Pebbles and Branching Programs

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Les Valiant's 60th Birthday Celebration
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Joint work with

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Les Valiant, STOC 1975
On Non-Linear Lower Bounds in Computational Complexity

(Constructs linear size superconcentrators)

Complexity Classes

$$\mathbf{AC}^0(6) \subseteq \mathbf{NC}^1 \subseteq \mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{LogCFL}$$
$$\subseteq \mathbf{AC}^1 \subseteq \mathbf{NC}^2 \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PH}$$

As far as is known, $\mathbf{AC}^0(6)$ cannot determine whether a majority of its input bits are ones.

Yet it is open whether $\mathbf{AC}^0(6) = \mathbf{PH}$.

Here we introduce the

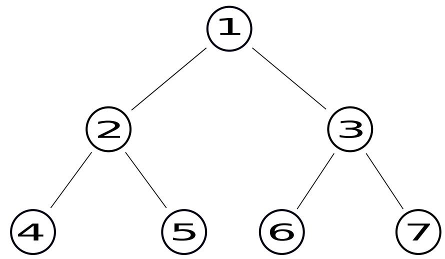
Tree Evaluation Problem (TEP)

We show TEP is in $\mathbf{LogDCFL}$.

We are trying to prove TEP $\notin \mathbf{L}$
(and TEP $\notin \mathbf{NL}$)

Tree Evaluation Problem

(Generalizes a problem in [Taitslin05])



Tree of height $h = 3$ with heap numbering

T_d^h: Balanced d-ary tree of height h
DEFAULT: $d = 2$
$[k] = \{1, ..., k\}$

TEP(h, k) Applies to T_2^h. Assume $h, k \geq 2$
Input:
$v_i \in [k]$ for each leaf i
Function $f_i : [k] \times [k] \to [k]$ for each internal node i
(Thus every node i gets a value $v_i \in [k]$)

Output: root value $v_1 \in [k]$

Decision Problem: Does $v_1 = 1$?

Claim: TEP$(h, k) \in \mathbf{LogDCFL}$

Space-efficient algorithms for TEP come from pebbling

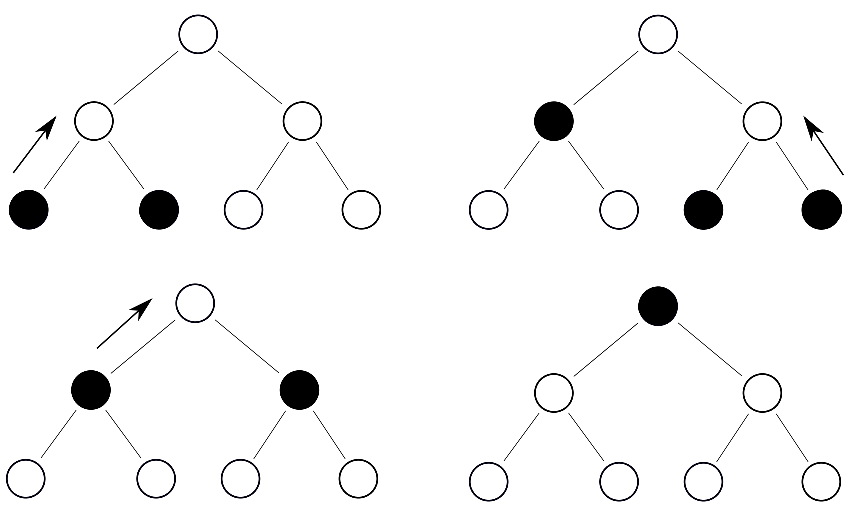
Deterministic algorithms come from 'black' pebbling. [Paterson/Hewitt70]

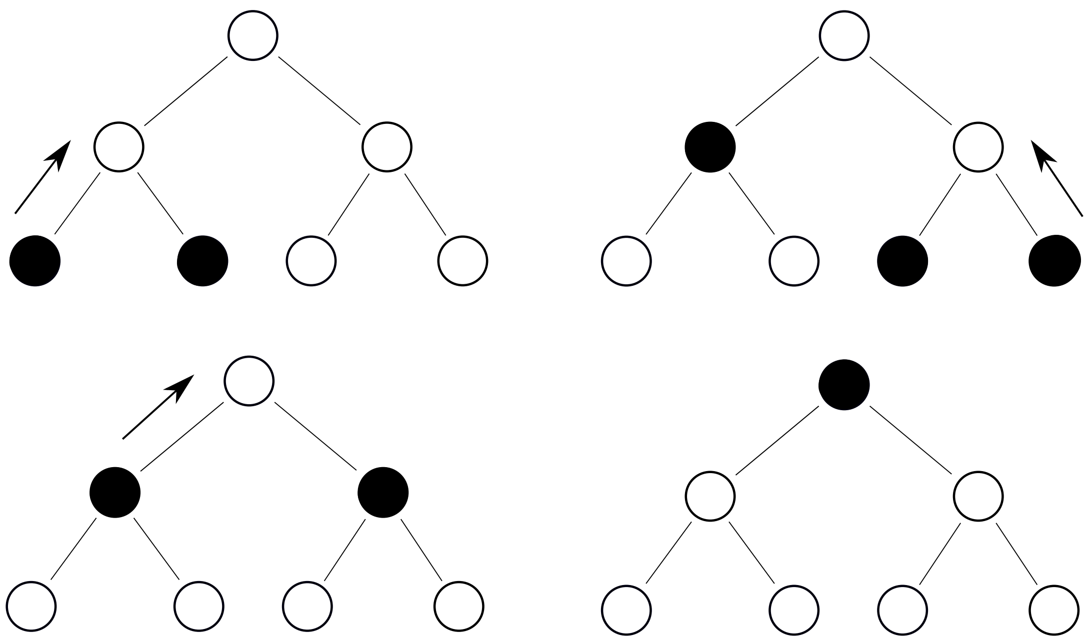
Rules:

- Place a pebble on any leaf.
- If both children of node i are pebbled, slide one of them to the parent.
- Remove any pebble at any time.

Goal: Pebble the root using a minimum number of pebbles.

Easy Theorem: T_2^h requires exactly h pebbles.





Recall: T_2^h requires exactly h pebbles.

Corollary: TEP$(h,k) \in$ DSPACE$(h \log k)$

This is NOT a log space algorithm.

Input size $n = (2^h - 1)k^2 \log k$
$\log n = \Theta(h + \log k)$

k-way Branching Programs

A k-way BP B solving TEP(h, k) is a directed multigraph with nodes called *states*. Each non-final state q is labeled either with a leaf node i, with k outedges from q labeled $1, ..., k$ indicating the possible values for v_i, or labeled with (i, x, y) where i is an internal node and the outedges are labeled with the possible values for $f_i(x, y)$. Each final state has a label from $[k]$ indicating the output v_1.

$Size(B)$ is the number of states in B.

A Turing machine M solving TEP(h, k) in space $s(h, k)$ can be simulated by a family of BPs of size $2^{O(s(h,k))}$ (the number of possible configurations of M).

$Size(h,k)$ is the number of states in the smallest deterministic BP solving TEP(h,k).

$Size_h(k) = Size(h,k)$ for fixed h.

Lemma $Size_h(k) = O(k^h)$
Proof: h pebbles suffice to pebble T_2^h, and for fixed h, the number of steps in the pebbling of T_2^h is constant.

This is the best upper bound known for the order of $Size_h(k)$.

Lemma: A lower bound of $Size_h(k) = \Omega(k^{r(h)})$ for some unbounded function $r(h)$ implies $L \neq \mathbf{LogDCFL}$.

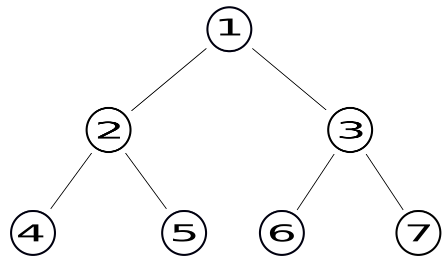
Recall best known upper bound:
$Size_h(k) = O(k^h)$

Best known lower bounds:
$Size_h(k) = \Omega(k^3)$ for each $h \geq 3$.
(Tight bounds are known for $h = 2$ and $h = 3$)

$h = 2$: $Size_2(k) = \Omega(k^2)$
This is obvious because each state of the BP can only make one query of the form (i, x, y), and there are k^2 possible values for (x, y).

$h = 3$: $Size_3(k) = \Omega(k^3)$
This is *not* obvious, because the number of input variables is only $O(k^2)$.
Proof I: Use Nečiporuk's method
Proof II: Use the "state sequence" method.

Nečiporuk's method counts the number of BPs on s states and compares this with the number of functions obtainable by various restrictions of $\text{TEP}_h(k)$. This method cannot beat $\Omega(n^2)$ states, and so cannot show TEP $\notin \mathbf{L}$.



Theorem: $Size_3(k) \geq k^3$

Proof: ("State Sequence" method)
For $r, s \in [k]$ let $E^{r,s}$ be the set of inputs I s.t.
- $f_1^I(x, y) = (x + y) \bmod k$
- $f_2^I(x, y) = f_3^I(x, y) = 0$ for all $(x, y) \neq (r, s)$
- $v_4^I = v_6^I = r$ and $v_5^I = v_7^I = s$

Thus $|E^{r,s}| = k^2$ because each $I \in E^{r,s}$ determined by v_2^I, v_3^I.

Let $\Gamma^{r,s}$ be the set of states which query either $f_2(r, s)$ or $f_3(r, s)$. It suffices to show
(*) $\qquad |\Gamma^{r,s}| \geq k$ for all $r, s \in [k]$.

Proof of (*): (γ^I, v_i^I) determines the output of $\mathcal{C}(I)$ (the computation on input I), where γ^I is the last state of $\mathcal{C}(I)$ in $\Gamma^{r,s}$, and i is the node queried by γ^I.

