$

β -reduction

- what it means for a λ -term to compute

- $\boxed{M \rightarrow_{\beta} N}$ "M β -reduces in one step to N"

$$\underline{(\lambda x. M) N} \xrightarrow{\beta} M[N/x] \quad (\beta)$$

β -redex

$$\frac{M \rightarrow_{\beta} M'}{MN \rightarrow_{\beta} M'N}$$

$$\frac{N \rightarrow_{\beta} N'}{MN \rightarrow_{\beta} MN'}$$

(congruence)

$$\frac{M \rightarrow_{\beta} M'}{\lambda x. M \rightarrow_{\beta} \lambda x. M'} \quad (\xi)$$

- $M \rightarrow_{\beta} N$ (also written $M \xrightarrow{\beta}^* N$)
 - reflexive, transitive closure of \rightarrow_{β}
- $M =_{\beta} N$
 - reflexive, symmetric, transitive closure of \rightarrow_{β}
(smallest equivalence relation containing \rightarrow_{β})
- M is in (β) normal form if there is no M' such that $M \rightarrow_{\beta} M'$.

example :

$$\begin{aligned}
 & (\lambda x.y)((\lambda z.zz)(\lambda w.w)) \\
 \rightarrow_{\beta} & (\lambda x.y) ((\lambda w.w)(\lambda w.w)) \\
 \rightarrow_{\beta} & (\lambda x.y) (\underline{\lambda w.w}) \\
 \rightarrow_{\beta} & y
 \end{aligned}$$

or

$$\begin{aligned}
 & (\lambda x.y) \overbrace{((\lambda z.zz)(\lambda w.w))}^N \\
 \rightarrow_{\beta} & y
 \end{aligned}
 \quad
 \begin{aligned}
 & (\lambda x.y) N \rightarrow_{\beta} \\
 & y[N/x] =_{\alpha} y
 \end{aligned}$$

Q: Does \rightarrow_{β} always terminate?

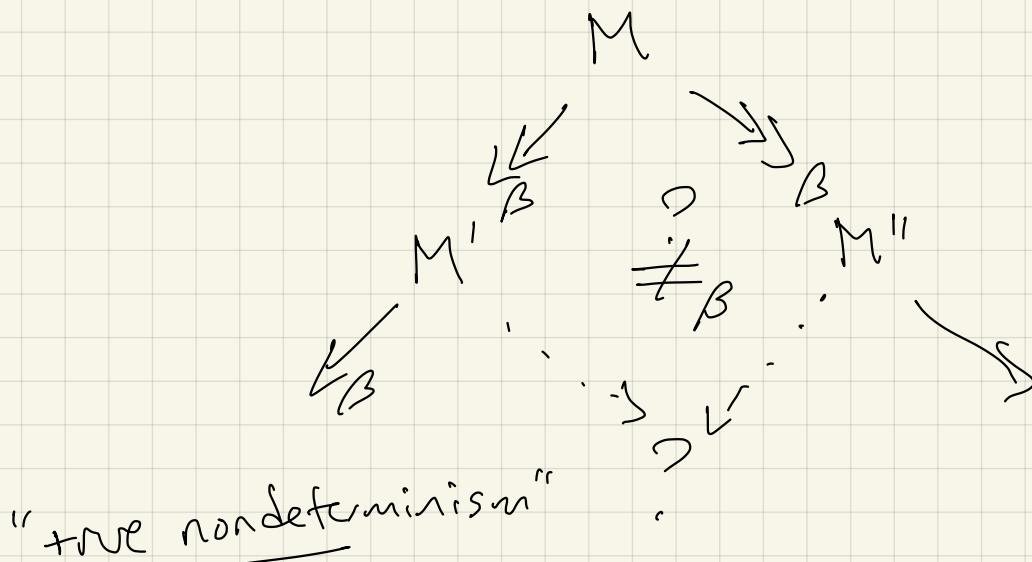
$$\omega = (\lambda x.xx)$$

$$\Omega = \omega\omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \Omega \rightarrow_{\beta} \dots$$

