Properties of the Catadioptric Fundamental Matrix

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Abstract. The geometry of two uncalibrated views obtained with a parabolic catadioptric device is the subject of this paper. We introduce the notion of circle space, a natural representation of line images, and the set of incidence preserving transformations on this circle space which happens to equal the Lorentz group. In this space, there is a bilinear constraint on transformed image coordinates in two parabolic catadioptric views involving what we call the catadioptric fundamental matrix. We prove that the angle between corresponding epipolar curves is preserved and that the transformed image of the absolute conic is in the kernel of that matrix, thus enabling a Euclidean reconstruction from two views. We establish the necessary and sufficient conditions for a matrix to be a catadioptric fundamental matrix.

1 Introduction

The geometry of perspective views has been extensively studied in the past decade. Two books [6] and [2] contain comprehensive treatments of the subject. At the same time, the need for a larger field of view in surveillance, robotics, and image based rendering motivated the design of omnidirectional cameras. Among several designs, the catadioptric systems with a single effective viewpoint, called central catadioptric [10], attracted special attention due to their elegant and useful geometric properties (see the collection [1]). Structure from motion given omnidirectional views is an evolving research area. Gluckman and Nayar [5] studied ego-motion estimation by mapping the catadioptric image to the sphere. Svoboda et al [14] first established the epipolar geometry for all central catadioptric systems. Kang [8] proposed a direct self-calibration by minimizing the epipolar constraint.

In this paper we study the geometry of two uncalibrated views obtained with a parabolic catadioptric device. We assume that the optical axes of the lens and the mirror are parallel and that the aspect ratio and skew parameter are known leaving only the focal length (combined scaling factor of mirror, lens, and CCD-chip) and the image center (intersection of the optical axis with the

* The authors are grateful for support through the following grants: NSF-IIS-0083209, SNF-EIA-0120565, NSF-IIS-0121293, NSF-EIA-9703220, a DARPA/ITO/NGI subcontract to UNC, and a Penn Research Foundation grant.

A. Heyden et al. (Eds.): ECCV 2002, LNCS 2351, pp. 140–154, 2002.

image plane) as unknown. The parabolic projection $\boldsymbol{x} = (u, v, 1)^T$ of a point $\boldsymbol{X} = (x, y, z, w)^T \in \mathbb{P}^3$ incorporates two steps: 1. intersecting the paraboloid and the ray from the paraboloid's focus through $\boldsymbol{X}$; and 2. orthographically projecting this intersection to the image plane. It reads [10,14,3] as follows

$$u = c_x + \frac{2fx}{-z + \sqrt{x^2 + y^2 + z^2}} \quad \text{and} \quad v = c_y + \frac{2fy}{-z + \sqrt{x^2 + y^2 + z^2}}, \tag{1}$$

where $(c_x, c_y, 1)$ is the intersection of the optical axis with the image plane and f is the projected focal length of the mirror, and where it is also assumed that the focus is $O = (0, 0, 0, 1)$, the origin, and the z-axis is parallel to the optical axis (1 left). The circle centered at $(c_x, c_y, 1)$ and whose imaginary radius is $2f$ will be named $\boldsymbol{\omega}'$ and is called the calibrating conic because it gives the three intrinsics c_x, c_y and f. Every image of a line is a circle which intersects $\boldsymbol{\omega}'$ antipodally [3].

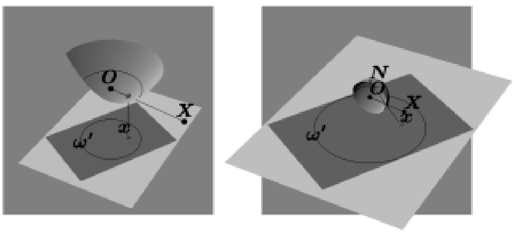

Fig. 1. The projection on a paraboloidal mirror with subsequent orthographic projection (left) and the equivalent model: spherical projection with subsequent stereographic projection (right).

It was shown in [3] that the parabolic projection described above is equivalent to another two step projection: project the point in space to the sphere and then project this point from the north pole to the plane of the equator; see figure 1 (right). This type of projection is equivalent to a parabolic projection in which the calibrating conic $\boldsymbol{\omega}'$ is identical to the projection of the equator. The second step in the two step projection is stereographic projection which has two properties which will be relevant to us: 1. it projects any circle on the sphere great or small to a circle in the plane; and 2. stereographic projection is conformal in that locally it preserves angles [11].

In [4] an extra coordinate is added to the image coordinates so that a general perspective projection becomes proportional to a linear transformation of the new image coordinates. The mechanism is achieved by "lifting" a point in the image plane to the surface of a paraboloid that is not necessarily equal to the physical paraboloid being used as the mirror. Once lifted to the paraboloid, a special class of linear transformations preserves the surface of the parabola while inducing translation and scaling in the image points. An appropriate transformation exists which maps lifted image points into rays which are calibrated and are collinear with the space point and the focus. This lifting space also has the advantage of being able to represent the images of lines (circles) in the image plane.

