CIS 551 / TCOM 401

Computer and Network Security

Spring 2007

Lecture 15

Announcements

- Prof. Zdancewic will be back next lecture

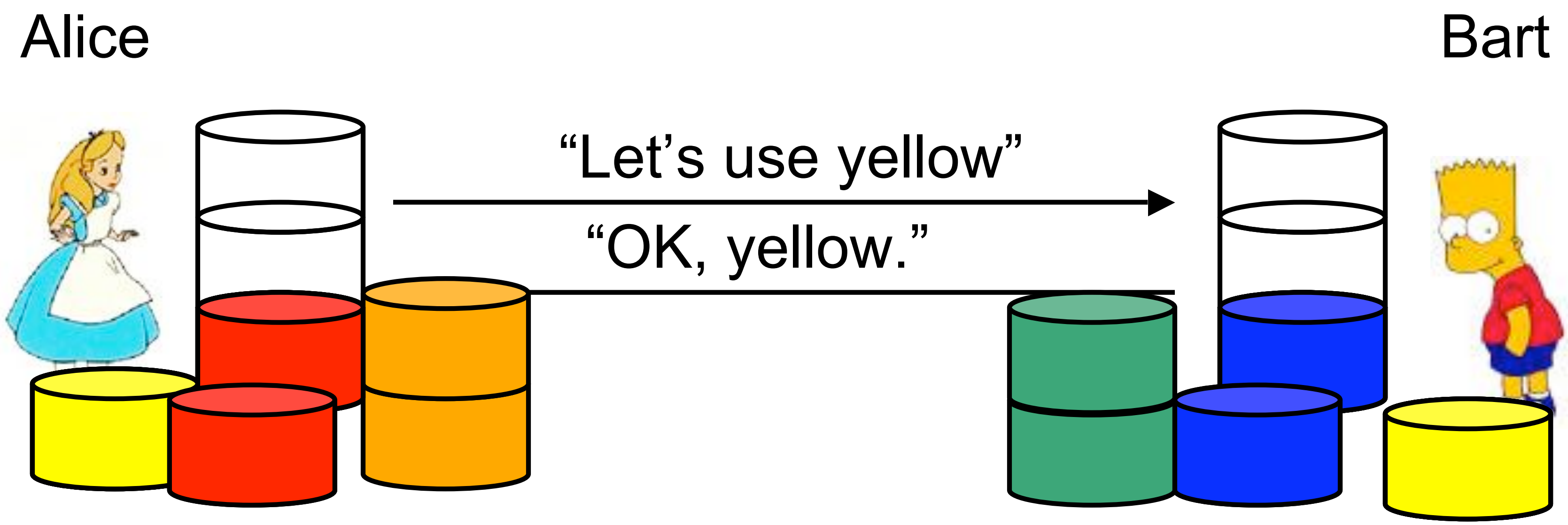
Problems with Shared Key Crypto

- Compromised key means interceptors can decrypt any ciphertext they've acquired.
  - Change keys frequently to limit damage
- Distribution of keys is problematic
  - Keys must be transmitted securely
  - Use couriers?
  - Distribute in pieces over separate channels?
- Number of keys is $O(n^2)$ where n is # of participants
- Potentially easier to break?

