Pricing and Resource Allocation
Game Theoretic Models

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December 8, 2009

Motivation

The framework with prices ignores a number of issues that are important for analysis of resource allocation in large-scale communication networks:

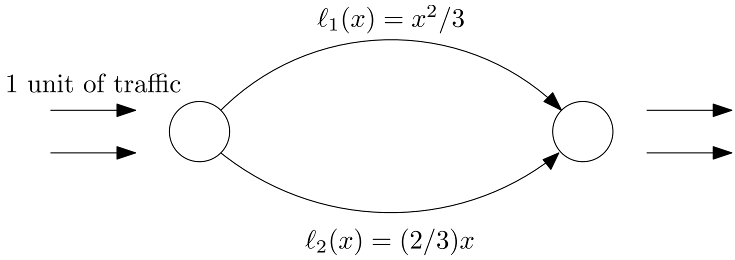
1. Centralized signals may be impractical or impossible
2. Prices are often set by multiple service providers with the objective of maximizing revenue

We investigate the implications of profit maximizing pricing by multiple decentralized service providers.

The model is of practical importance for a number of settings:

1. Transportation and communication networks
2. Markets in which there are snob effects

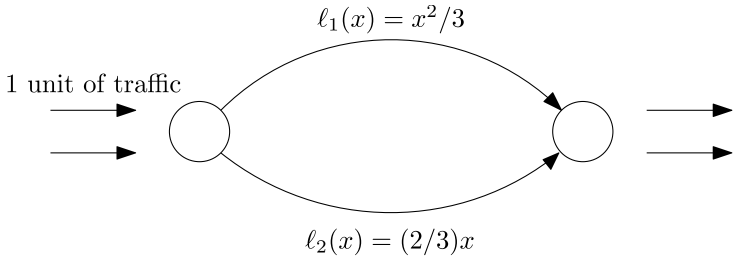
Example



1. The efficient allocation that minimizes the total delay cost $\sum_i \ell_i(x_i)x_i$ is $x_1^{opt} = 2/3$ and $x_2^{opt} = 1/3$
2. The equilibrium allocation that equates delay on the two paths is $x_1^{eq} \approx .73$ and $x_2^{eq} \approx .27$

- The equilibrium of traffic assignment without prices can be inefficient.

Example (Cont'd)



1. Monopolist will set prices $p_1^m = (2/3)^3$ and $p_2^m = (2/3^2)$. The resulting traffic in equilibrium will be $x_1^m = 2/3$ and $x_2^m = 1/3$
2. Duopoly situation results in $p_1^d \approx .61$ and $p_2^d \approx .44$. The resulting traffic in equilibrium will be $x_1^d \approx .58$ and $x_2^d \approx .42$

- Increasing competition can increase inefficiency

Intuition for the Inefficiency of Duopoly

The inefficiency is related to a new source of monopoly power for each duopolist, which they exploit by distorting the pattern of traffic:

1. Provider 1 charges higher price
2. Some traffic is pushed from route 1 to route 2
3. The congestion on route 2 is raised
4. The remaining traffic on route 1 become more “locked-in”

Model

We are interested in the problem of routing d units of flow across I links.

1. $\mathcal{I} = \{1, \ldots, I\}$, set of links
2. $x = [x_1, \ldots, x_I]$, where x_j denotes total flow on link j
3. $\ell_j(x_j)$, a convex, non-decreasing, and continuously differentiable flow-dependent latency function for each link j in the network, $\ell_j(0) = 0$ for all j
4. p_j, price per unit flow of link j
5. The cost per unit of traffic is the sum of price and latency $\ell_j + p_j$
6. We assume that this is the aggregate flow of many "small" users, who have a homogeneous reservation utility R and decide not to send their flow if the effective costs exceeds the reservation utility

Wardrop Equilibrium

We adopt the Wardrop's principle in characterizing the flow distribution on the network. For a given price vector $p \geq 0$, a vector $x^{eq} \in \mathbb{R}^I_+$ is a Wardrop equilibrium if

$$\begin{aligned} \ell_i(x_i^{eq}) + p_i = \min_j \{\ell_j(x_j^{eq}) + p_j\} \ &, \ \forall i \text{ with } x_i^{eq} > 0 \ , \\ \ell_i(x_i^{eq}) + p_i \leq R \ &, \ \forall i \text{ with } x_i^{eq} > 0 \ , \\ \sum_{i \in \mathcal{I}} x_i^{eq} \leq d \ &, \end{aligned}$$

with $\sum_{i \in \mathcal{I}} x_i^{eq} = d$ if $\min_j \{\ell_j(x_j^{eq}) + p_j\} < R$. We denote the set of equilibriums at a given price vector p by $W(p)$.

