Untyped λ-Calculus - functions

terms $N, M := x \mid (\lambda x. M) \mid (M\, N)$

α-equivalence $M =_\alpha M'$ (rename bound variables)

Free Variables:

$$fv(x) = \{x\}$$
$$fv(\lambda x. M) = fv(M) \setminus \{x\}$$
$$fv(M\, N) = fv(M) \cup fv(N)$$

Capture avoiding Substitution: $M[N/x]$

- replaces free occurrences of x in M by N
- renames bound variables in M to avoid capturing the free variables of N

e.g. $(\lambda x. y\, x)[\lambda z. x / y] =_\alpha$
$(\lambda w. (\lambda z. x)\, w)$

β-reduction - what it means for a λ-term to compute

- $\boxed{M \to_\beta N}$ " M β-reduces in one step to N"

$$(\lambda x . M)\, N \to_\beta M[N/x] \qquad (\beta)$$

β-redex

$$\frac{M \to_\beta M'}{M\,N \to_\beta M'\,N} \qquad \frac{N \to_\beta N'}{M\,N \to_\beta M\,N'} \qquad \text{(congruence)}$$

$$\frac{M \to_\beta M'}{\lambda x. M \to_\beta \lambda x. M'} \qquad (\xi)$$

- $M \twoheadrightarrow_\beta N$ (also written $M \to_\beta^* N$)
  - reflexive, transitive closure of $\to_\beta$

- $M =_\beta N$
  - reflexive, symmetric, transitive closure of $\to_\beta$.

    (smallest equivalence relation containing $\to_\beta$)

- M is in (β) <u>normal</u> <u>form</u> if there is no M' such that $M \to_\beta M'$.

example:

$$(\lambda x.y)((\lambda z.zz)(\lambda w.w))$$
$$\to_\beta (\lambda x.y)((\lambda w.w)(\lambda w.w))$$
$$\to_\beta (\lambda x.y)(\lambda w.w)$$
$$\to_\beta y$$

or

$$(\lambda x.y)\overbrace{((\lambda z.zz)(\lambda w.w))}^{N}$$
$$\to_\beta y$$

$(\lambda x.y)N \to_\beta$
$y[N/x] =_\alpha y$

Q: Does $\twoheadrightarrow_\beta$ always terminate?

$$\omega = (\lambda x.xx)$$
$$\Omega = \omega\omega \to_\beta \Omega \to_\beta \Omega \to_\beta \cdots$$

$$\begin{array}{ccc} (\lambda y.x)\,\underline{\Omega} & \xrightarrow{\ \beta\ } & x \\ \downarrow_\beta & \nearrow_\beta & \\ (\lambda y.x)\,\Omega & & \\ \downarrow & & \\ \vdots & & \end{array}$$

M terminates iff $\exists N$, normal(N) s.t. $M \twoheadrightarrow_\beta N$

$$
\begin{tikzcd}
 & & M \arrow[dl, ->>, "\beta"'] \arrow[dr, ->>, "\beta"] & & \\
 & M' \arrow[dl, "\beta"'] \arrow[dr, dotted, ->>] & \overset{?}{\not=}_\beta & M'' \arrow[dr] \arrow[dl, dotted] & \\
 {} & & ? & & {}
\end{tikzcd}
$$

"true nondeterminism"

$$e_1 \oplus e_2 \to e_1 \qquad\qquad x \oplus y \to x$$

$$e_1 \oplus e_2 \to e_2 \qquad\qquad x \oplus y \to y$$

$$
\begin{tikzcd}
M \arrow[r, ->>, "\beta"] \arrow[d] & N \not\twoheadrightarrow_\beta \\
M' \arrow[ur, ->>, "\beta"'] &
\end{tikzcd}
\qquad\qquad M =_\beta N
$$

"weak normalization"

$$
\begin{tikzcd}
 & M \arrow[dl, ->>, "\beta"'] \arrow[dr, ->>, "\beta"] & \\
M' \arrow[rr, phantom, "=_\alpha"] & & M'' \\
\not\to_\beta & & \not\to_\beta
\end{tikzcd}
$$

examples of $=_\beta$

$$\lambda x.\lambda y.x \neq_\beta \lambda x.\lambda y.y$$

$$(\lambda x.y)M =_\beta y \qquad \text{(for any } M)$$

Q: If $M \to_\beta N$ can N be "bigger" than M?