$$(\lambda y.x) \underbrace{\Omega}_{\downarrow \beta} \rightarrow_{\beta} x$$

$$(\lambda y.x) \Omega \xrightarrow{\beta} \dots$$

M
terminates iff
 $\exists N, \text{normal}(N)$
s.t. $M \rightarrow_{\beta} N$



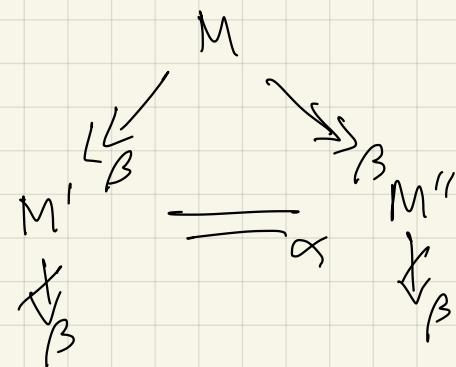
"true nondeterminism"

$$\begin{array}{ll}
 e_1 \oplus e_2 \rightarrow e_1 & x \oplus y \rightarrow x \\
 e_1 \oplus e_2 \rightarrow e_2 & x \oplus y \rightarrow y
 \end{array}$$

$$\begin{array}{c}
 M \xrightarrow{\beta} N \not\xrightarrow{\beta} \\
 \downarrow \quad \nearrow \beta \\
 M' \xrightarrow{\beta} N
 \end{array}$$

$$M =_{\beta} N$$

"weak normalization"



examples of $=_\beta$

$$\lambda x. \lambda y. x \neq_\beta \lambda x. \lambda y. y$$

$$(\lambda x. y) M =_\beta y \quad (\text{for any } M)$$

Q: If $M \rightarrow_\beta N$ can N be "bigger" than M ?

Q: If M, N both don't terminate

$$M \rightarrowr_\beta M' \rightarrow \dots$$

$=_\beta$

$$N \rightarrowr_\beta N' \rightarrow \dots$$

$$M =_\beta N ?$$

$$\Omega \rightarrow_\beta \Omega$$

$$(\lambda x. x \times x)(\underbrace{\lambda x. x \times x}_v) \rightarrow_\beta (\lambda v) v \rightarrow_\beta vvvv \rightarrow$$

Extensionality (in general)

Two functions f and g are extensionally equal

if for all v , $(fv) = (gv)$

e.g. $f : \mathbb{R} \rightarrow \mathbb{R}$ $g : \mathbb{R} \rightarrow \mathbb{R}$

$$\forall x \in \mathbb{R}, f(x) = g(x)$$

"they have the same graph"

$$f(x) = x^2 + 2x + 1 \quad g(x) = (x+1)^2$$

λ -calculus two terms M and N are extensionally equal if, for all A , $(MA) \underset{\beta}{=} (NA)$

What about: y vs. $\lambda x. y x$?

$$(\lambda x. y x) A \rightarrow_{\beta} y A$$

(But note that $y \neq_{\beta} \lambda x. y x$)

how to prove it?)

$$(\lambda w. M) N \stackrel{w}{=} M$$

when $w \notin FV(M)$

Lemma

$(\forall A. MA =_{\beta} M'A) \text{ iff } Mx =_{\beta} M'x$

for some $x \notin \text{fv}(M) \cup \text{fv}(M')$ (i.e. x is fresh)

Proof

• \Rightarrow assume $(\forall A. MA =_{\beta} M'A)$ let x be fresh
then $Mx =_{\beta} M'x$ by instantiting $A := x$

• \Leftarrow suppose $Mx =_{\beta} M'x$ (for x fresh)
Let A be given.

then $MA =_{\beta} (\lambda x. Mx) A =_{\beta} (\lambda x. M'x) A$
 $=_{\beta} M'A$

□

Eta reduction

$$\lambda x. M_x \rightarrow_{\eta} M \quad (x \notin \text{fv}(M)) \quad (\eta)$$

- $\rightarrow \rightarrow_{\eta}$ (or \rightarrow^*_{η})

- $\rightarrow_{\beta\eta}$ both β and η rules

- $=_{\eta}$
- $=_{\beta\eta}$

induced equivalences

equivalent to extensionality principle.