Diffie-Hellman Key Exchange

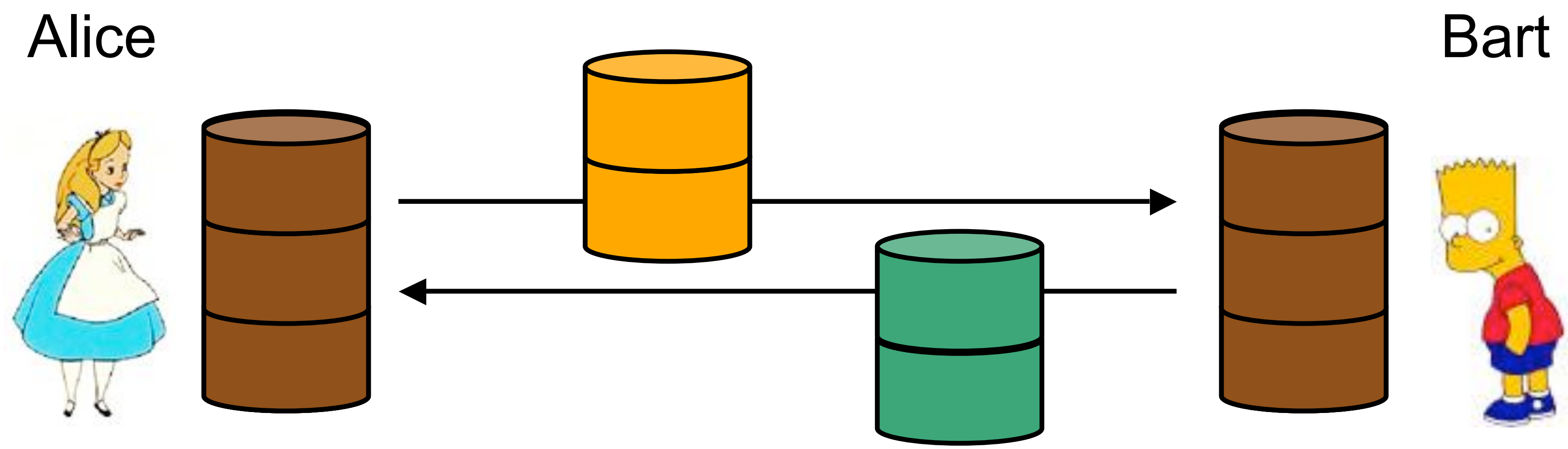
- Problem with shared-key systems: Distributing the shared key
- Suppose that Alice and Bart want to agree on a secret (i.e. a key)
  - Communication link is public
  - They don't already share any secrets

Diffie-Hellman by Analogy: Paint



1. Alice & Bart decide on a public color, and mix one liter of that color.
2. They each choose a random secret color, and mix two liters of their secret color.
3. They keep one liter of their secret color, and mix the other with the public color.

Diffie-Hellman by Analogy: Paint



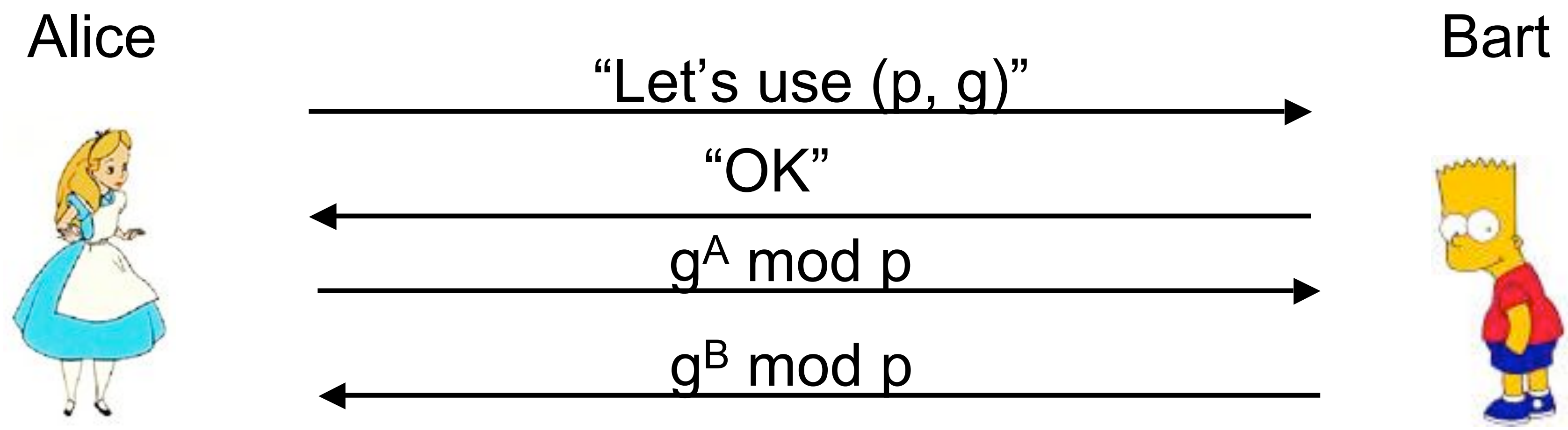
4. They exchange the mixtures over the public channel.
5. When they get the other person’s mixture, they combine it with their retained secret color.
6. The secret is the resulting color: Public + Alice’s + Bart’s