Q: If M, N both don't terminate

$$M \twoheadrightarrow_\beta M' \to \ldots$$

$$N \twoheadrightarrow_\beta N' \to \ldots$$

$$\boxed{=_\beta}$$

$$M =_\beta N \;?$$

$$\Omega \to_\beta \Omega$$

$$(\lambda x.xxx)\underbrace{(\lambda x.xxx)}_{\nu} \to_\beta (\nu\nu)\nu \to_\beta \nu\nu\nu\nu \to$$

Extensionality (in general)

Two functions f and g are extensionally equal

if for all v, $(f\,v) = (g\,v)$

e.g. $f : \mathbb{R} \to \mathbb{R}$ $\quad g : \mathbb{R} \to \mathbb{R}$

$\forall x \in \mathbb{R},\ f(x) = g(x)$

"they have the same graph"

$f(x) = x^2 + 2x + 1 \qquad g(x) = (x+1)^2$

λ-calculus two terms M and N are extensionally equal if, for all A, $(M\,A) =_\beta (N\,A)$

What about: y vs. $\lambda x.\, y\, x$?

$$(\lambda x.\, y\, x)\, A \to_\beta y\, A$$

(But note that $y \neq_\beta \lambda x.\, y\, x$

how to prove it?)

$(\lambda w.\, M)\, N \overset{\Leftarrow}{=}_\beta M$

when $w \notin FV(M)$

Lemma

$$(\forall A.\ MA =_\beta M'A) \quad \text{iff} \quad Mx =_\beta M'x$$

for some $x \notin fv(M) \cup fv(M')$ (i.e. x is fresh)

Proof

- $\Rightarrow$ assume $(\forall A.\ MA =_\beta M'A)$ let x be fresh

  then $Mx =_\beta M'x$ by instantiting $A := x$

- $\Leftarrow$ suppose $Mx =_\beta M'x$ (for x fresh)

  Let A be given.

  then $MA =_\beta (\lambda x. Mx) A =_\beta (\lambda x. M'x) A$

  $=_\beta M'A$

$\square$

Eta reduction

$$\lambda x. M x \to_\eta M \quad (x \notin fv(M)) \qquad (\eta)$$

- $\twoheadrightarrow_\eta$ (or $\xrightarrow{*}_\eta$)
- $\to_{\beta\eta}$ both β and η rules
- $=_\eta$
- $=_{\beta\eta}$

induced equivalences

equivalent to extensionality principle.

$M =_{\beta\eta} N$ iff $\forall A,\ M A =_\beta N A$

Eta expansion: $M \to_{\eta^+} \lambda x. M x \quad (x \notin fv(M))$

Confluence (or, the Church Rosser property)

Theorem: (Church Rosser)

If M, N, and P are λ-terms and $M \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta P$ then $\exists Z$, s.t. $N \twoheadrightarrow_\beta Z$, $P \twoheadrightarrow_\beta Z$

$$\begin{tikzcd} & M \arrow[dl, twoheadrightarrow, "\beta"'] \arrow[dr, twoheadrightarrow, "\beta"] & \\ N \arrow[dr, dotted, twoheadrightarrow, "\beta"'] & & P \arrow[dl, dotted, twoheadrightarrow, "\beta"] \\ & Z & \end{tikzcd}$$

$$\boxed{\begin{tikzcd} & M \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\ N \arrow[dr, dotted, twoheadrightarrow, "\beta"'] & & P \arrow[dl, dotted, twoheadrightarrow, "\beta"] \\ & Z & \end{tikzcd}}$$

$$\begin{tikzcd} & M \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\ N \arrow[dr, dotted, "\beta"'] & & P \arrow[dl, dotted, "\beta"] \\ & Z & \end{tikzcd}$$