$$M =_{\beta\eta} N \text{ iff } \forall A, MA =_{\beta} NA$$

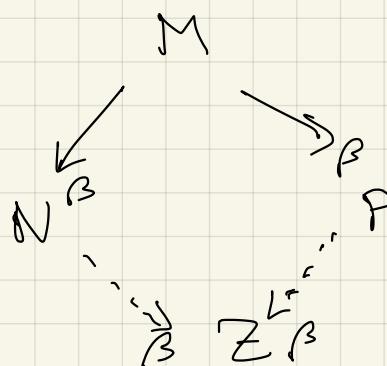
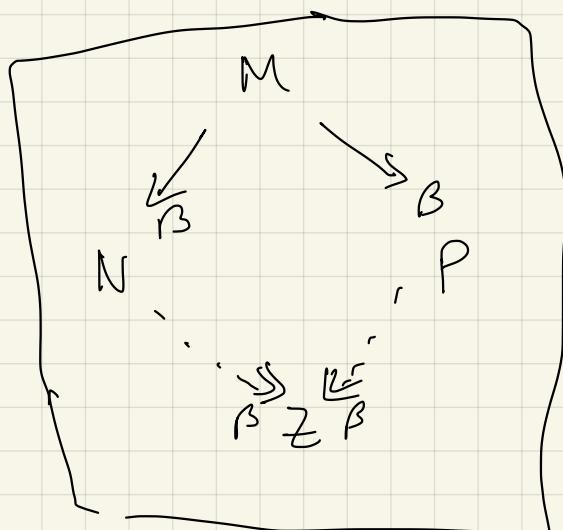
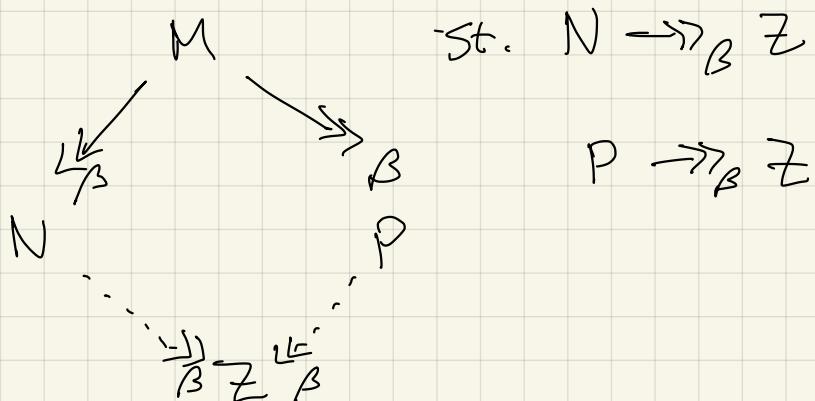
Eta expansion: $M \rightarrow_{\eta+} \lambda x. M_x \quad (x \notin \text{fv}(M))$

Confluence (or, the Church Rosser property)

Theorem: (Church Rosser)

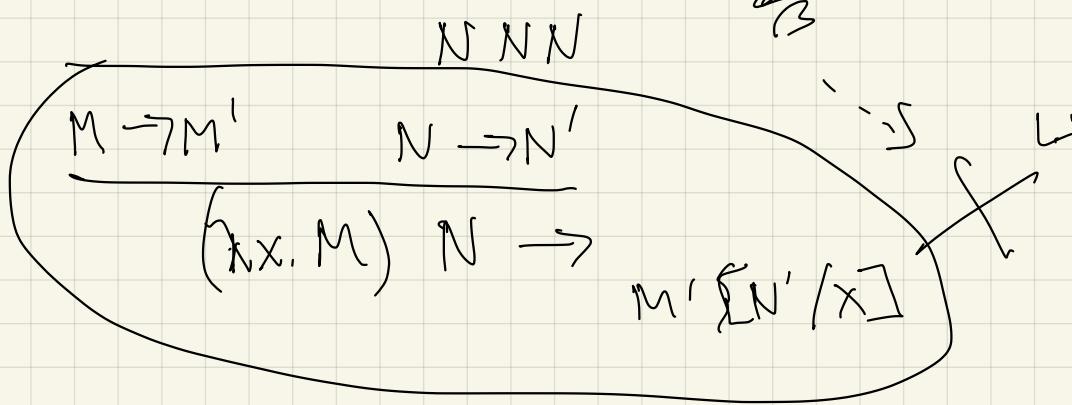
If M , N , and P are 2-terms and

$M \rightarrow_{\beta} N$ and $M \rightarrow_{\beta} P$ then $\exists Z,$



($\lambda x. x \times x$)