Diffie-Hellman Key Exchange

- Choose a prime p (publicly known)
  - Should be about 512 bits or more
- Pick g < p (also public)
  - g must be a *primitive root* of p.
  - A primitive root *generates* the finite field p.
  - Every n in {1, 2, …, p-1} can be written as g^k mod p
  - Example: 2 is a primitive root of 5
  - $2^0 = 1 \quad 2^1 = 2 \quad 2^2 = 4 \quad 2^3 = 3 \pmod 5$

  - Intuitively means that it's hard to take logarithms base g because there are many candidates.

Diffie-Hellman



1. Alice & Bart decide on a public prime p and primitive root g.
2. Alice chooses secret number A. Bart chooses secret number B
3. Alice sends Bart $g^A \bmod p$.
4. The shared secret is $g^{AB} \bmod p$.

Details of Diffie-Hellman

- Alice computes g^{AB} mod p because she knows A:
  - $g^{AB} \bmod p = (g^B \bmod p)^A \bmod p$

- An eavesdropper gets g^A mod p and g^B mod p
  - They can easily calculate g^{A+B} mod p but that doesn't help.
  - The problem of computing discrete logarithms (to recover A from g^A mod p is hard.

Example

- Alice and Bart agree that $q=71$ and $g=7$.
- Alice selects a private key $A=5$ and calculates a public key $g^A \equiv 7^5 \equiv 51 \pmod{71}$. She sends this to Bart.
- Bart selects a private key $B=12$ and calculates a public key $g^B \equiv 7^{12} \equiv 4 \pmod{71}$. He sends this to Alice.
- Alice calculates the shared secret:
  $S \equiv (g^B)^A \equiv 4^5 \equiv 30 \pmod{71}$
- Bart calculates the shared secret
  $S \equiv (g^A)^B \equiv 51^{12} \equiv 30 \pmod{71}$

Why Does it Work?

- Security is provided by the difficulty of calculating discrete logarithms.
- Feasibility is provided by
  - The ability to find large primes.
  - The ability to find primitive roots for large primes.
  - The ability to do efficient modular arithmetic.
- Correctness is an immediate consequence of basic facts about modular arithmetic.

One-way Functions

- A function is one-way if it's
  - Easy to compute
  - Hard to invert (in the average case)

- Examples
  - Exponentiation vs. Discrete Log
  - Multiplication vs. Factoring
  - Knapsack Packing
    - Given a set of numbers {1, 3, 6, 8, 12} find the sum of a subset
    - Given a target sum, find a subset that adds to it

- Trapdoor functions
  - Easy to invert given some extra information
  - E.g. factoring p*q given q

Public Key Cryptography

- Sender encrypts using a *public key*
- Receiver decrypts using a *private key*
- Only the private key must be kept secret
  - Public key can be distributed at will
- Also called *asymmetric* cryptography
- Can be used for digital signatures
- Examples: RSA, El Gamal, DSA, various algorithms based on elliptic curves

- Used in SSL, ssh, PGP, …

Public Key Notation

- Encryption algorithm

  E : keyPub x plain $\rightarrow$ cipher

  Notation: K{msg} = E(K, msg)
- Decryption algorithm

  D : keyPriv x cipher $\rightarrow$ plain

  Notation: k{msg} = D(k,msg)
- D inverts E

  D(k, E(K, msg)) = msg
- Use capital “K” for public keys
- Use lower case “k” for private keys
- Sometimes E is the same algorithm as D

Secure Channel: Private Key

Alice

Bart

K_B\{Hello!\} →

← K_A\{Hi!\}

K_A, K_B
k_A

K_A, K_B
k_B

Trade-offs for Public Key Crypto

- More computationally expensive than shared key crypto
  - Algorithms are harder to implement
  - Require more complex machinery
- More formal justification of difficulty
  - Hardness based on complexity-theoretic results
- A principal needs one private key and one public key
  - Number of keys for pair-wise communication is O(n)

