Solutions for Homework 2
Networked Life, Fall 2014
Prof Michael Kearns
Due as hardcopy at the start of class, Tuesday December 9

Problem 1 (15 points: Graded by Shahin) Recall the network structure of our in-class trading experiment shown in Figure 1

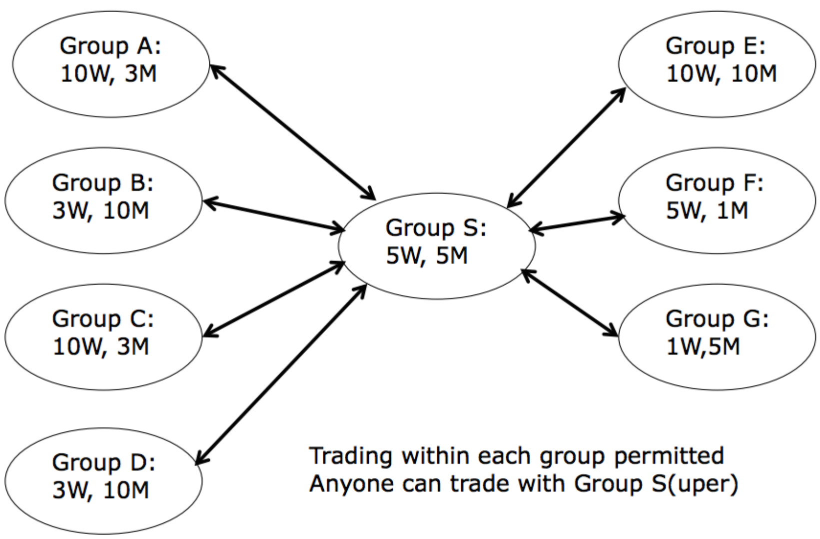

Figure 1: Network Trading - In Class Example

a. (3 points) Carefully apply the theory relating network structure to equilibrium to this network. You must describe exactly how the theory yields the resulting equilibrium wealths/prices, and what parties will trade with each other. No credit will be given for simply describing the equilibrium; you must show how it is derived.

   Solution. The equilibrium is as shown in Slides 26 of the lecture on Trading in Networks. □

b. (3 points) What is the smallest number of Wheat players that can be added to Group F that will cause the Milk players in Group S to trade only with Group F Wheat players? Justify your answer.

   Solution. Let x denote the number of Wheat players added to group F. Then, we want the ratio of $(5+x)/6$ be the highest ratio of nodes to neighbors in the network. So in particular, it should also be bigger than the second highest ratio which is $(15+x)/8$ (adding Wheat players of group F and A together).

$$\frac{5+x}{6} > \frac{15+x}{8} \rightarrow x > 15. \tag{1}$$

So at least 16 wheat players should be added to group F. Note that it is not simply sufficient to beat the 25/12 ratio because this ratio will also get bigger when adding more players to group F. The same logic applies to parts c and e. □

c. (3 points) What is the smallest number of Milk players that can be added to Group B that will cause the Wheat players in Group S to trade only with Group B Milk players? Justify your answer.

Solution. Similar to part b we want

$$\frac{10+x}{8} > \frac{15+x}{9} \to x > 30. \tag{2}$$

So at least 31 Milk players should be added to group B. □

d. (3 points) What is the smallest number of edges that can be added to the network so that all equilibrium wealths are equal? Describe in detail which edges should be added.

Solution. The smallest number of edges that needs to be added is 13. To see this, let group E's Wheat and Milk players trade together. Let 5 of the Wheat players in group A trade with the 5 Milk players in S and 3 with group A's Milk players. Connect the other 2 to 2 of the Milk players in group B. Similarly for group B, let 5 of the Milk players in group B trade with the 5 Wheat players in S and 3 with group B's Wheat players. The remaining 2 are now trading with the 2 Wheat players of group A. For group C let 3 of the Wheat players to internally trade with group C's Milk players. Do the opposite in group D. Now connect the 7 remaining Wheat players from group C to 7 remaining Milk players in group D. For group E let 1 of the Wheat players to internally trade with the sole group F's Milk players. Do the opposite in group F. Now connect the 4 remaining Wheat players from group E to the 4 remaining Milk players in group F. In this way we added $2 + 7 + 4 = 13$ edges and arrived at a perfect matching.

