Structured Hedging for Resource Allocations with Leverage

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Motivating Example

**Problem**: Given a student salary $\$, i.e., a small amount, how do we invest in the stock market to make a lot of money?

Some questions:

- How big of a loan?
- What about risk?
- How can we design an algorithm to do this from data?
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  - How can we design an algorithm to do this from data?
Problem: Given $n$ objects the goal is to find a strategy $p$ which determines how to split up a resource over the $n$ objects.

- Resources: people, CPUs, money, and products.
- Objects: software teams, server requests, assets, and warehouses.

Examples: Job scheduling for compute servers, advertisement and recommendation serving, portfolio selection.

Such problems often need to be solved online.
Key Aspects

- Leverage
- Long and short positions
- Structured hedging
Resource allocation problems have focused on budgeting resources currently in possession.

Many problems allow borrowing resources to use as leverage to increase performance.

- Contractors to help on software projects.
- Compute servers to handle high demand.
- Money to increase investment power.
Long and Short Positions

Many problems have different allocation types or positions. For example, in finance we can hold long and short positions.

- **Long positions**: Purchase shares of stock using cash. Profit if price of shares increases.
- **Short positions**: Borrow shares from bank, sell shares at price $X$, purchase shares at price $Y$, and give back shares to bank. Profit if $Y < X$, i.e., price of shares decreases.
Hedging

- Long and short positions are opposing positions which we can use to alleviate risk in structurally dependent assets, i.e., hedging.
- Example: hold a long position in \( s_1 \) and a short position in \( s_2 \). If both crash, there will be a loss in the long position \( s_1 \) but a gain in the short position \( s_2 \).
- Question: How to determine which positions to hold and in which stocks?
- Answer: Consider structurally dependent stocks described via a graph.
We consider $n$ assets where the goal is to find an allocation $\mathbf{p} \in \mathcal{P} \subset \mathbb{R}^{2n}$ over the long/short positions and assets such that a convex objective $f(\mathbf{p})$ is minimized.

Example: With $n = 2$ assets, we describe $\mathbf{p}$ as

$$
\begin{aligned}
\text{long positions} & \quad \begin{bmatrix} q_{\ell}(1) \\ q_{\ell}(2) \end{bmatrix} \\
\text{short positions} & \quad \begin{bmatrix} q_s(1) \\ q_s(2) \end{bmatrix}
\end{aligned}
= \begin{bmatrix} q_{\ell}(1) \\ q_{\ell}(2) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ q_s(1) \\ q_s(2) \end{bmatrix}
$$

where $q_{\ell} \geq 0$ and $q_s \leq 0$. 

$\mathcal{P}$ denotes the set of all such allocations.
Structured Hedging

Given graph $G$ with weighted adjacency matrix $W \in \mathbb{R}^{2n \times 2n}$ and long/short positional vectors $q_\ell, q_s \in \mathbb{R}^{2n}$, we introduce a strongly convex hedging penalty function

$$
\Omega_h = \sum_{i=1}^{n} \sum_{\substack{j=1+n \atop j \neq i+n}}^{2n} W_{ij} (q_\ell(i) + q_s(j))^2
= p^T L p
$$
Thus, for the purposes of structured hedging in resource allocation, we will consider problems of the form

\[
\min_{p \in \mathcal{P}} \ell(p) + \beta \Omega_h(p)
\]  

(1)

where \( \Omega_h(p) = p^T L p \).
Many resource allocation problems need to be solved dynamically and repeatedly, i.e., online.

We can pose this as an Online Convex Optimization (OCO) problem with objective function $f_t : \mathcal{P} \rightarrow \mathbb{R}$.

In OCO, optimization proceeds in rounds where at round $t$ the algorithm has to pick a solution from a feasible set $p_t \in \mathcal{P}$ without knowing $f_t(\cdot)$ and incur a loss of $f_t(p_t)$.

The problem is then

$$\min_{p \in \mathcal{P}} \ell_t(p) + \beta \Omega_h(p).$$ (2)
Regret

- In order to accomplish a sub-linear regret, we consider solving a linearized version using a first-order Taylor expansion of $f_t$ at $p_t$ along with a proximal term

$$p_{t+1} := \arg\min_{p \in P} \langle p, \nabla \ell_t(p_t) \rangle + \beta \Omega_h(p) + d(p, p_t), \quad (3)$$

where $d(p, p_t) = \frac{1}{2\eta_t} \|p - p_t\|^2_2$. 

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Consider a stock market consisting of \( n \) stocks \( \{s(1), \ldots, s(n)\} \) over a span of \( T \) days.

**Price relatives:** \( x_t(i) = \frac{\text{closing price}}{\text{opening price}} \), \( \hat{x}_t = [x_t(1) \ldots x_t(n)]^\top \), and \( x_t = [\hat{x}_t, \hat{x}_t]^\top \).

**Portfolio:** \( p_t = [p_t(1) \ldots p_t(2n)]^\top \in \mathcal{P} \) where the first \( |\ell| \) elements are long-only positions, i.e., \( p_t(i) \geq 0 \) and the last \( |s| \) elements are short-only positions, i.e., \( p_t(i) \leq 0 \).

