Analysis of Algorithms

Time and space

- To analyze an algorithm means:
  - developing a formula for predicting *how fast* an algorithm is, based on the *size of the input* (time complexity), and/or
  - developing a formula for predicting *how much memory* an algorithm requires, based on the *size of the input* (space complexity)
- Usually **time** is our biggest concern
  - Most algorithms require a fixed amount of space

What does “size of the input” mean?

- If we are searching an array, the “size” of the input could be the size of the array
- If we are merging two arrays, the “size” could be the sum of the two array sizes
- If we are computing the n^{th} Fibonacci number, or the n^{th} factorial, the “size” is n
- We choose the “size” to be a parameter that determines the actual time (or space) required
  - It is *usually* obvious what this parameter is
  - Sometimes we need two or more parameters

Characteristic operations

- In computing time complexity, one good approach is to count characteristic operations
  - What a "characteristic operation" is depends on the particular problem
  - If searching, it might be comparing two values
  - If sorting an array, it might be:
    - comparing two values
    - swapping the contents of two array locations
    - both of the above
  - Sometimes we just look at how many times the *innermost loop* is executed

Exact values

- It is sometimes possible, *in assembly language*, to compute *exact* time and space requirements
  - We know exactly how many bytes and how many cycles each machine instruction takes
  - For a problem with a known sequence of steps (factorial, Fibonacci), we can determine how many instructions of each type are required
- However, often the exact sequence of steps cannot be known in advance
  - The steps required to sort an array depend on the actual numbers in the array (which we do not know in advance)

Higher-level languages

- In a higher-level language (such as Java), we *do not know* how long each operation takes
  - Which is faster, `x < 10` or `x <= 9` ?
  - We don’t know exactly what the compiler does with this
  - The compiler almost certainly optimizes the test anyway (replacing the slower version with the faster one)
- In a higher-level language we *cannot* do an exact analysis
  - Our timing analyses will use *major* oversimplifications
  - Nevertheless, we can get some very useful results

Average, best, and worst cases

- Usually we would like to find the *average* time to perform an algorithm
- However,
  - Sometimes the “average” isn’t well defined
    - Example: Sorting an “average” array
      - Time typically depends on how out of order the array is
      - How out of order is the “average” unsorted array?
  - Sometimes finding the average is too difficult
- Often we have to be satisfied with finding the *worst* (longest) time required
  - Sometimes this is even what we want (say, for time-critical operations)
- The *best* (fastest) case is seldom of interest

Constant time

- *Constant time* means there is some constant `k` such that this operation always takes `k` nanoseconds
- A Java statement takes constant time if:
  - It does not include a loop
  - It does not include calling a method whose time is unknown or is not a constant
- If a statement involves a choice (`if` or `switch`) among operations, each of which takes constant time, we consider the statement to take constant time
  - This is consistent with *worst-case analysis*

Linear time

- We may not be able to predict to the nanosecond how long a Java program will take, but do know *some* things about timing:

```
for (i = 0, j = 1; i < n; i++) {
    j = j * i;
}
```

  - This loop takes time `k*n + c`, for some constants `k` and `c`

    `k` : How long it takes to go through the loop once
    (the time for `j = j * i`, plus loop overhead)

    `n` : The number of times through the loop
    (we can use this as the "size" of the problem)

    `c` : The time it takes to initialize the loop

  - The total time `k*n + c` is *linear in n*

Constant time is (usually) better than linear time

- Suppose we have two algorithms to solve a task:
  - Algorithm A takes `5000` time units
  - Algorithm B takes `100*n` time units
- Which is better?
  - Clearly, algorithm B is better if our problem size is small, that is, if `n < 50`
  - Algorithm A is better for larger problems, with `n > 50`
  - So B is better on small problems that are quick anyway
  - But A is better for large problems, *where it matters more*
- We usually care most about very large problems
  - But not always!