Note that the naive solution of adding an edge between the 7 Wheat players from group A (and also C) to 7 Milk players in group B (and also D), then adding 4 edges between the 4 Wheat players from group F to the 4 Milk players in group G will result in $7 + 7 + 4 = 18$ edges with is obviously not the smallest number of edges needed for the perfect matching.

Finally note that adding an edge between two groups will count as more than one edge! For example the edge between group A and S represents 65 edges: 5×10 edges between the Milk players of group S and the Wheat players of group A and 5×3 edges between the Wheat players of group S and the Milk players of group A. □

e. (3 points) Suppose we add another Group H to the network. Let us denote the number of Milk and Wheat players in group H as m_H and w_H, respectively. Assume $w_H > m_H$. Precisely determine the relationship between m_H and w_H such that the Group S Milk players trade only with Group H Wheat players.

Solution.

$$\frac{w_H}{m_H+5} > \frac{w_H+5}{m_H+6} \to w_H > 5m_H + 25. \tag{3}$$

□

Note that the answer to some part of this question is different than what I discussed in the review session. (Thanks to Lulu for pointing this out!) So I graded this question *very leniently* especially on parts c and e.

Problem 2 (9 points: Graded by Ryan) Consider the competitive contagion experiments in which you participated. Suppose that for some graph G, the distribution of seed pairs chosen by the participants is P. We say that P is an equilibrium if every seed pair (i, j) appearing in P receives the same average payoff (call it x) against P, and there is no seed pair (i, j) not appearing in P that receives a payoff greater than x against P.

a. (3 points) Suppose the graph G is generated by Preferential Attachment. Describe what you think the equilibrium distribution P would look like. Justify your answer.

   Solution. The 2 highest degree nodes would be the only seed pairs to choose. So all players would play the same seed pairs and hope the randomization worked in their favor where they won the seed pair. The heavy tail degree distribution characteristic of PA graphs means that there are few nodes with degree many times more than the average degree. Thus, the expected number of times a player could win a high degree node, even if everyone is playing it, would still lead to a better payoff than choosing a small degree node. □

b. (3 points) Describe, as precisely as possible, a network G for which the equilibrium distribution P will *not* have all players choose the same seed pair. Justify your answer.

   Solution. The network with 3 disjoint connected components with equal number of nodes in each component. □

c. (3 points) Describe, as precisely as possible, a network G for with the equilibrium distribution P requires that every player play a different seed pair. Justify your answer.

   Solution. The network with $2n$ disjoint connected components with equal number of nodes in each component, where n is the number of players. □

Common issues with this problem: you needed to make sure that the network you made did not have **an equilibrium** where everyone played the same seed pair. This would happen in the cycle graph. You must consider the **expected** payoff under the distribution P.

Problem 3 (10 points: Graded by Shahin) Consider a 2-player game where both the row and column players have two actions.

a. (5 points) Suppose the utility of each pair of actions is summarized as in Table 1. Complete the empty cells of the utility matrix such that the game has no pure strategy equilibrium.

Row Player/Column Player	a	b
c	+1, -1	
d		

Table 1: Utility Matrix

Solution. Here are two of the many possible ways to fill the entries. □

Row Player/Column Player	a	b
c	+1, -1	2, 0
d	0, 2	3, 1

Row Player/Column Player	a	b
c	+1, -1	-1, +1
d	-1, +1	+1, -1

Table 2: Games with no pure strategy Nash-equilibrium.

b. (5 points) Suppose the utility of each pair of actions is summarized as in Table 3. Complete the empty cells of the utility matrix such that the game has exactly two pure strategy equilibria. Does your game have a mixed strategy Nash-equilibrium?