In this paper we combine the lifting idea, which has the effect of factoring out some portion of the non-linearity of the problem, with the use of stereographic projection. Thus instead of using the paraboloid as a lifting surface, we intend to use the sphere, where we will apply the inverse of stereographic projection to lift image points to the sphere. Though this can be seen to be the same as using the paraboloid, using the sphere has the advantage of being more symmetric and drastically simplifies our derivations.

We summarize here the original contributions of this paper:

1. A new representation of image points and line images for parabolic catadioptric images is defined using inverse stereographic projection.
2. The equivalent class of linear transformations of this space is shown to preserve angles and is equal to the Lorentz group.
3. A projection formula analogous to the perspective projection formula is derived. Using this projection formula we reformulate the multiple view matrix and the rank deficiency condition remains from the perspective case. Mixed sensor types can be included in the multiple view matrix.
4. From this catadioptric multiple view matrix the catadioptric fundamental matrix is derived. We prove that the lifted images of the absolute conic of the left (right) camera belong to the two-dimensional left (right) null-space of the catadioptric fundamental matrix. Self-calibration becomes, thus, the intersection of two null-spaces. It is possible with two parabolic views as opposed to three views required in the perspective case (even with known aspect ratio and skew).
5. Because of the stereographic projection involved in the parabolic projection, angles between epipolar circles are preserved. We prove the equivalent algebraic condition on the singular vectors of the catadioptric fundamental matrix.
6. Based on the last two facts, we derive the necessary and sufficient conditions for a given matrix to be a catadioptric fundamental matrix.

2 The Spherical Representation of Points and Circles: Circle Space

A unit sphere centered at the origin has the quadratic form

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2}$$

Given a point $\boldsymbol{x} = (u, v, 0, 1)$ we wish to find the point $\widetilde{\boldsymbol{x}}$ on the sphere which when stereographically[1] projected from $\boldsymbol{N} = (0, 0, 1, 1)$ would give $\boldsymbol{x}$. It is easy to verify that the point

[1] This is **not** necessarily the same stereographic projection which was used to generate the image point from a point in space.

$$\widetilde{\boldsymbol{x}} = \left(2u, 2v, u^2+v^2-1, u^2+v^2+1\right)^T \tag{3}$$

lies on the sphere and is collinear with $\boldsymbol{N}$ and $\boldsymbol{x}$. The point $\widetilde{\boldsymbol{p}}$ will be called the "lifting" of the point $\boldsymbol{x}$, whereas $\boldsymbol{x}$ is the stereographic projection of $\widetilde{\boldsymbol{x}}$.

Circles can also be represented in this framework due to the following fact. Stereographic projection maps points on the sphere to co-circular points in the plane if and only if the points on the sphere also lie on a plane. We represent a circle in the image plane with the polar point of the plane containing the lifted image points lying on the circle. Recall from projective geometry that the polar point of a plane is the vertex of the cone tangent to the sphere (or any quadric surface) at the intersection of the plane with the sphere. The polar plane of a point has the reverse relationship.

Let $\boldsymbol{\gamma}$ be a circle centered in the image plane at $(c_x, c_y, 1)$ with radius r. We claim that the plane containing the lifted points of $\boldsymbol{\gamma}$ is

$$\boldsymbol{\pi} = \left(2c_x, 2c_y, c_x^2+c_y^2-r^2-1, -c_x^2-c_y^2+r^2-1\right)^T .$$

The polar point of this plane $\boldsymbol{\pi}$ will be the point representation $\widetilde{\boldsymbol{\gamma}}$ (Fig. 2 (left)) of the circle $\boldsymbol{\gamma}$, where

$$\widetilde{\boldsymbol{\gamma}} = \boldsymbol{Q}\boldsymbol{\pi} = \left(2c_x, 2c_y, c_x^2+c_y^2-r^2-1, c_x^2+c_y^2-r^2+1\right) . \tag{4}$$

As a result it can be shown that $\boldsymbol{p} \in \boldsymbol{\gamma}$ if and only if $\widetilde{\boldsymbol{p}}^T\boldsymbol{Q}\widetilde{\boldsymbol{\gamma}} = 0$. This has dual interpretations: 1. the set of points $\boldsymbol{p}$ lying on $\boldsymbol{\gamma}$ have liftings lying on the plane $\boldsymbol{Q}\widetilde{\boldsymbol{\gamma}}$; and 2. the set of circles $\boldsymbol{\gamma}$ containing a point $\boldsymbol{p}$ have point representations lying on the plane $\boldsymbol{Q}\widetilde{\boldsymbol{p}}$. We claim that definition (4) also applies when r is imaginary.