$$\overbrace{((\lambda y.y)(\lambda w.w))}^N$$



Reduction vs. Evaluation

$$(\lambda x. M) \underline{N} \rightarrow_{\beta} M[N/x]$$

$$\frac{M \rightarrow_{\beta} M'}{MN \rightarrow_{\beta} M'N}$$

$$\frac{N \rightarrow N'}{MN \rightarrow_{\beta} M N'}$$

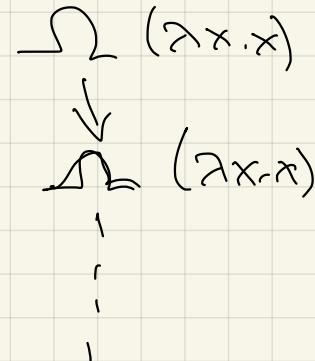
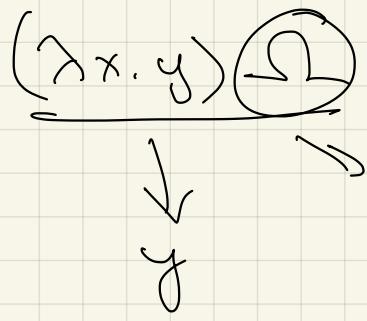
$$\frac{M \rightarrow_{\beta} M'}{\lambda x. M \rightarrow_{\beta} \lambda x. M'}$$

(don't do this!)

① Can you make a deterministic version?

② How does that choice affect the theory?

$$\rightarrow_n \quad \rightarrow_{bv} \quad \rightarrow_{bn}$$



Normal Order / Call by name evaluation

Call - by name

$$(\lambda x. M) N \xrightarrow{bn} M[N/x] \quad (\beta)$$

$$\frac{M \xrightarrow{bn} M'}{MN \xrightarrow{bn} M'N} \quad x \xrightarrow{bn} x$$

$$M \xrightarrow{bn} M' \xrightarrow{bn} M''$$

"by-name normal" weak head normal

$$\boxed{-x} \quad \boxed{\lambda x. M} \quad \boxed{(x \ M_1 \ M_2) \dots M_n} \quad =_{bn} \quad =\beta$$

Normal order evaluation - "leftmost outermost"

- Coincides with $=\beta$ for terms that have a normal

[lazy evaluation / call-by-need]

Normal order evaluation

CBN

$$\text{whnf} = \rightarrow_{bn}$$

$$(\lambda x. M) N \rightarrow_{bn} M[N/x]$$

$$\rightarrow_\beta \Rightarrow \rightarrow_{bn}$$

$$\frac{M \rightarrow_n M'}{M N \rightarrow_{bn} M' N}$$

$$\rightarrow_{bn} \subseteq \rightarrow_\beta$$

$$\rightarrow_n \subseteq \rightarrow_\beta$$

Normal order

$$\frac{M \rightarrow_n M'}{\lambda x. M \rightarrow_n \lambda x. M'}$$

$$(\lambda x. M) N \rightarrow_n M[N/x]$$

$$\frac{N \rightarrow_n N'}{x N \xrightarrow[\text{hnf}]{} x N'}$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_n M' N}$$

$$\frac{M \rightarrow \text{hnf} M'}{M N \rightarrow_n M' N}$$

Lemma $M \not\rightarrow_{bn} \Gamma \vdash M : A \Rightarrow \text{whnf}(M)$ induction on M

- $M = x \quad \checkmark$

- $M = \lambda x : B. M' \quad \checkmark$

- $M = M_1 M_2 \quad M_1 \not\rightarrow_{bn} \quad \text{so } \text{whnf}(M_1)$
 - $M_1 : B \rightarrow C \quad \left\{ \begin{array}{l} M_1 = \lambda x : B. M'_1 \quad \times \\ M_1 = M \end{array} \right.$

Applicative | Call-by-value | Eager

(β_{bv})

$(\lambda x. M) V \xrightarrow{bv} M[V/x] \quad (V \text{ "by value normal"})$

$$\frac{M \rightarrow M'}{MN \rightarrow_{bv} M'N}$$

$$\frac{N \rightarrow N'}{VN \rightarrow VN'}$$

$(V \text{ "by value normal"})$

b.v. $V := x$

normal $\lambda x. M$

$=_{bv}$

$(x V_1 \dots V_n)$

$(\lambda x. x) \underset{\text{ap}}{\underset{\curvearrowleft}{\rightarrow}} \neq_{bv} y$

- $(\lambda x. M_x) \neq_{bv} M$