Row Player/Column Player	a	b
c		-1, 2
d	2, -1	

Table 3: Utility Matrix

Solution. There are many possibilities for this part as well. The equilibria can be the initial filled cells (Table 4), or the initial empty cells (Table 5) or a combination of one initially empty cell and one initially filled cell (Table 6). The game always have a mixed strategy Nash-equilibrium (and by that we mean a mixed strategy that is different than the pure strategies!).

□

Row Player/Column Player	a	b
c	+1, -1	-1, 2
d	2, -1	-4, -3

Table 4: The equilibria are the initial filled cells. The game has a mixed strategy.

Row Player/Column Player	a	b
c	5, 5	-1, 2
d	2, -1	5, 5

Table 5: The equilibria are the initial empty cells. The game has a mixed strategy equilibrium.

Row Player/Column Player	a	b
c	2, 4	-1, 2
d	2, -1	-1, -3

Table 6: The equilibria are one initially filled cell and one initially empty cell. The game has a mixed strategy.

Problem 4 (12 points: Graded by Shahin) Remember the attendance dynamic where the horizontal axis denotes the percentage of students who attended the class today and the vertical axis denotes the percentage of students who are going to attend the next class. If possible, plot an attendance dynamic with the properties mentioned in each part. If not, briefly describe why plotting such attendance dynamic is impossible.

a. (3 points) An attendance dynamic with 5 equilibria such that 3 equilibria are stable and 2 are unstable.

b. (3 points) An attendance dynamic with 5 equilibria such that 2 equilibria are stable and 3 are unstable.

c. (3 points) An attendance dynamic with 5 equilibria such that only 1 equilibrium is stable and 4 are unstable.

d. (3 points) An attendance dynamic with 5 equilibria such that all equilibria are unstable.

Solution. It is possible to have an attendance dynamic in all parts except the last part.[1] For grading on part (d), unless you draw something with a clearly stable equilibrium or your graph does not have support for some parts of 0-100 range, you received full credit. The dynamics are drawn in Figure 2. □

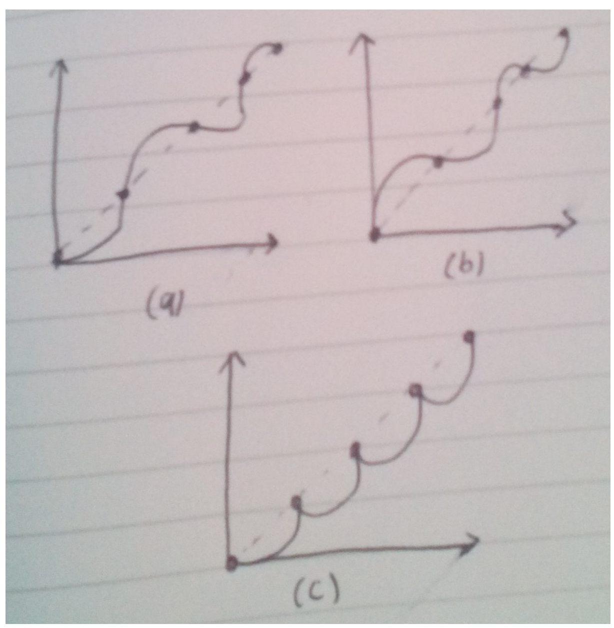

Figure 2: Problem 4

[1]It is actually possible to get an attendance dynamic for the last part. But the solution is out of scope of this course.

Problem 5 (5 points: Graded by Shahin) Pick one of the experiments from the assigned papers on behavioral experiments that we have not discussed in class (*e.g.,* Independent Set or Network Bargaining). Briefly but precisely describe the equilibria in the experiment. Then compare these equilibria in terms of the pay-offs that players receive and describe why some equilibria might be more preferable to players than others.