RSA Algorithm

- Ron Rivest, Adi Shamir, Leonard Adleman
  - Proposed in 1979
  - They won the 2002 Turing award for this work
- Has withstood years of cryptanalysis
  - Not a guarantee of security!
  - But a strong vote of confidence.
- Hardware implementations: 1000 x slower than DES

RSA at a High Level

- Public and private key are derived from secret prime numbers
  - Keys are typically ≥ 1024 bits
- Plaintext message (a sequence of bits)
  - Treated as a (large!) binary number
- Encryption is modular exponentiation
- To break the encryption, conjectured that one must be able to factor large numbers
  - Not known to be in P (polynomial time algorithms)

Number Theory: Modular Arithmetic

- Examples:
  - 10 mod 12 = 10
  - 13 mod 12 = 1
  - (10 + 13) mod 12 = 23 mod 12 = 11 mod 12
  - $23 \equiv 11 \pmod{12}$
  - "23 is congruent to 11 (mod 12)"

- $a \equiv b \pmod{n}$ iff $a = b + kn$ (for some integer k)
- The *residue* of a number modulo n is a number in the range 0…n-1

Number Theory: Prime Numbers

- A *prime number* is an integer > 1 whose only factors are 1 and itself.
- Two integers are *relatively prime* if their only common factor is 1
  - gcd = greatest common divisor
  - gcd(a,b) = 1
  - gcd(15,12) = 3, so they're not relatively prime
  - gcd(15,8) = 1, so they are relatively prime
- Easy to compute GCD using Euclid's Algorithm

Finite Fields (Galois Fields)

- For a prime p, the set of integers mod p forms a *finite field*
- Addition + Additive unit 0
- Multiplication * Multiplicative unit 1
- Inverses: $n * n^{-1} = 1$ for $n \neq 0$
  - Suppose p = 5, then the finite field is {0,1,2,3,4}
  - $2^{-1} = 3$ because $2 * 3 \equiv 1 \bmod 5$
  - $4^{-1} = 4$ because $4 * 4 \equiv 1 \bmod 5$

- Usual laws of arithmetic hold for modular arithmetic:
  - Commutativity, associativity, distributivity of * over +

RSA Key Generation

- Choose large, distinct primes p and q.
  - Should be roughly equal length (in bits)
- Let $n = p*q$
- Choose a random encryption exponent e
  - With requirement: e and (p-1)*(q-1) are relatively prime.
- Derive the decryption exponent d
  - $d = e^{-1} \bmod ((p-1)*(q-1))$
  - d is e's inverse mod ((p-1)*(q-1))
- Public key: $K = (e,n)$ pair of e and n
- Private key: $k = (d,n)$
- Discard primes p and q (they're not needed anymore)

RSA Encryption and Decryption

- Message: m
- Assume m < n
  - If not, break up message into smaller chunks
  - Good choice: largest power of 2 smaller than n

- Encryption: $E((e,n), m) = m^e \bmod n$
- Decryption: $D((d,n), c) = c^d \bmod n$

Example RSA

- Choose p = 47, q = 71
- n = p * q = 3337
- (p-1)*(q-1) = 3220
- Choose e relatively prime with 3220: e = 79
  - Public key is (79, 3337)
- Find d = 79^{-1} mod 3220 = 1019
  - Private key is (1019, 3337)
- To encrypt m = 688232687966683
  - Break into chunks < 3337
  - 688 232 687 966 683
- Encrypt: E((79, 3337), 688) = 688^{79} mod 3337 = 1570
- Decrypt: D((1019, 3337), 1570) = 1570^{1019} mod 3337 = 688

Euler's *totient* function: $\phi(n)$

- $\phi(n)$ is the number of positive integers less than n that are relatively prime to n
  - $\phi(12) = 4$
  - Relative primes of 12 (less than 12): {1, 5, 7, 11}