The value of $\rho = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ determines whether $\boldsymbol{x}$ lies inside ($\rho < 0$), outside ($\rho > 0$), or on the surface of the sphere ($\rho = 0$). We find that under the condition that $\widetilde{\boldsymbol{\gamma}}$ have not been scaled from their definition in (4)[2] then $\rho = \widetilde{\boldsymbol{\gamma}}^T\boldsymbol{Q}\widetilde{\boldsymbol{\gamma}} = 4r^2$, implying that if $\widetilde{\boldsymbol{\gamma}}$ lies inside the sphere then it represents a circle with imaginary radius since ρ must be negative; if $\widetilde{\boldsymbol{\gamma}}$ lies on the sphere then $\rho = 0$ which implies that $\boldsymbol{\gamma}$ is a circle of zero radius or a point, which we already knew since it is then of the form (3); otherwise $\widetilde{\boldsymbol{\gamma}}$ lies outside the sphere and represents a circle with real radius. Hence $\widetilde{\boldsymbol{\omega}}$, representing an imaginary circle, must lie inside the sphere and $\widetilde{\boldsymbol{\omega}}$ must lie outside the sphere because it represents a real circle.

In particular from the definition in (4) we determine that

$$\widetilde{\boldsymbol{\omega}}' = \left(2c_x, 2c_y, c_x^2+c_y^2-4f^2-1, c_x^2+c_y^2-4f^2+1\right)^T \tag{5}$$

is the point representation of the calibrating conic. Similarly $\boldsymbol{\omega}$, the image of the absolute conic [13], has point representation

$$\widetilde{\boldsymbol{\omega}} = \left(2c_x, 2c_y, c_x^2+c_y^2+4f^2-1, c_x^2+c_y^2+4f^2+1\right)^T . \tag{6}$$

[2] The circle space representation lies in $\mathbb{P}^3$ and so is a homogeneous space, but in some rare instances like this one we will require that $\widetilde{\boldsymbol{\gamma}}$ is exactly of the form in (4).

Their geometric interpretation will be elucidated in Proposition 2.

We now state without proof some miscellaneous facts. We define $\boldsymbol{\pi}_x = \boldsymbol{Q}\boldsymbol{x}$ to be the polar plane of the point $\boldsymbol{x}$ with respect to $\boldsymbol{Q}$. The first fact is that a circle on the sphere projects to a line if and only if the circle contains $\boldsymbol{N}$. All points on $\boldsymbol{\pi}_N$, in this case the plane tangent to the sphere at $\boldsymbol{N}$, are points whose polar planes must contain $\boldsymbol{N}$. Therefore $\boldsymbol{\pi}_N$ contains the point representations of lines in the image plane.

The second fact is that points on $\boldsymbol{\pi}_\infty$ have polar planes going through the origin and therefore yield great circles. Thus the points at infinity represent exactly the lines of $\mathbb{P}^2$ as they are represented on the sphere. Is there a linear transformation of circle space which maps the point representations of line images to $\boldsymbol{\pi}_\infty$ so that they represent line images in $\mathbb{P}^2$?

The third fact is a cautionary note. Unlike in perspective geometry where the line image between two image points is uniquely defined, this is not the case in circle space. For any two image points there is a one parameter family of circles, a line in circle space, going through them. The correct circle for a given parabolic catadioptric image is the one which intersects $\widetilde{\boldsymbol{\omega}}'$ antipodally.

Since we will be dealing with the angle of intersection of two circles we need a well-defined way to determine this angle. If two circles $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ are centered respectively at $\boldsymbol{g}$ and $\boldsymbol{h}$, have radii r_1 and r_2, and intersect at $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, we define the angle between them to be the angle $\angle \boldsymbol{g}\boldsymbol{p}_1\boldsymbol{h}$. This angle is the same as π minus the angle between the tangent vectors as can be seen from Figure 2 (right). Let $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_Q = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{y}$ and $\|\boldsymbol{x}\|_Q = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle_Q}$[3].

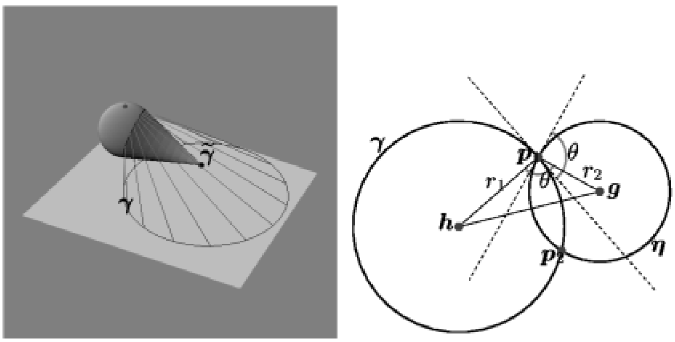

Fig. 2. On the left, the lifting of a circle γ to the point $\widetilde{\gamma}$ in spherical circle space. On the right, the angle of intersection of two circles $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ is defined to be the angle $\angle \boldsymbol{g}\boldsymbol{p}_1\boldsymbol{h}$ since this is the same as at least one of the two angles between the tangent vectors.