Solution. Open ended. There are more than one equilibria in all of these experiments for example in independent set, any independent set is an equilibrium. However, players will receive more pay off in total (in terms of social welfare) if they find the largest independent set (instead of only an independent set). So players might prefer to find the largest independent set instead of any independent set if they want to maximize the social welfare. Similar observations can be made for other experiments like network bargaining. □

Problem 6 (10 points: Graded by Ryan) Recall the Online Coloring Experiment on 30 graphs and the results that were covered in class. There were three types of graph families (Random, Preferential Attachment, and Small Worlds) that were used in the online experiments and each had the same number of nodes and edges.

a. Many students claimed that one strategy they used was to color the maximum degree node first and then color its neighbors in as few colors as possible. Why might this heuristic lead students to faster finish times in Preferential Attachment graphs? Why might this same heuristic cause students to have worse finish times in Small World graphs?

   Solution. PA has few nodes of very high degree and few links between its neighbors. That means that we could color the highly connected node, say Red, and then color all of its neighbors Blue because the chance that two Blues would now be connected was low. This was because the clustering coefficient for PA graphs is small.

   In SW, we do not have very high degree nodes, but we have lots of nodes that are close in distance to one another. Hence, there would be lots of nodes that have neighbors who are also neighbors of each other, i.e. triangles due to the high clustering coefficient. This makes it hard to color the neighbors of a highly connected node with few different colors. Also the nodes with highest degree are not much higher than most of the nodes. □

b. Another strategy that students claimed to use was to find triangles and then color all the nodes in the triangle differently. What family of graphs (Preferential Attachment or Small World) would you think this would be a better heuristic for? Explain your reason.

   Solution. I would argue that SW would be, because there is a large clustering coefficient for SW (i.e. friends of friends happen to also be friends of eachother). Hence, we would image lots of cliques, e.g. triangles, in the network. Dealing with the triangles first would nearly color the whole graph. □

Problem 7 (9 points: Graded by Ryan) Recall Network Trading where there are people with one unit of Milk (Red) and people with one unit of Wheat (Blue) that want to trade their good for the other good.

a. (6 points) What is the equilibrium for the Network Trading example in Figure 3? Remember that you must state the set of prices *and* the trades. Does this lead to wealth variation among the players?

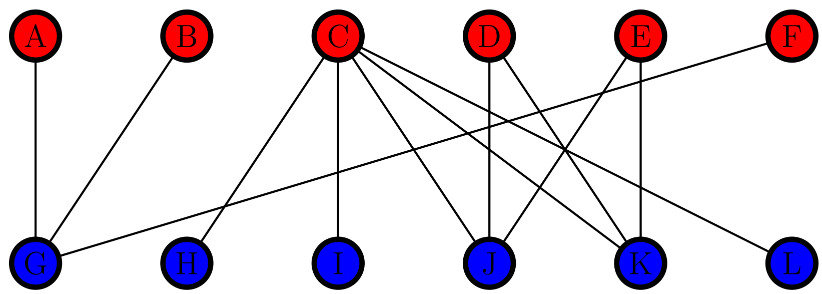

Figure 3: Network Trading

Solution. Prices: $A, B, F, H, I, L = 1/3$, $G, C = 3$, and $D = E = J = K = 1$. Trades: G splits her good into thirds and gives each fraction to A, B, F, while A, B, F give their entire good to G. Similarly for C and the ones it trades with H, I, L. Then D trades 1 for 1 with J as does E with K. □

b. (3 points) What is the fewest number of edges you need to add in order for the equilibrium to have no wealth variation? What edges did you add? Why do you know that there is no wealth variation in the new graph?

Solution. 2 edges, (B, H) and (F, L). We know there is no wealth variation because there is a perfect matching. □

Problem 8 (9 points: Graded by Ryan) Price of Anarchy - We refer to the social welfare SW of a pair of strategies for the players as the total payoff to both players under that strategy. As an example, $SW(\text{Confess}, \text{Defect}) = 0 + 10 = 10$ in the Prisoner's Dilemma given in Table 7.

a. (6 points) Consider the following Prisoner's Dilemma given in Table 7 where for each pair of action the entries of the table show the satisfaction of each prisoner after hearing their verdict (so prisoners prefer higher numbers). What is the social welfare SW for the Nash equilibrium for this game? We will call this value $SW(NE)$. Find the strategy OPT that maximizes SW, and calculate its social welfare, *i.e.* $SW(OPT)$. Calculate the price of anarchy POA of the Prisoner's Dilemma by computing $POA = \frac{SW(OPT)}{SW(NE)}$. Explain in non-technical terms what it means to have a POA much larger than 1.