- For p a prime, $\phi(p) = p-1$. Why?
- For p,q two distinct primes, $\phi(p*q) = (p-1)*(q-1)$
  - There's p*q-1 numbers less than p*q
  - Factors of p*q =
    - {1*p, 2*p, …, q*p} for a total of q of them
    - {1*q, 2*q, …, p*q} for another of of them
    - No other numbers
    - $\phi(p*q) = (p*q) - (p + q - 1) = pq - p - q + 1 = (p-1)*(q-1)$

q many multiples of p

All #s ≤ p*q

don't double count p*q

P many multiples of q

Fermat's Little Theorem

- Generalized by Euler.
- Theorem: If p is a prime then $a^p \equiv a \bmod p$.
- Corollary: If gcd(a,n) = 1 then $a^{\phi(n)} \equiv 1 \bmod n$.
- Easy to compute $a^{-1} \bmod n$
  - $a^{-1} \bmod n = a^{\phi(n)-1} \bmod n$
  - Why? $a * a^{\phi(n)-1} \bmod n$
    $= a^{\phi(n)-1+1} \bmod n$
    $= a^{\phi(n)} \bmod n$
    $= 1$

Example of Fermat's Little Theorem

- What is the inverse of 5, modulo 7?
- 7 is prime, so $\phi(7) = 6$
- $5^{-1} \bmod 7$ $= 5^{6-1} \bmod 7$

$$
\begin{aligned}
&= 5^5 \bmod 7 \\
&= (5^2 * 5^2 * 5) \bmod 7 \\
&= ((5^2 \bmod 7) * (5^2 \bmod 7) * (5 \bmod 7)) \bmod 7 \\
&= ((4 \bmod 7) * (4 \bmod 7) * (5 \bmod 7)) \bmod 7 \\
&= ((16 \bmod 7) * (5 \bmod 7)) \bmod 7 \\
&= ((2 \bmod 7) * (5 \bmod 7)) \bmod 7 \\
&= (10 \bmod 7) \bmod 7 \\
&= 3 \bmod 7 \\
&= 3
\end{aligned}
$$

Chinese Remainder Theorem

- (Or, enough of it for our purposes…)
- Suppose:
  - p and q are relatively prime
  - $a \equiv b \pmod{p}$
  - $a \equiv b \pmod{q}$
- Then: $a \equiv b \pmod{p*q}$
- Proof:
  - p divides (a-b) (because a mod p = b mod p)
  - q divides (a-b)
  - Since p, q are relatively prime, p*q divides (a-b)
  - But that is the same as: $a \equiv b \pmod{p*q}$

Proof that D inverts E

$c^d \bmod n$	
$= (m^e)^d \bmod n$	(definition of c)
$= m^{ed} \bmod n$	(arithmetic)
$= m^{k*(p-1)*(q-1)+1} \bmod n$	(d inverts e)
$= m*m^{k*(p-1)*(q-1)} \bmod n$	(arithmetic)
$= m \bmod n$	(C. R. theorem)
$= m$	($m < n$)

$e*d \equiv 1 \bmod (p-1)*(q-1)$

Finished Proof

- Note: $m^{p-1} \equiv 1 \bmod p$ (if p doesn't divide m)
  - Why? Fermat's little theorem.
- Same argument yields: $m^{q-1} \equiv 1 \bmod q$

- Implies: $m^{k*\phi(n)+1} \equiv m \bmod p$
- And $m^{k*\phi(n)+1} \equiv m \bmod q$

- Chinese Remainder Theorem implies:
  $m^{k*\phi(n)+1} \equiv m \bmod n$

How to Generate Prime Numbers

- Many strategies, but *Rabin-Miller* primality test is often used in practice.
  - $a^{p-1} \equiv 1 \bmod p$
- Efficiently checkable test that, with probability ¾, verifies that a number p is prime.
  - Iterate the Rabin-Miller primality test t times.
  - Probability that a composite number will slip through the test is $(¼)^t$
  - These are worst-case assumptions.
- In practice (takes several seconds to find a 512 bit prime):
  1. Generate a random n-bit number, p
  2. Set the high and low bits to 1 (to ensure it is the right number of bits and odd)
  3. Check that p isn't divisible by any "small" primes 3,5,7,…,<2000
  4. Perform the Rabin-Miller test at least 5 times.

Rabin-Miller Primality Test

- Is n prime?
- Write n as $n = (2^r)*s + 1$
- Pick random number a, with $1 \le a \le n - 1$
- If
  - $a^s \equiv 1 \mod n$ and
  - for all j in {0 … r-1}, $a^{2^j s} \equiv -1 \mod n$
- Then return composite
- Else return probably prime