Then \( p \) prescribes investing \( p_t(i) \) fraction of the current cash including leverage in stock \( s(i) \).
For portfolios that allow both long and short positions with leverage, the multiplicative gain in wealth at the end of the day $t$ is

$$q^\top \ell x_t + (1 - q^\top \ell 1)(1 + r) + q^\top s (x_t - 1) + q^\top s 1 r$$

- **Market change in wealth**
- **Cash borrowed or not invested**
- **Market change in wealth**
- **Interest owed on borrowed shares**

**Logarithmic gain in wealth:**

$$LS_T(p_{1:T}, x_{1:T}) = \sum_{t=1}^{T} \log(q^\top \ell x_t + q^\top s (x_t - 1 + r) + (1 - q^\top \ell 1)(1 + r))$$
To avoid financial ruin we make an assumption on the price relatives: for $B_\ell, B_s \geq 0$, $0 < 1 - B_\ell < x_t < 1 + B_s < \infty$.

We do not assume $B_\ell = B_s$ therefore we define a halfspace that guarantees no-ruin as $\|q_\ell\|_1 + \frac{B_s + r}{B_\ell + r} \|q_s\|_1 \leq \frac{1+r}{B_\ell + r} \equiv a^\top p \leq \frac{1+r}{B_\ell + r}$. 
Online Portfolio Selection is a special case of (3) with

\[ \ell_t(p_t) = -\log \left( \alpha_1 q_{\ell}^T x_t + \alpha_2 q_s^T (x_t - 1 + r) + (1 - q_{\ell}^T 1)(1 + r) \right) \]

where \( \alpha_1, \alpha_2 \) are parameters that control the importance of long and short positions respectively.

Letting \( \eta_t = \eta \) and multiplying each term in (3) by \( \eta \) so that \( \lambda = \eta \beta \), then the online portfolio selection with structured hedging problem is

\[ \min_{q_{\ell} \geq 0} \, q_s \leq 0 \, a^T p \leq \frac{1+r}{B_{\ell}+r} \]

\[ \eta \langle p, \nabla \ell_t(p_t) \rangle + \lambda p^T L p + \frac{1}{2} \| p - p_t \|_2^2. \]
We propose a **projected gradient descent algorithm** for solving (4) by computing the gradient, setting it equal to zero, and solving for \( p \):

\[
p_{t+1} = \prod_{\mathcal{P}} (\eta \nabla \ell_t(p_t) + p_t) \left( \lambda (L + L^T) + I \right)^{-1}
\]  

(5)

where

\[
\nabla \ell_t(p_t) = \frac{\alpha_1 D^\top \ell_t x_t + \alpha_2 D^\top_s (x_t - 1 + r) - D^\top \ell \mathbf{1}(1 + r)}{\alpha_1 p^\top D^\top \ell x_t + \alpha_2 p^\top D^\top_s (x_t - 1 + r) + (1 - p^\top D^\top \ell \mathbf{1})(1 + r)}
\]

(6)

and \( \prod_{\mathcal{P}} \) is a projection onto the convex constraint set.
Initial investment of $1, invested uniformly with maximum leverage over the positions.

Daily interest rate $r = 0.000245$ which is equivalent to a yearly interest rate of 6.3%.

Graph $G$ is constructed by computing the linear correlation coefficient between all stocks using the previous $\delta$ days of $x_t$. 
Exponentiated Gradient (EG)

- One of the first algorithms in this class used for online portfolio selection.
- Solves the problem

\[ p_{t+1} = \arg\min_{p \in \Delta_n} \eta \langle p, \nabla \ell_t(p_t) \rangle + d(p, p_t) \]

where \( d(p, p_t) = KL(p || p_t) \) and has solution

\[ p_{t+1}(i) \propto p_t(i) \exp(-\eta \nabla \ell_t(i)) \].

Leveraged Long and Short EG

- Developed various EG* variant portfolios and experimented to observe affect of:
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- Long positions

EG uses KL divergence as the proximal term where $p \in \Delta_n = \{ p(i) \geq 0 \forall i, \sum_i p(i) = 1 \}$.

To allow $p(i) < 0$, we instead use $d(p, p_t) = \frac{1}{2} \| p - p_t \|_2^2$ with $p \in P$ for the EG* variants.
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Datasets

- Standard & Poor’s 500 (S&P500\textsuperscript{a}): 25 stocks, 1276 trading days between 1998-2003 (dot-com crash).
- Standard & Poor’s 500 (S&P500\textsuperscript{b}): 263 stocks, 505 trading days between 2007-2009 (financial/housing crash).

Datasets very different in nature:
- DJIA: 25 out of 30 stocks (83%) lost value.
- NYSE: every stock gained value.
- S&P500\textsuperscript{a}: 7 out of 25 stocks (28%) lost value.
- S&P500\textsuperscript{b}: 253 out of 263 stocks (96%) lost value.
- TSX: 32 out of 88 stocks (36%) lost value.
## Leveraged Long and Short EG Results

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**Table:** Cumulative wealth of EG and EG* variants for each of the five datasets.

**EG* (LO) earns $690 trillion!**
Comparison with Benchmark Algorithms

- Benchmark algorithms:
  - Uniform buy-and-hold (U-BAH)
  - Uniform constant rebalanced portfolio (U-CRP)
  - Universal portfolio (UP)
  - Online lazy updates (OLU)

Benchmark algorithm leveraged long and short variants:
- U-BAH LO, SO, LS
- U-CRP LO, SO, LS
- Best stock LO, SO
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Thank you!

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