Row Player/Column Player	Cooperate	Defect
Cooperate	9, 9	0, 10
Defect	10,0	1,1

Table 7: Prisoner's Dilemma

Solution.

$$SW(NE) = 2 \qquad SW(OPT) = 18 \qquad POA = 9.$$

This means that selfish individuals end up doing a lot worse than if they followed the advice of some overseer. □

b. (3 points) Recall that the Nash equilibrium for the game Rock-Paper-Scissors (given in Table 8 is mixed where each player should choose to play Rock, Paper, or Scissors with equal probability. Calculate the POA for Rock-Paper-Scissors. Be sure to write all of your work. Write in non-technical terms what your value of POA means.

Row Player/Column Player	Rock	Paper	Scissors
Rock	1, 1	0,2	2,0
Paper	2,0	1,1	0,2
Scissors	0,2	2,0	1,1

Table 8: Rock-Paper-Scissors

Solution.

$$SW(NE) = 9 \times \frac{1}{3} \times \frac{1}{3} \times (2) = 2$$
$$SW(OPT) = 2 \implies POA = 1.$$

Selfish players can do just as well as the optimal strategy. □

Problem 9 (12 points: Graded by Ryan) Selfish Routing - Suppose one unit of flow that is fully divisible wants to route from node S to node T in the network given in Figure 4. The latency functions are given on each edge in terms of the fraction of people using that edge. Recall that an equilibrium is a way to route the unit flow such that if any fraction of people deviate, they only get a longer total commute time from S to T.

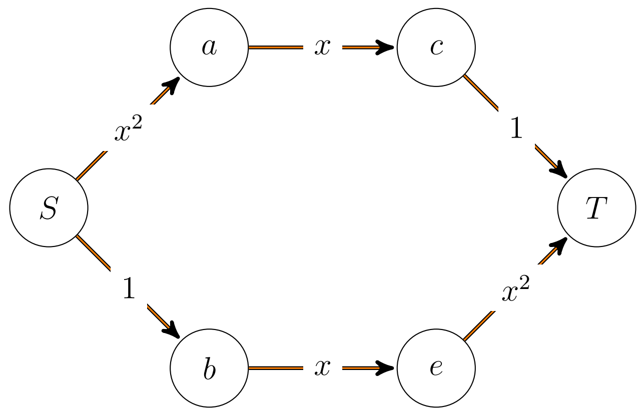

Figure 4: Selfish Routing - Part (a)

a. (4 points) Which of the following is an equilibrium in Figure 4? Give a reason for each one. Also give the total payoff for each strategy. What is the optimal strategy?

- All unit of flow is sent from S to T along the top route ($S \to a \to c \to T$).

  Solution. Not, because half of the players can take the bottom route and do better. Its payoff (or total cost) is 3. □

- Splitting the unit of flow evenly across the top route and the bottom route.

  Solution. Is an equilibrium. If any fraction diverts, they will make the route they were on lower and the new route have higher total cost. The total cost is 7/4, which is also optimal. □

b. (4 points) Which of the following is an equilibrium in Figure 5? Give a reason for each one. Also give the total payoff for each strategy. What is the optimal strategy?

- All unit of flow is sent from S to T along the zig-zag route ($S \to b \to c \to T$).

  Solution. Is not an equilibrium. Payoff is 2. □

- Splitting the unit of flow evenly across the top route and the bottom route.