The array subset problem

- Suppose you have two sets, represented as unsorted arrays:

```
int[] sub = { 7, 1, 3, 2, 5 };
int[] super = { 8, 4, 7, 1, 2, 3, 9 };
```

  and you want to test whether every element of the first set (sub) also occurs in the second set (super):

```
System.out.println(subset(sub, super));
```

- (The answer in this case should be false, because sub contains the integer 5, and super doesn't)
- We are going to write method subset and compute its time complexity (how fast it is)
- Let's start with a helper function, member, to test whether *one* number is in an array

member

```
static boolean member(int x, int[] a) {
    int n = a.length;
    for (int i = 0; i < n; i++) {
        if (x == a[i]) return true;
    }
    return false;
}
```

- If x is *not* in a, the loop executes n times, where n = a.length
  - This is the worst case
- If x *is* in a, the loop executes n/2 times *on average*
- Either way, linear time is required: k*n+c

subset

- ```
  static boolean subset(int[] sub, int[] super) {
      int m = sub.length;
      for (int i = 0; i < m; i++)
          if (!member(sub[i], super) return false;
      return true;
  }
  ```
- The loop (and the call to member) will execute:
  - m = sub.length times, if sub is a subset of super
    - This is the worst case, and therefore the one we are most interested in
  - Fewer than sub.length times (but we don't know how many)
    - We would need to figure this out in order to compute *average* time complexity
- The worst case is a linear number of times through the loop
