CIS 551 / TCOM 401

Computer and Network Security

Spring 2006

Lecture 9

Announcements

- MIDTERM HAS BEEN POSTPONED:
  - Midterm is now next Tuesday (2/14/2006)
  - If this causes problems for you, see me after class.

Problems with Shared Key Crypto

- Compromised key means interceptors can decrypt any ciphertext they've acquired.
  - Change keys frequently to limit damage
- Distribution of keys is problematic
  - Keys must be transmitted securely
  - Use couriers?
  - Distribute in pieces over separate channels?
- Number of keys is $O(n^2)$ where n is # of participants
- Potentially easier to break?

Public Key Cryptography

- Sender encrypts using a *public key*
- Receiver decrypts using a *private key*
- Only the private key must be kept secret
  - Public key can be distributed at will
- Also called *asymmetric* cryptography
- Can be used for digital signatures
- Examples: RSA, El Gamal, DSA, various algorithms based on elliptic curves

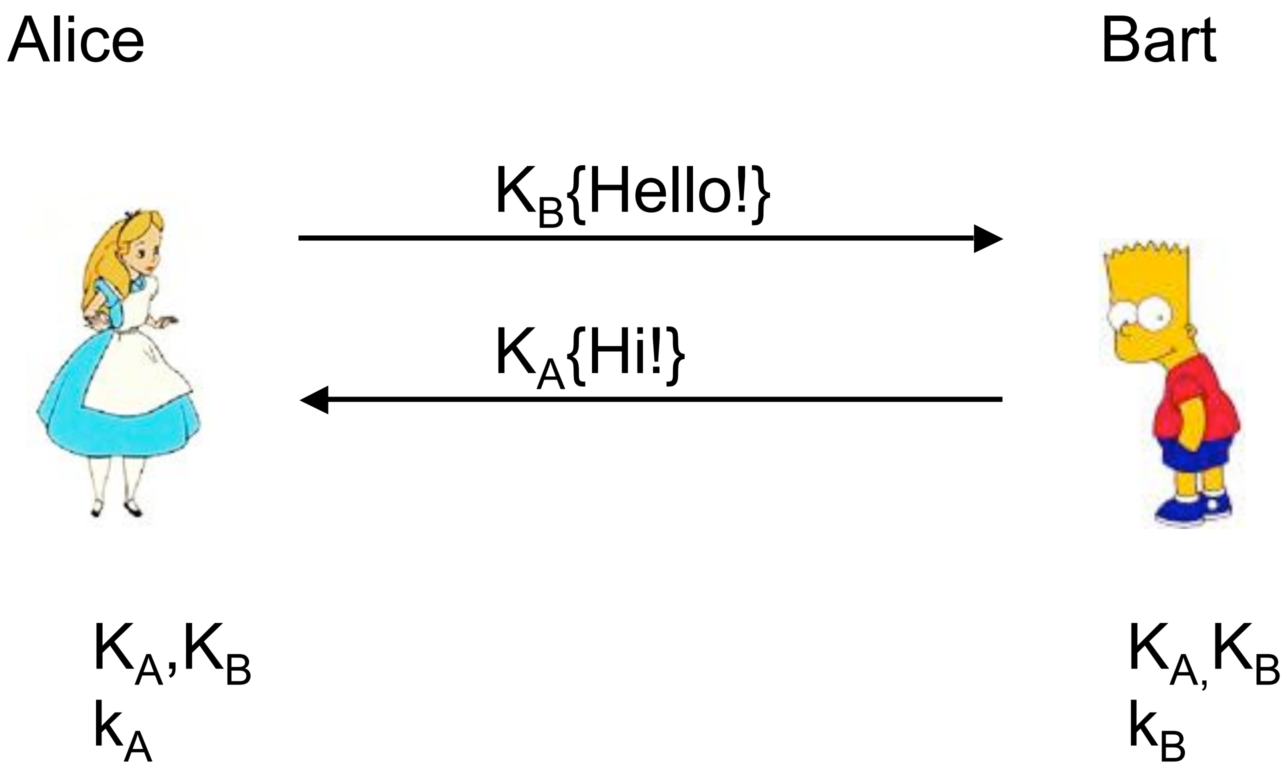
- Used in SSL, ssh, PGP, …

Public Key Notation

- Encryption algorithm
  - E : keyPub x plain $\rightarrow$ cipher
  - Notation: K{msg} = E(K, msg)
- Decryption algorithm
  - D : keyPriv x cipher $\rightarrow$ plain
  - Notation: k{msg} = D(k,msg)
- D inverts E
  - D(k, E(K, msg)) = msg
- Use capital "K" for public keys
- Use lower case "k" for private keys