  Solution. Yes, this is still an equilibrium with a cost of 7/4. Finding the optimal is a bit tricky here. We could evenly distribute the players among the 3 routes (i.e. 1/3 take top, 1/3 take bottom, and 1/3 take the zig-zag). This gives a total cost of $1.6296 < 1.75$. We will accept students saying that strategy 2 is the optimal though. □

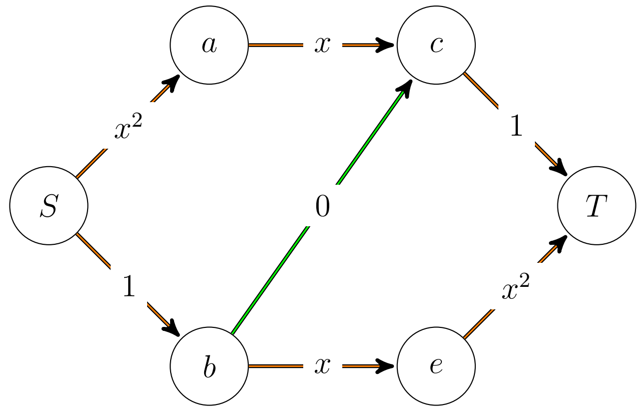

Figure 5: Selfish Routing - Part (b)

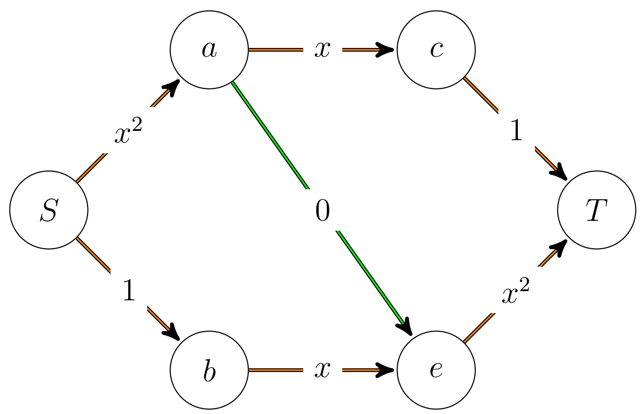

Figure 6: Selfish Routing - Part (c)

c. (4 points) Which of the following is an equilibrium in Figure 6? Give a reason for each one. Also give the total payoff for each strategy. What is the optimal strategy?

- All unit of flow is sent from S to T along the zig-zag route ($S \to a \to e \to T$).

  Solution. This is an equilibrium with cost 2. □

- Splitting the unit of flow evenly across the top route and the bottom route.

  Solution. This is not an equilibrium. An ϵ fraction of people can do better. The cost is 7/4. The optimal solution for this problem is actually quite complicated to find (because we have an x^2 cost on an edge which is not linear). However, you could find strategies that were close to optimal (or better than the strategies given). Consider putting 1/4 of the people on the top route, 1/4 on the bottom, and 1/2 on the zig-zag. This gives a total cost of

  $$1/4[(3/4)^2+1/4+1]+1/2[(3/4)^2+(3/4)^2]+1/4[(3/4)^2+1/4+1] = 1.46875 < 1.5.$$

  Note that you can do a little better by sending flows (.3, .4, .3) along (top route, zig-zag, bottom route). The optimal value is actually 1.46329. □

Problem 10 (8 points: Graded by Shahin) The command *traceroute* determines the route taken by a packet to reach a destination *e.g.* Figure 7 shows the route a packet takes to reach Stanford's Computer Science Department.

```
seas581:~ Shahin$ traceroute cs.stanford.edu
traceroute to cs.stanford.edu (171.64.64.64), 64 hops max, 52 byte packets
 1  seas-apn-gw.router.upenn.edu (158.130.104.1)  1.660 ms  0.935 ms  1.935 ms
 2  huntsman.hnt-brdr.router.upenn.edu (128.91.241.90)  18.406 ms  1.657 ms  1.130 ms
 3  0.hnt-brdr.vag-brdr.router.upenn.edu (128.91.238.205)  1.656 ms  1.068 ms  23.376 ms
 4  vag-brdr.i2-wash.router.upenn.edu (128.91.240.182)  5.100 ms  5.135 ms  5.079 ms
 5  et-5-0-0.104.rtr.atla.net.internet2.edu (198.71.45.6)  18.430 ms  18.769 ms  18.716 ms
 6  et-10-2-0.105.rtr.hous.net.internet2.edu (198.71.45.13)  42.304 ms  42.309 ms  43.198 ms
 7  et-5-0-0.111.rtr.losa.net.internet2.edu (198.71.45.21)  74.867 ms  74.999 ms  74.903 ms
 8  hpr-lax-hpr2--i2-r&e.cenic.net (137.164.26.200)  74.966 ms  75.098 ms  76.112 ms
 9  svl-hpr2--lax-hpr2-10g.cenic.net (137.164.25.38)  85.285 ms
    svl-hpr2--lax-hpr2-10g-2.cenic.net (137.164.25.50)  85.220 ms  90.475 ms
10  hpr-stanford--svl-hpr2-10ge.cenic.net (137.164.27.62)  86.118 ms  85.684 ms  85.863 ms
11  csmx-west-rtr-vl9.sunet (171.66.255.214)  87.900 ms  92.676 ms  87.791 ms
12  cs.stanford.edu (171.64.64.64)  85.996 ms  85.833 ms  85.585 ms
```