Proposition 1. The angle θ between two circles $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ can be obtained from the "dot product" in circle space:

[3] $\langle \cdot, \cdot \rangle_Q$ is *not* a real dot product nor is $\| \cdot \|_Q$ a real norm. $\langle \cdot, \cdot \rangle_Q$ is a symmetric bilinear form but since Q is not positive definite it does not officially qualify as a dot product.

$$\cos^2\theta = \frac{\langle\widetilde{\gamma},\widetilde{\boldsymbol{\eta}}\rangle_Q^2}{\|\widetilde{\gamma}\|_Q^2\|\widetilde{\boldsymbol{\eta}}\|_Q^2}\,. \tag{7}$$

Proof: As shown in Fig. 2 let $\boldsymbol{g}$ be the center and r_1 the radius of the circle $\boldsymbol{\gamma}$, and let $\boldsymbol{h}$ be the center and r_2 the radius of $\boldsymbol{\eta}$. Let $\boldsymbol{p}_1$ be one of the intersections of the two circles. By solving for $\cos\theta$ in the law of cosines the angle $\theta = \angle \boldsymbol{g}\boldsymbol{p}_1\boldsymbol{h}$ satisfies

$$\cos^2\theta = \frac{\left(r_1^2+r_2^2-\|\boldsymbol{g}-\boldsymbol{h}\|^2\right)^2}{4r_1^2r_2^2} = \frac{\left(r_1^2+r_2^2-\boldsymbol{g}^T\boldsymbol{g}-\boldsymbol{h}^T\boldsymbol{h}+2\boldsymbol{g}^T\boldsymbol{h}\right)^2}{4r_1^2r_2^2}\,.$$

According to the assumptions we must have

$$\begin{aligned}\widetilde{\gamma} &= \lambda\left(2\boldsymbol{g}^T,\boldsymbol{g}^T\boldsymbol{g}-r_1^2-1,\boldsymbol{g}^T\boldsymbol{g}-r_1^2+1\right)^T\\ \widetilde{\boldsymbol{\eta}} &= \mu\left(2\boldsymbol{h}^T,\boldsymbol{h}^T\boldsymbol{h}-r_2^2-1,\boldsymbol{h}^T\boldsymbol{h}-r_2^2+1\right)^T.\end{aligned} \tag{8}$$

First notice that by calculating $\widetilde{\gamma}^T\boldsymbol{Q}\widetilde{\boldsymbol{\eta}}$ one finds that

$$\boldsymbol{g}^T\boldsymbol{h} = \boldsymbol{g}^T\boldsymbol{g}+\boldsymbol{h}^T\boldsymbol{h}-r_1^2-r_2^2+\frac{\langle\widetilde{\gamma},\widetilde{\boldsymbol{\eta}}\rangle_Q}{2\lambda\mu}\,, \tag{9}$$

and also that $r_1^2 = \|\widetilde{\gamma}\|_Q^2/4\lambda^2$ and $r_2^2 = \|\widetilde{\boldsymbol{\eta}}\|_Q^2/4\mu^2$. Substituting (9) into (8) and then substitutions for r_1^2 and r_2^2 yields (7). □

The square in $\cos^2\theta$ is necessary because $\widetilde{\gamma}$ and $\widetilde{\boldsymbol{\eta}}$ are homogeneous and the scale factors λ and μ could be negative. The corollary follows immediately from the proposition.

Corollary 1. Two circles $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ are orthogonal if and only if $\widetilde{\gamma}^T\boldsymbol{Q}\widetilde{\boldsymbol{\eta}} = 0$.

Lemma 1. Two circles $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ are centered at the same point and have a ratio of radii equal to i (one is imaginary, the other is real, but excluding complex circles) if and only if they are orthogonal and their polar planes intersect in a line on $\boldsymbol{\pi}_N$.

Proof: The forward and reverse directions can be verified by direct calculation. Verify that the first conditions imply $\widetilde{\gamma}\boldsymbol{Q}\widetilde{\boldsymbol{\eta}} = 0$ and that the three planes are linearly dependent (the 3×3 sub-determinants of the matrix $(Q\widetilde{\gamma},Q\widetilde{\boldsymbol{\eta}},\boldsymbol{\pi}_N)^T$ are zero). The converse can be shown by solving for the center and radius of $\boldsymbol{\eta}$ in terms of $\boldsymbol{\gamma}$. □

Lemma 2. A set of circles $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ are coaxal if and only if their point representations $\{\widetilde{\gamma}_\lambda\}_{\lambda\in\Lambda}$ are collinear.