- Sometimes E is the same algorithm as D

Secure Channel: Private Key



Trade-offs for Public Key Crypto

- More computationally expensive than shared key crypto
  - Algorithms are harder to implement
  - Require more complex machinery
- More formal justification of difficulty
  - Hardness based on complexity-theoretic results
- A principal needs one private key and one public key
  - Number of keys for pair-wise communication is O(n)

RSA Algorithm

- Ron Rivest, Adi Shamir, Leonard Adleman
  - Proposed in 1979
  - They won the 2002 Turing award for this work

- Has withstood years of cryptanalysis
  - Not a guarantee of security!
  - But a strong vote of confidence.

- Hardware implementations: 1000 x slower than DES

RSA at a High Level

- Public and private key are derived from secret prime numbers
  - Keys are typically ≥ 1024 bits
- Plaintext message (a sequence of bits)
  - Treated as a (large!) binary number
- Encryption is modular exponentiation
- To break the encryption, conjectured that one must be able to factor large numbers
  - Not known to be in P (polynomial time algorithms)

Number Theory: Modular Arithmetic

- Examples:
  - 10 mod 12 = 10
  - 13 mod 12 = 1
  - (10 + 13) mod 12 = 23 mod 12 = 11 mod 12
  - $23 \equiv 11 \pmod{12}$
  - “23 is congruent to 11 (mod 12)”

- $a \equiv b \pmod{n}$ iff $a = b + kn$ (for some integer k)
- The *residue* of a number modulo n is a number in the range 0…n-1

Number Theory: Prime Numbers

- A *prime number* is an integer > 1 whose only factors are 1 and itself.

- Two integers are *relatively prime* if their only common factor is 1
  - gcd = greatest common divisor
  - gcd(a,b) = 1
  - gcd(15,12) = 3, so they're not relatively prime
  - gcd(15,8) = 1, so they are relatively prime

Finite Fields (Galois Fields)

- For a prime p, the set of integers mod p forms a *finite field*
- Addition + Additive unit 0
- Multiplication * Multiplicative unit 1
- Inverses: $n * n^{-1} = 1$ for $n \neq 0$
  - Suppose p = 5, then the finite field is {0,1,2,3,4}
  - $2^{-1} = 3$ because $2 * 3 \equiv 1 \bmod 5$
  - $4^{-1} = 4$ because $4 * 4 \equiv 1 \bmod 5$

- Usual laws of arithmetic hold for modular arithmetic:
  - Commutativity, associativity, distributivity of * over +

RSA Key Generation

- Choose large, distinct primes p and q.
  - Should be roughly equal length (in bits)
- Let $n = p*q$
- Choose a random encryption exponent e
  - With requirement: e and (p-1)*(q-1) are relatively prime.
- Derive the decryption exponent d
  - $d = e^{-1} \bmod ((p-1)*(q-1))$
  - d is e's inverse mod ((p-1)*(q-1))
- Public key: K = (e,n)   pair of e and n
- Private key: k = (d,n)
- Discard primes p and q (they're not needed anymore)

RSA Encryption and Decryption

- Message: m
- Assume m < n
  - If not, break up message into smaller chunks
  - Good choice: largest power of 2 smaller than n

- Encryption: $E((e,n), m) = m^e \bmod n$
- Decryption: $D((d,n), c) = c^d \bmod n$

Example RSA

- Choose p = 47, q = 71
- n = p * q = 3337
- (p-1)*(q-1) = 3220
- Choose e relatively prime with 3220: e = 79
  - Public key is (79, 3337)
- Find d = 79^{-1} mod 3220 = 1019
  - Private key is (1019, 3337)
- To encrypt m = 688232687966683
  - Break into chunks < 3337
  - 688 232 687 966 683
- Encrypt: E((79, 3337), 688) = 688^{79} mod 3337 = 1570
- Decrypt: D((1019, 3337), 1570) = 1570^{1019} mod 3337 = 688