Figure 7: Output of command *traceroute*

a. (4 points) Briefly describe the technological, geographical and economical inferences you can have from Figure 7.

   Solution. **Technological**: The packet takes 12 hops to reach from Penn to Stanford. The first few hops are in Penn and then the packet leaves from the border router at Penn. Note that the delays are different for each router along the path.
   Geographical: The packet is moving from Philadelphia to Washington, Atlanta, Houston, Los Angeles and finally to Stanford. You can infer that from the url names.
   Economical: It seems like the packet is going from Penn to a network controlled by Internet2, then the Cenic network till it reaches Stanford. So there can be a financial relationship between each two consecutive entries in the sequence mentioned above. Also it turns out that Internet2 and Cenic are both non-profit organization. So it seems like the communication is happening through a path that only facilitates relationships between academic entities in the network.

   □

b. (4 points) Consider the network in Figure 8 where user A, who has Comcast as its provider, tries to send a packet to user B, who has Sprint as its provider. Comcast and Sprint operate separately and each uses the shortest path to route the packet through its network. Does this routing scheme result in the packet from user A to user B to travel through the shortest path in the network? If so, briefly describe why. If not, plot a simple internal network for Comcast and Sprint to illustrate a counter example.

   Solution. Since the providers only know their networks, the routing scheme may not result in the shortest path in the entire network. For example in Figure 9, the

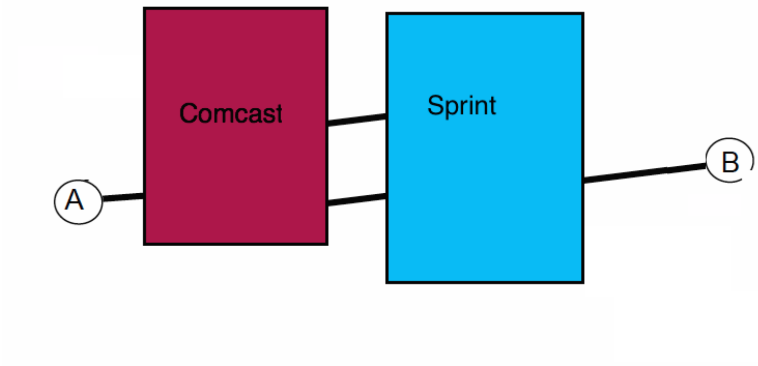

Figure 8: Online Packet Routing

shortest path in Comcast's network to send A to Sprint's network is through the top connection. However, the shortest path in the entire network is through the bottom connection between Comcast and Sprint. $\square$

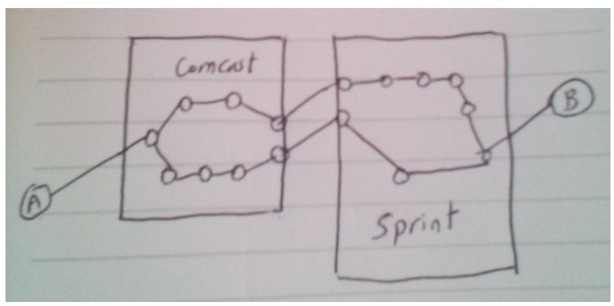

Figure 9