See [12] for a proof when $\boldsymbol{Q}$ is the parabola instead of the sphere. The same reasoning applies.

Proposition 2. Let $\boldsymbol{\omega}'$ be a circle representing the calibrating conic. The set of circles intersecting $\boldsymbol{\omega}'$ antipodally, i.e. the set of line images, lie on a plane whose polar point with respect to $\boldsymbol{Q}$ is $\boldsymbol{\omega}$.

Proof: All lines through the center of $\boldsymbol{\omega}'$ intersect $\boldsymbol{\omega}'$ antipodally and are also orthogonal to $\boldsymbol{\omega}'$, therefore these lines' point representations lie on the line $\boldsymbol{\ell}$

which is the intersection of the plane $\boldsymbol{\pi}_N$ (containing all point representations of lines) and the plane $\boldsymbol{\pi}_{\widetilde{\omega}'} = \boldsymbol{Q}\widetilde{\boldsymbol{\omega}'}$ (containing all point representations of circles or lines orthogonal to $\boldsymbol{\omega}'$). Any circle $\boldsymbol{\gamma}$ intersecting $\boldsymbol{\omega}'$ antipodally in points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is coaxal with $\boldsymbol{\omega}$ and the line through $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, which also goes through the center of $\boldsymbol{\omega}'$. Thus by Lemma 2 their point representations are collinear. Hence $\widetilde{\boldsymbol{\gamma}}$, the representation of an arbitrary circle antipodal to $\widetilde{\boldsymbol{\omega}}'$, lies on the plane $\boldsymbol{\pi}$ through $\boldsymbol{\ell}$ and $\widetilde{\boldsymbol{\omega}}'$.

Now we show that the polar point of $\boldsymbol{\pi}$ must equal $\widetilde{\boldsymbol{\omega}}$. Let $\boldsymbol{A} = \boldsymbol{Q}^{-1}\boldsymbol{\pi}$ to be the polar point of $\boldsymbol{\pi}$. The circle represented by $\boldsymbol{A}$ is orthogonal to $\boldsymbol{\omega}'$ since

$$\boldsymbol{A}^T\boldsymbol{Q}\widetilde{\boldsymbol{\omega}}' = \boldsymbol{\pi}^T\boldsymbol{Q}^{-T}\boldsymbol{Q}\widetilde{\boldsymbol{\omega}}' = \boldsymbol{\pi}^T\widetilde{\boldsymbol{\omega}}' = 0\,,$$

the last equality following by the definition of $\boldsymbol{\pi}$. Since they are orthogonal and their polar planes intersect in the line ℓ on $\boldsymbol{\pi}_N$, by Lemma 1, $\widetilde{\boldsymbol{\omega}}'$ and the circle represented by $\boldsymbol{A}$ must have the same center and have a ratio of radii equal to i. Therefore $\boldsymbol{A} = \widetilde{\boldsymbol{\omega}}$. □

2.1 The Lorentz Group and Plane Preserving Subgroups

In a perspective image a natural class of transformations on image points is the set of collineations, projective transformations specified by non-singular 3×3 matrices. We would like to find an equivalent structure for parabolic catadioptric images under the requirement that the transformation operate linearly on the circle space. Therefore this class must consist of some subset of 4×4 matrices. These transformations also should not act in a way which happens to transform a point into a circle or vice versa, for this would inviolate incidence relationships in the image plane. Thus the surface of the sphere must remain invariant under any such transformation. This is the barest of conditions necessary to determine the set of transformations and we therefore investigate the set $\mathcal{L} = \{\boldsymbol{A} : \boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A} = \boldsymbol{Q}\}$. This is a group since it is closed under multiplication and inversion and contains the identity. As it turns out this is a well known six dimensional[4] Lie group from the study of physics called the Lorentz group [7]. Any transformation from this group preserves angles between circles, for if $\boldsymbol{A} \in \mathcal{L}$ then $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_Q = \langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}\boldsymbol{y}\rangle_Q$. Since two circles can be constructed to form any angle, these transformations must preserve all angles when they transform the image plane. Angles replace the cross ratio as the invariance under these transformations. It also implies that general projective transformations applied to image points that do not preserve angles, such as shearing or change of aspect ratio, can not be represented as a linear transformation of circle space, at least not in a way which preserves incidence relationships.