Euler's *totient* function: $\phi(n)$

- $\phi(n)$ is the number of positive integers less than n that are relatively prime to n
  - $\phi(12) = 4$
  - Relative primes of 12 (less than 12): {1, 5, 7, 11}

- For p a prime, $\phi(p) = p-1$. Why?
- For p,q two distinct primes, $\phi(p*q) = (p-1)*(q-1)$

Fermat's Little Theorem

- Generalized by Euler.
- Theorem: If p is a prime then $a^p \equiv a \bmod p$.
- Corollary: If gcd(a,n) = 1 then $a^{\phi(n)} \equiv 1 \bmod n$.
- Easy to compute $a^{-1} \bmod n$
  - $a^{-1} \bmod n = a^{\phi(n)-1} \bmod n$
  - Why? $a * a^{\phi(n)-1} \bmod n$

    $= a^{\phi(n)-1+1} \bmod n$

    $= a^{\phi(n)} \bmod n$

    $= 1$

Example of Fermat's Little Theorem

- What is the inverse of 5, modulo 7?
- 7 is prime, so $\phi(7) = 6$
- $5^{-1} \bmod 7$
  $= 5^{6-1} \bmod 7$
  $= 5^{5} \bmod 7$
  $= ((25 \bmod 7 * 5^{3} \bmod 7) \bmod 7$
  $= (4 \bmod 7 * 5^{3} \bmod 7) \bmod 7$
  $= ((4 \bmod 7 * 4 \bmod 7) \bmod 7 * 5 \bmod 7) \bmod 7$
  $= (16 \bmod 7 * 5 \bmod 7) \bmod 7$
  $= (2 * 5) \bmod 7$
  $= 10 \bmod 7$
  $= 3$

Chinese Remainder Theorem

- (Or, enough of it for our purposes…)
- Suppose:
  - p and q are relatively prime
  - $a \equiv b \pmod{p}$
  - $a \equiv b \pmod{q}$
- Then: $a \equiv b \pmod{p*q}$
- Proof:
  - p divides (a-b) (because a mod p = b mod p)
  - q divides (a-b)
  - Since p, q are relatively prime, p*q divides (a-b)
  - But that is the same as: $a \equiv b \pmod{p*q}$

Proof that D inverts E

$c^d \bmod n$	
$= (m^e)^d \bmod n$	(definition of c)
$= m^{ed} \bmod n$	(arithmetic)
$= m^{k*(p-1)*(q-1) + 1} \bmod n$	(d inverts e)
$= m*m^{k*(p-1)*(q-1)} \bmod n$	(arithmetic)
$= m \bmod n$	(C. R. theorem)
$= m$	($m < n$)

Finished Proof

- Note: $m^{p-1} \equiv 1 \bmod p$ (if p doesn't divide m)
  - Why? Fermat's little theorem.
- Same argument yields: $m^{q-1} \equiv 1 \bmod q$

- Implies: $m^{k*\phi(n)+1} \equiv m \bmod p$
- And $m^{k*\phi(n)+1} \equiv m \bmod q$

- Chinese Remainder Theorem implies:
  $m^{k*\phi(n)+1} \equiv m \bmod n$

How to Generate Prime Numbers

- Many strategies, but *Rabin-Miller* primality test is often used in practice.
  - $a^{p-1} \equiv 1 \bmod p$
- Efficiently checkable test that, with probability ¾, verifies that a number p is prime.
  - Iterate the Rabin-Miller primality test t times.
  - Probability that a composite number will slip through the test is $(1/4)^t$
  - These are worst-case assumptions.
- In practice (takes several seconds to find a 512 bit prime):
  1. Generate a random n-bit number, p
  2. Set the high and low bits to 1 (to ensure it is the right number of bits and odd)
  3. Check that p isn't divisible by any "small" primes 3,5,7,…,<2000
  4. Perform the Rabin-Miller test at least 5 times.