Wardrop Equilibrium – Monotonicity

Wardrop Equilibriums satisfy intuitive monotonicity properties:

Proposition

1. *For some* $\tilde{p}_j < p_j$, *let* $\tilde{x} \in W(\tilde{p}_j, p_{-j})$ *and* $x \in W(p_j, p_{-j})$, *then* $\tilde{x}_j \geq x_j$ *and* $\tilde{x}_i \leq x_i$ *for all* $i \neq j$.
2. *For some* $\tilde{\mathcal{I}} \subseteq \mathcal{I}$, *suppose that* $\tilde{p}_j < p_j$ *for all* $j \in \tilde{\mathcal{I}}$ *and* $\tilde{p}_j = p_j$ *for all* $j \notin \tilde{\mathcal{I}}$, *and* $\tilde{x} \in W(\tilde{p})$ *and* $x \in W(p)$, *then* $\sum_{j \in \tilde{\mathcal{I}}} \tilde{x}_j \geq \sum_{j \in \tilde{\mathcal{I}}} x_j$.

Social Optimum

A flow vector x^{opt} is a social optimum if it is an optimal solution for the social problem

$$\max_{x \geq 0, \sum_{i \in \mathcal{I}} x_i \leq d} \sum_{i \in \mathcal{I}} (R - \ell_i(x_i)) x_i \ .$$

1. When $\sum_{i \in \mathcal{I}} x_i = d$, the above social problem is equivalent as to minimize $\sum_{i \in \mathcal{I}} \ell_i(x_i) x_i$. When $\sum_{i \in \mathcal{I}} x_i < d$, we charge a penalty of R for each unit of undelivered traffic
2. The social problem maximizes the social surplus, i.e., the difference between users' willingness to pay and total latency

For a given vector $x \geq 0$, we define the value of the objective function in the social problem

$$\mathbb{S}(x) = \sum_{i \in \mathcal{I}} (R - \ell_i(x_i)) x_i \ .$$

Social Optimum and Pricing

The social optimum solution x^{opt} which maximizes $\sum_{i \in \mathcal{I}} (R - \ell_i(x_i))x_i$ subject to $x \geq 0$ and $\sum_{i \in \mathcal{I}} x_i \leq d$ shall satisfies that

$$\forall x_i^{opt}, x_j^{opt} > 0, \ell_i(x_i^{opt}) + \ell_i'(x_i^{opt})x_i^{opt} = \ell_j(x_j^{opt}) + \ell_j'(x_j^{opt})x_j^{opt} \quad .$$

So the pricing $p_i = \ell_i'(x_i^{opt})x_i^{opt} + c$ for any constant c achieves social optimum solution.

Monopoly Pricing and Equilibrium

The monopolist sets the prices to maximize his profit given by

$$\Pi(p, x) = \sum_{i \in \mathcal{I}} p_i x_i \ , \quad x \in W(p) \ .$$

This is a two-stage dynamic pricing-congestion game:

1. The monopolist anticipates the demand of users, and sets the prices p
2. The users choose their flow vectors x according to the Wardrop equilibrium given the prices p

Definition (Monopoly Equilibrium)

A pair (p^{opt}, x^{opt}) is a monopoly equilibrium if $x^{opt} \in W(p^{opt})$ and

$$\Pi(p^{opt}, x^{opt}) \geq \Pi(p, x) \ , \quad \forall p \geq 0, \forall x \in W(p) \ .$$

Wardrop Equilibrium – Uniqueness

Proposition (Uniqueness for Strictly Increasing Latencies)

Assume ℓ_i is strictly increasing for all i. For any price vector $p \geq 0$, the set of Wardrop Equilibriums, $W(p)$, is a singleton.

Proposition (Weak Uniqueness for General Case)

For any price vector $p \geq 0$, for any Wardrop Equilibriums $x, \hat{x} \in W(p)$, $\Pi(p, x) = \Pi(p, \hat{x})$.

Monopoly Equilibrium and Subgame-Perfect Equilibrium

Definition

A pair (p^*, x^*) is a subgame-perfect equilibrium (SPE) of the pricing congestion game if $x^* \in W(p^*)$ and for all $p \geq 0$, there exists $x \in W(p)$ such that

$$\Pi(p^*, x^*) \geq \Pi(p, x) \quad .$$

The definition of the monopoly equilibrium is stronger than the definition of subgame-perfect equilibrium. However, given the weak uniqueness property, a pair (p^{eq}, x^{eq}) is an monopoly equilibrium if and only if it is an subgame-perfect equilibrium of the pricing congestion game.

Monopoly Equilibrium and Social Optimum

Theorem (Acemoglu and Ozdaglar '06)

The price-setting by monopolists achieves efficiency.

Proof Sketch.