- But the loop body doesn't take constant time, since it calls member, which takes linear time
  ```

Analysis of array subset algorithm

- We've seen that the loop in subset executes m = sub.length times (in the worst case)
- Also, the loop in subset calls member, which executes in time linear in n = super.length
- Hence, the execution time of the array subset method is m*n, along with assorted constants
  - We go through the loop in subset m times, calling member each time
  - We go through the loop in member n times
  - If m and n are similar, this is roughly quadratic, i.e., n^2

What about the constants?

- An added constant, f(n)+c, becomes less and less important as n gets larger
- A constant multiplier, k*f(n), does *not* get less important, but...
  - Improving k gives a *linear* speedup (cutting k in half cuts the time required in half)
  - Improving k is usually accomplished by careful code optimization, not by better algorithms
  - We aren't that concerned with *only* linear speedups!
- Bottom line: ***Forget the constants!***

Simplifying the formulae

- Throwing out the constants is one of *two* things we do in analysis of algorithms
  - By throwing out constants, we simplify $12n^2 + 35$ to just n^2
- Our timing formula is a polynomial, and may have terms of various orders (constant, linear, quadratic, cubic, etc.)
  - We usually discard all but the *highest-order* term
    - We simplify $n^2 + 3n + 5$ to just n^2

Big O notation

- When we have a polynomial that describes the time requirements of an algorithm, we simplify it by:
  - Throwing out all but the highest-order term
  - Throwing out all the constants