In the previous section it was said that the set of line images of a given parabolic projection have point representations lying on a plane in circle space. The plane on which they lie is polar to the point representation of the image of the absolute conic, $\widetilde{\boldsymbol{\omega}}$. What is the set of transformations preserving this plane and what meaning does this have? In order that a transformation preserve the

[4] The inclusion of scale yields an additional dimension, and then $\boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A} = \lambda\boldsymbol{Q}$.

plane it must preserve $\widetilde{\boldsymbol{\omega}}$. Therefore $\widetilde{\boldsymbol{\omega}}$ must be an eigenvector of the transformation for any eigenvalue (since $\widetilde{\boldsymbol{\omega}}$ is homogeneous). Let

$$\mathcal{L}_{\widetilde{\omega}} = \{\boldsymbol{A} : \boldsymbol{A}^T \boldsymbol{Q} \boldsymbol{A} = \boldsymbol{Q} \text{ and } A\widetilde{\boldsymbol{\omega}} = \lambda \widetilde{\boldsymbol{\omega}} \text{ for some } \lambda\}\,.$$

This is a group since it is also closed under multiplication and inversion.

We examine two subcases, $\widetilde{\boldsymbol{\omega}} = (0,0,0,1)$ and $\widetilde{\boldsymbol{\omega}} = \boldsymbol{N}$. We will calculate the Lie algebra for the connected component containing the identity. If $\boldsymbol{A}(t)$ is a continuous parameterization of matrices in $\mathcal{L}_{\widetilde{\omega}}$ such that $\boldsymbol{A}(0) = I$, then the first condition gives

$$\frac{d}{dt}\boldsymbol{A}(t)^T \boldsymbol{Q} \boldsymbol{A}(t)\bigg|_{t=0} = \frac{d}{dt}\boldsymbol{Q}\bigg|_{t=0} \quad \text{and} \quad \boldsymbol{A}'(0)^T \boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A}'(0) = 0\,.$$

The second condition is equivalent to the 2×2 sub-determinants of the matrix $(\widetilde{\boldsymbol{\omega}}, \boldsymbol{A}(t)\widetilde{\boldsymbol{\omega}})^T$ being zero. Each of the six equations for the sub-determinants can be differentiated with respect to t and evaluated at $t = 0$ and then one can solve for the entries $\boldsymbol{A}'(0)$. When $\widetilde{\boldsymbol{\omega}} = (0,0,0,1)$, this yields

$$\boldsymbol{A}'(0) = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ -a_{12} & 0 & a_{23} & 0 \\ -a_{13} & -a_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is just the set of matrices which are skew symmetric in the first three rows and columns and zero elsewhere. Therefore $\mathcal{L}_{(0,0,0,1)}$ is the set of rotations in $\mathbb{P}^3$.

If $\widetilde{\boldsymbol{\omega}} = \boldsymbol{N}$, the north pole, then $\boldsymbol{A}'(0)$ must be of the form

$$\boldsymbol{A}'(0) = \begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{13} \\ a_{12} & 0 & -a_{23} & a_{23} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{13} & a_{23} & a_{34} & 0 \end{pmatrix}.$$

The Lie group generated by this Lie algebra preserves $\boldsymbol{N}$ and therefore it preserves the plane tangent to $\boldsymbol{N}$ on which lie the point representations of lines. They therefore sends lines to lines while also by default preserving angles. Therefore this subgroup corresponds to affine transformations in the plane. An important subcase and reparameterization of $\mathcal{L}_N$ is defined under the following substitutions, $a_{12} = 0, a_{34} = -\log 2f, a_{13} = -c_x \frac{\log 2f}{2f-1}, a_{23} = -c_y \frac{\log 2f}{2f-1}$. Upon exponentiation we have the following matrix dependent on $\widetilde{\boldsymbol{\omega}}$,

$$\boldsymbol{K}_{\widetilde{\omega}} = \begin{pmatrix} 1 & 0 & c_x & -c_x \\ 0 & 1 & c_y & -c_y \\ -\frac{c_x}{2f} & -\frac{c_y}{2f} & \frac{1-c_x^2-c_y^2+4f^2}{2f} & \frac{1+c_x^2+c_y^2-4f^2}{2f} \\ -\frac{c_x}{2f} & -\frac{c_y}{2f} & \frac{1-c_x^2-c_y^2-4f^2}{2f} & \frac{1+c_x^2+c_y^2+4f^2}{2f} \end{pmatrix}. \tag{10}$$

This has the effect on the image points of translating by $(-c_x, -c_y)$ and then scaling by $\frac{1}{2f}$. Also notice that $\boldsymbol{K}_{\widetilde{\omega}}\widetilde{\boldsymbol{\omega}} = \lambda O$ and $\boldsymbol{K}_{\widetilde{\omega}}\widetilde{\boldsymbol{\omega}}' = (0,0,\lambda,0)$. We will use this matrix in the next section.