Thrifty Branching Programs

A deterministic BP solving $\mathsf{TEP}_h(k)$ is *thrifty* if for every query $f_i(x, y)$ (for every input), (x, y) are the values of the children of node i.

Thrifty BPs can implement black pebbling, and hence solve $\mathsf{TEP}_h(k)$ with $O(k^h)$ states. It turns out that this is also a lower bound.

Theorem: Thrifty deterministic BPs solving $\mathsf{TEP}_h(k)$ have $\Omega(k^h)$ states.

The proof is nontrivial.

Thus any BP beating the $O(k^h)$ upper bound must make queries $f_i(x, y)$ which are irrelevant to the value v_i of the node i.

Thrifty Hypothesis: Thrifty BPs are optimal among deterministic BPs solving $\mathsf{TEP}_h(k)$.

(Not true for solving the decision version of TEP)

Nondeterministic Branching Programs

Black/White Pebbling: A white pebble can be placed on any node at any time (representing a guess as to the value). The pebble can be removed if the node is a leaf, or both children have pebbles.

T_2^h can be B/W pebbled with $\lceil h/2 \rceil + 1$ pebbles. (This is optimal.)

Recall T_2^h requires h pebbles to black pebble it.

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T_2^h can be B/W pebbled with $\lceil h/2 \rceil + 1$ pebbles. (This is optimal.)

Recall T_2^h requires h pebbles to black pebble it.

Nondeterministic BPs implement B/W pebbling, so $NSize_h(k) = O(k^{\lceil h/2 \rceil + 1})$.

For $h = 3$ this gives $O(k^3)$ states, but best lower bound is $k^{2.5}$ states (via both Nečiporuk and 'state-sequence' methods).

This led us to discover "fractional pebbling".

T_2^3 can be B/W pebbled with 2.5 pebbles, so

$NSize_3(k) = \Theta(k^{2.5})$.

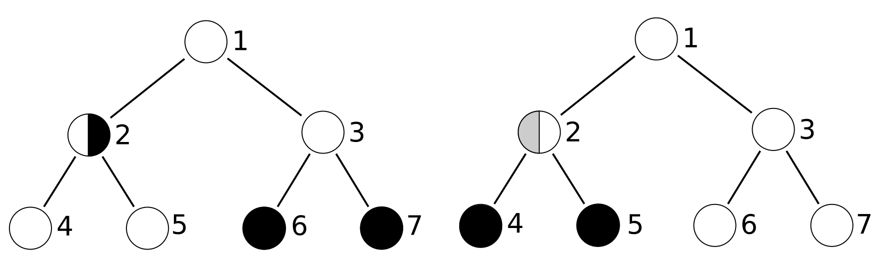
Fractional Pebbling

Fractional pebbling is like B/W pebbling, except now a node i can have a pair $(b(i), w(i))$ of real values, where

$$0 \leq b(i), w(i) \qquad b(i) + w(i) \leq 1$$

If both children of node i have total pebble value 1, then $w(i)$ can be set to 0, and any black fraction can be slid up from the children to increase $b(i)$.

The tree T_2^3 can be fractionally pebbled with 2.5 pebbles.



Theorem Thrifty nondeterministic BPs can implement fractional pebbling to solve $\mathsf{TEP}_h(k)$.

Theorem Bounds on fractional pebbling.
$\#\text{FRpebbles}(T_2^3) = 2.5$
$\#\text{FRpebbles}(T_2^4) = 3$
$h/2 - 1 \leq \#\text{FRpebbles}(T_2^h) \leq h/2 + 1$

Theorem(Repeat) Thrifty nondeterministic BPs can implement fractional pebbling.

Corollary $NSize_3(k) = \Theta(k^{2.5})$
$NSize_4(k) = O(k^3)$
$NSize_h(k) = O(k^{h/2+1}), h \geq 2$
(All upper bounds use thrifty BPs)

Theorem $ThriftyNSize_4(k) = \Theta(k^3)$

Open Question: Can nondeterministic Thrifty BPs beat fractional pebbling bound for $h > 4$?

(Recall that black pebbling is optimal for deterministic thrifty BPs.)

Conclusion

Thrifty Hypothesis: Thrifty BPs are optimal among deterministic k-way BPs solving $\mathsf{TEP}_h(k)$.
(i.e. $Size_h(k) = \Omega(k^h)$.)

In other words, the black pebbling method is the most space-efficient deterministic method for solving $\mathsf{TEP}_h(k)$.

A proof implies $\mathbf{L} \neq \mathbf{LogDCFL}$
(so $\mathbf{NC}^1 \subsetneq \mathbf{NC}^2$).

A disproof would involve a new space-efficient algorithm and would also be interesting
(think superconcentrators).

Next Step: Prove or disprove $Size_4(k) = \Omega(k^4)$

(Best known bound: $Size_4(k) = \Omega(k^3)$.)

Separating $\mathbf{L}$ from P is important!