$$\begin{tikzcd} & (\lambda x. xxx)\ \overbrace{((\lambda y.y)(\lambda w.w))}^{N} \arrow[dl, "\beta"'] \arrow[dr] & \\ NNN \arrow[dr, dotted] & & (\lambda x.xxx)(\lambda w.w) \arrow[dl, dotted] \\ & \cancel{\ } & \end{tikzcd}$$

$$\frac{M \to M' \qquad N \to N'}{(\lambda x. M)\ N \to M'[N'/x]}$$

$M \Rrightarrow N$

$M \rhd N$

Reduction vs. Evaluation

$$(\lambda x. M)\ N \longrightarrow_\beta M[N/x]$$

$$\frac{M \longrightarrow_\beta M'}{MN \longrightarrow_\beta M'N} \qquad \frac{N \longrightarrow N'}{MN \longrightarrow_\beta MN'}$$

$$\frac{M \longrightarrow_\beta M'}{\lambda x. M \longrightarrow_\beta \lambda x. M'}$$ (don't do this!)

① Can you make a deterministic version?

② How does that choice affect the theory?

$$\longrightarrow_n \qquad \longrightarrow_{bv} \qquad \longrightarrow_{bn}$$

$$\begin{array}{ccc} (\lambda x. y)\, \Omega & \qquad & \Omega\, (\lambda x. x) \\ \downarrow & & \downarrow \\ y & & \Omega\, (\lambda x. x) \\ & & \vdots \end{array}$$

Normal Order / Call by name evaluation

Call-by name

$$(\lambda x. M)\ N \to_{bn} M[N/x] \qquad (\beta)$$

$$\frac{M \to_{bn} M'}{M N \to_{bn} M' N} \qquad x \to_{bn} x$$

$$M \twoheadrightarrow_{bn} M' \twoheadrightarrow_{bn'}$$

"by-name normal" weak head normal

- x
- $\boxed{\lambda x. M}$
- $((x\ M_1)\ M_2) \dots M_n$

$=_{bn}$

$=_{\beta}$

Normal order evaluation – "leftmost outermost"

- Coincides with $=_\beta$ for terms that have a normal

[lazy evaluation / call-by-need]

Normal order evaluation

CBN

whnf $= \ \not\to_{bn}$

$$(\lambda x.M)\,N \to_{bn} M[N/x]$$

$$\frac{M \to_{bn} M'}{M\,N \to_{bn} M'N}$$

$\not\to_{\beta} \Rightarrow \not\to_{bn}$

$\to_{bn} \subseteq \to_{\beta}$

$\to_{n} \subseteq \to_{\beta}$

Normal order

$$(\lambda x.M)\,N \to_{n} M[N/x]$$

$$\frac{M \to_n M'}{\lambda x\,M \to_n \lambda x.M'} \qquad \frac{N \to_n N'}{x\,N \to_{hnf} x\,N'}$$

$$\frac{M \to_{bn} M'}{M\,N \to_n M'N} \qquad \frac{M \to_{hnf} M'}{M\,N \to_n M'N}$$

<u>Lemma</u> $M \not\to_{bn}$ $\quad \Gamma \vdash M : A \Rightarrow \mathrm{whnf}(M)$ $\qquad$ induction on M

- $M = x$ ✓
- $M = \lambda x:B.M'$ ✓
- $M = M_1 M_2$ $\quad M_1 \not\to_{bn}$ so whnf(M_1)
  - $M_1 : B \to C$ $\begin{cases} M_1 = \lambda x:B.M_1' & \times \\ M_1 = M \end{cases}$

Applicative / Call-by-value / Eager

(β_{bv}) $\quad (\lambda x. M)\ V \to_{bv} M[V/x]$ $\quad$ (V "by value normal")

$$\frac{M \to_{bv} M'}{M\,N \to_{bv} M'\,N} \qquad \frac{N \to N'}{V\,N \to V\,N'}$$ (V "by value normal")

b.v. normal $V ::= x$

$\lambda x. M$

$(x\ V_1 \dots V_n)$

$=_{bv}$

$(\lambda x. y)\ \Omega \neq_{bv} y$

- $(\lambda x. M\,x) \neq_{bv} M$