- If an algorithm takes $12n^3+4n^2+8n+35$ time, we simplify this formula to just n^3
- We say the algorithm requires $O(n^3)$ time
- We call this Big O notation
  - Later on we will talk about related Big Ω and Big θ

Big O for subset algorithm

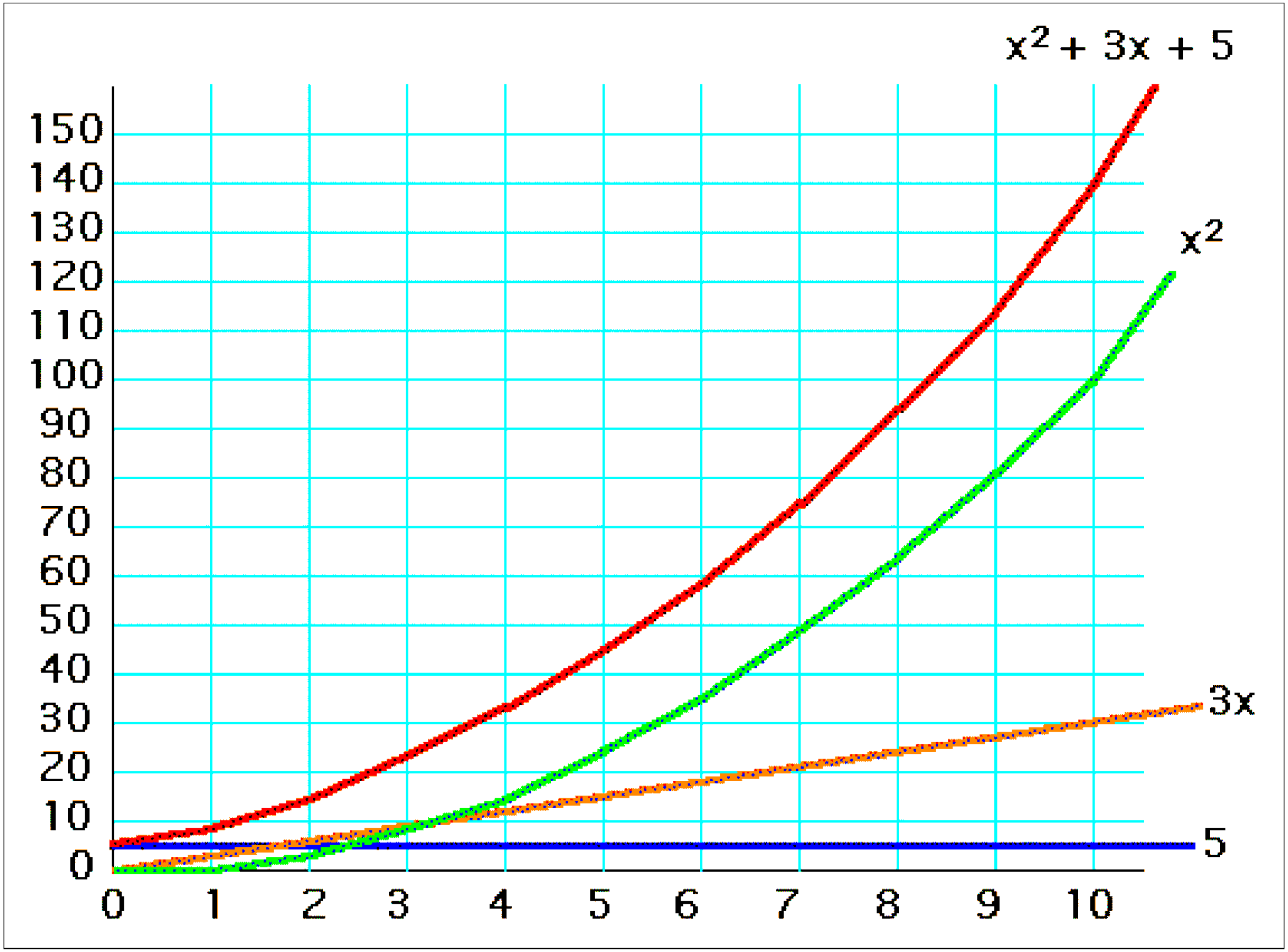
- Recall that, if n is the size of the set, and m is the size of the (possible) subset:
  - We go through the loop in subset m times, calling member each time
  - We go through the loop in member n times
- Hence, the actual running time should be k*(m*n) + c, for some constants k and c
- We say that subset takes O(m*n) time

Can we justify Big O notation?

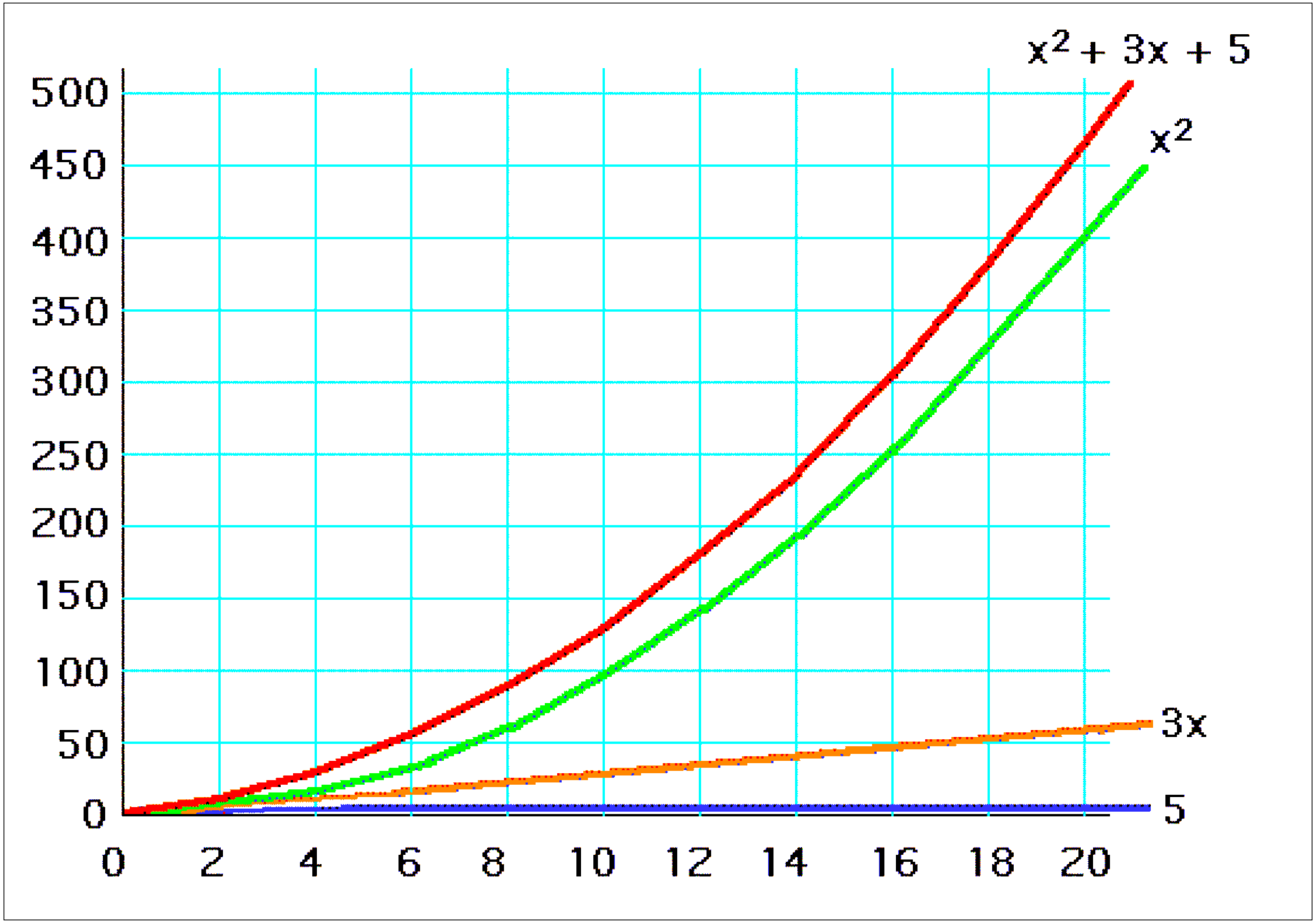
- Big O notation is a *huge* simplification; can we justify it?
  - It only makes sense for *large* problem sizes
  - **For sufficiently large problem sizes, the highest-order term swamps all the rest!**
- Consider $R = x^2 + 3x + 5$ as x varies:

$x = 0$	$x^2 = 0$	$3x = 0$	$5 = 5$	$R = 5$
$x = 10$	$x^2 = 100$	$3x = 30$	$5 = 5$	$R = 135$
$x = 100$	$x^2 = 10000$	$3x = 300$	$5 = 5$	$R = 10,305$
$x = 1000$	$x^2 = 1000000$	$3x = 3000$	$5 = 5$	$R = 1,003,005$
$x = 10,000$	$x^2 = 10^8$	$3x = 3*10^4$	$5 = 5$	$R = 100,030,005$
$x = 100,000$	$x^2 = 10^{10}$	$3x = 3*10^5$	$5 = 5$	$R = 10,000,300,005$

$y = x^2 + 3x + 5$, for x=1..10



$y = x^2 + 3x + 5$, for x=1..20



Common time complexities

BETTER

- $O(1)$ constant time
- $O(\log n)$ log time
- $O(n)$ linear time
- $O(n \log n)$ log linear time
- $O(n^2)$ quadratic time
- $O(n^3)$ cubic time
- $O(n^k)$ polynomial time
- $O(2^n)$ exponential time

WORSE

The End

(for now)