Suppose p is the prices by monopolists. The corresponding x satisfies that

$$\ell_i(x_i) + p_i = \min_j \{\ell_j(x_j) + p_j\} \le R, \forall x_i > 0 \ .$$

If $\ell_i(x_i) + p_i < R$ for some $x_i > 0$, then the monopolist could raise the all prices by the same amount so that the set of equilibriums does not change but the revenue is increased. So $\ell_i(x_i) + p_i = R$ for all $x_i > 0$. So

$$\mathbb{S}(x) = \sum_{i \in \mathcal{I}} (R - \ell_i(x_i))x_i = \sum_{i \in \mathcal{I}} p_i x_i \ .$$

coincides the monopolists' objective function. □

Oligopoly Pricing and Equilibrium

1. There is a set $\mathcal{S}$ of S service providers
2. Each service provider $s \in \mathcal{S}$ owns a difference subset $\mathcal{I}_s$ of the links
3. Service provider s charges a price p_i per unit on link $i \in \mathcal{I}_s$
4. Given the vector of prices of links owned by other services providers p_{-s}, the payoff of service provider s is

$$\Pi_s(p_s, p_{-s}, x) = \sum_{i \in \mathcal{I}_s} p_i x_i \ , \ \forall x \in W(p_s, p_{-s}) \ .$$

We adopt the notion of Nash equilibrium and define a vector $(p^{eq}, x^{eq}) \geq 0$ to be a Oligopoly Equilibrium if for all $s \in \mathcal{S}$, $x^{eq} \in W(p_s^{eq}, p_{-s}^{eq})$ and

$$\Pi_s(p_s^{eq}, p_{-s}^{eq}, x^{eq}) \geq \Pi_s(p_s, p_{-s}^{eq}, x) \ , \ \forall p_s \geq 0, \forall x \in W(p_s, p_{-s}^{eq}) \ .$$

Oligopoly Equilibrium and Subgame-Perfect Equilibrium

Definition

A pair (p^*, x^*) is a subgame-perfect equilibrium of the price competition game if $x^* \in W(p^*)$ and there exists a function $x : \mathbb{R}^I_+ \mapsto \mathbb{R}^I_+$ such that $x(p) \in W(p)$ for all $p \geq 0$ and for all $s \in \mathcal{S}$,

$$\Pi_s(p_s^*, p_{-s}^*, x^*) \geq \Pi_s(p_s, p_{-s}^*, x(p_s, p_{-s}^*)) \ , \ \forall p_s \geq 0 \ .$$

Similar to the monopoly case, a pair (p^{eq}, x^{eq}) is an oligopoly equilibrium if and only if it is an subgame-perfect equilibrium of the price competition game.

Efficiency Metric

Given a price competition game with latency function $\{\ell_i\}_{i \in \mathcal{I}}$, we define the efficiency metric at some oligopoly equilibrium flow x^{eq} as the ratio of the social surplus in x^{eq} to the social surplus in x^{opt}:

$$\frac{\mathbb{S}(x^{eq})}{\mathbb{S}(x^{opt})} \quad .$$

This efficiency metric coincides the notion of the "price of anarchy" by [Koutsoupias and Papadimitriou '99].

Tight Bound on the Efficiency Metric

Theorem (Acemoglu and Ozdaglar '07)

Consider a general parallel link network with $I \geq 2$ links and S service providers, where provider s owns a set of links $\mathcal{I}_s \subset \mathcal{I}$. Then, for all price competition games with pure strategy equilibrium flow x^{eq}, we have

$$\frac{\mathbb{S}(x^{eq})}{\mathbb{S}(x^{opt})} \geq \frac{5}{6} \ ,$$

and the bound is tight.

Upper Bound Example

1. A network with I links, each is owned by a different provider
2. The total flow is $d = 1$
3. The reservation utility is $R = 1$
4. The latency functions are given by

$$\ell_1(x) = 0 \ , \ \ \ell_i(x) = \frac{3}{2}(I-1)x \ , \ \ i = 2, \ldots, I \ .$$

The unique social optimum for this example is $x^{opt} = [1, 0, \ldots, 0]$. The oligopoly equilibrium is $p^{eq} = [1, \frac{1}{2}, \ldots, \frac{1}{2}]$, $x^{eq} = [\frac{2}{3}, \frac{1}{3(I-1)}, \ldots, \frac{1}{3(I-1)}]$. Hence, the efficiency metric for this example is $\frac{5}{6}$.

Summary

We investigate a model with simple network structure, where there are a single source and a single destination and each route is a single edge between the source and the destination. Here is a few take-home points:

1. The equilibrium of traffic assignment without prices can be inefficient
2. Increasing competition can increase inefficiency
3. The extent of inefficiency in the presence of oligopoly competition is bounded by $\frac{5}{6}$