In general when $\widetilde{\boldsymbol{\omega}}$ does not lie on the sphere, the dimension of $\mathcal{L}_{\widetilde{\omega}}$ is three because the sub-determinants give three independent constraints; this Lie group corresponds to rotations about the viewpoint. When $\widetilde{\boldsymbol{\omega}}$ lies on the sphere an additional dimension arises because the number of independent constraints decreases by one; this Lie group leaves the image point corresponding to $\widetilde{\boldsymbol{\omega}}$ invariant. One additional comment, since $\exp A^T = (\exp A)^T$, and since the Lie algebra of $\mathcal{L}$ can be seen to contain $\boldsymbol{A}$ if and only if it contains $\boldsymbol{A}^T$, then $\boldsymbol{B} \in \mathcal{L}$ if and only if $\boldsymbol{B}^T \in \mathcal{L}$.

3 Multiple Parabolic Views

We now wish to find a parabolic projection equation more closely resembling the perspective projection formula $\boldsymbol{\Pi X} = \lambda \boldsymbol{x}$, where $\boldsymbol{x} \in \mathbb{P}^2$ is the image of $\boldsymbol{X} \in \mathbb{P}^3$, $\boldsymbol{\Pi}$ is the 4×3 camera matrix, and λ is the projective depth depending on $\boldsymbol{\Pi}$, $\boldsymbol{X}$ and $\boldsymbol{x}$. As it stands, because of the non-linearity of the definition in (1) it is not trivial to apply the multiview results found for perspective cameras to the parabolic catadioptric case.

First we apply $\boldsymbol{K}_{\widetilde{\omega}}$ to the lifting of point $\boldsymbol{x}$ in (1), obtaining $\frac{4f}{r-z}(x, y, z, 4fr)^T$, where $r = \sqrt{x^2 + y^2 + z^2}$. This is a point collinear with $\boldsymbol{O}$ (the origin) and $\boldsymbol{X} = (x, y, z, w)$. Hence for some λ and μ, $\lambda \boldsymbol{O} + \mu \boldsymbol{K}_{\widetilde{\omega}} \widetilde{\boldsymbol{x}} = \boldsymbol{X}$. Because one of the four equations in this vector equation are redundant we can multiply on both sides by

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

from which we find that $\mu \boldsymbol{P K}_{\widetilde{\omega}} \widetilde{\boldsymbol{x}} = \boldsymbol{P X}$.. Upon performing the multiplication on the left hand side, one finds that in fact

$$\boldsymbol{P K}_{\widetilde{\omega}} \widetilde{\boldsymbol{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{c_x}{2f} & -\frac{c_y}{2f} & \frac{1}{2f} \end{pmatrix} (\boldsymbol{P} \widetilde{\boldsymbol{x}} - \boldsymbol{P} \widetilde{\boldsymbol{\omega}}) = \boldsymbol{J}_{\widetilde{\omega}} (\boldsymbol{P} \widetilde{\boldsymbol{x}} - \boldsymbol{P} \widetilde{\boldsymbol{\omega}}) ,$$

but this is satisfied only under the condition that $\widetilde{\boldsymbol{x}}$ and $\widetilde{\boldsymbol{\omega}}$ have not been arbitrarily scaled from their respective definitions in (3) and (6).

Now assume that $\boldsymbol{X}$ lies in a coordinate system translated by $\boldsymbol{t}$ and rotated by $\boldsymbol{R}$. Introduce a projection matrix $\boldsymbol{\Pi} = \boldsymbol{J}_{\widetilde{\omega}}^{-1}(\boldsymbol{R}, \boldsymbol{t})$ similar to the standard perspective projection matrix and define $\check{\boldsymbol{x}} = P\widetilde{\boldsymbol{x}}$, then equation (3) becomes

$$\lambda(\check{\boldsymbol{x}} - \check{\boldsymbol{\omega}}) = \boldsymbol{\Pi X} . \tag{11}$$

The vector $\check{\boldsymbol{\omega}}$ can not be incorporated into the projection matrix $\boldsymbol{\Pi}$ because the subtraction is dependent on the non-homogeneity of $\check{\boldsymbol{x}}$. It is interesting to note that the matrix $\boldsymbol{J}_{\widetilde{\omega}}^{-1}$ which fills the role of a calibration matrix is lower triangular as opposed to the perspective calibration matrix which is upper triangular. With equation (11) we can now reformulate the multiple view matrix.

Assume that n parabolic catadioptric cameras image the same point $\boldsymbol{X} \in \mathbb{P}^3$ so that there are n equations of the form (11). This implies that each of the n matrices $(\boldsymbol{\Pi}_i, \check{\boldsymbol{x}}_i - \check{\boldsymbol{\omega}}_i)$ is rank deficient because within each nullspace must respectively lie the vector $(\boldsymbol{X}, \lambda_i)^T$. We can combine all of these matrices into the single matrix

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{\Pi}_1 & \breve{\boldsymbol{x}}_1 - \breve{\boldsymbol{\omega}}_1 & 0 & \cdots & 0 \\ \boldsymbol{\Pi}_2 & 0 & \breve{\boldsymbol{x}}_2 - \breve{\boldsymbol{\omega}}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Pi}_n & 0 & 0 & \cdots & \breve{\boldsymbol{x}}_n - \breve{\boldsymbol{\omega}}_n \end{pmatrix} \tag{12}$$

which again must be rank deficient because within its nullspace lies the vector $(\boldsymbol{X}, -\lambda_1, -\lambda_2, \ldots, -\lambda_n)^T$. In the perspective formulation $\boldsymbol{M}$ is known as the multiple view matrix [6]. By manipulating its columns and rows its rank deficiency has been used by [9] to show that the only independent constraints between multiple views are at most trilinear, all others are redundant. The same method can be applied to this parabolic catadioptric multiple view matrix to show that the only independent constraints among multiple parabolic catadioptric views are trilinear. In the next section we derive the bilinear constraints and find a form of the parabolic catadioptric fundamental matrix.

Notice that it is possible to mix different point features from different camera types. This only changes the form of one triplet of rows of the matrix $\boldsymbol{M}$. In each row the difference will be in the form of $\boldsymbol{\Pi}_i$, the presence or absence of an $\breve{\boldsymbol{\omega}}_i$ as well as lifting or not of $\boldsymbol{x}$. If all sensors image the same point in space, the multiple view matrix will be rank deficient regardless of the type of sensors.

3.1 Deriving the Catadioptric Fundamental Matrix

We now derive the constraint on two parabolic catadioptric views. For two views $\boldsymbol{M}$ becomes

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{\Pi}_1 & \breve{\boldsymbol{x}}_1 - \breve{\boldsymbol{\omega}}_1 & 0 \\ \boldsymbol{\Pi}_2 & 0 & \breve{\boldsymbol{x}}_2 - \breve{\boldsymbol{\omega}}_2 \end{pmatrix},$$

where we assume $\boldsymbol{\Pi}_1 = \boldsymbol{J}_{\widetilde{\omega}_1}^{-1}(\boldsymbol{I}, 0)$ and $\boldsymbol{\Pi}_2 = \boldsymbol{J}_{\widetilde{\omega}_2}^{-1}(\boldsymbol{R}, \boldsymbol{t})$. This a square matrix and its rank deficiency implies that its determinant is zero:

$$0 = \det \boldsymbol{M} = (\breve{\boldsymbol{x}}_1 - \breve{\boldsymbol{\omega}}_1)^T \boldsymbol{G} (\breve{\boldsymbol{x}}_2 - \breve{\boldsymbol{\omega}}_2) = \begin{pmatrix} \breve{\boldsymbol{x}}_1^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{G} & -\boldsymbol{G}\breve{\boldsymbol{\omega}}_2 \\ -\breve{\boldsymbol{\omega}}_1^T \boldsymbol{G} & \breve{\boldsymbol{\omega}}_1^T \boldsymbol{G} \breve{\boldsymbol{\omega}}_2 \end{pmatrix} \begin{pmatrix} \breve{\boldsymbol{x}}_2 \\ 1 \end{pmatrix}, \tag{13}$$

where we know from previous results for perspective cameras [6] that $\boldsymbol{G} = \boldsymbol{J}_{\widetilde{\omega}_1}^T \boldsymbol{E} \boldsymbol{J}_{\widetilde{\omega}_2}$ for the essential matrix $\boldsymbol{E} = [t]_\times \boldsymbol{R}$. Unfortunately expression (13) is a constraint on $\breve{\boldsymbol{x}}_1$ and $\breve{\boldsymbol{x}}_2$ and not $\widetilde{\boldsymbol{x}}_1$ and $\widetilde{\boldsymbol{x}}_2$, however note that

$$\breve{\boldsymbol{x}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \widetilde{\boldsymbol{x}} = \boldsymbol{H}\widetilde{\boldsymbol{x}}$$

Thus we can rewrite equation (13) as

$$\widetilde{\boldsymbol{x}}_1^T \boldsymbol{F} \widetilde{\boldsymbol{x}}_2 = 0 \quad \text{where} \quad \boldsymbol{F} = \boldsymbol{H}^T \begin{pmatrix} \boldsymbol{G} & -\boldsymbol{G}\breve{\boldsymbol{\omega}}_2 \\ -\breve{\boldsymbol{\omega}}_1^T \boldsymbol{G} & \breve{\boldsymbol{\omega}}_1^T \boldsymbol{G} \breve{\boldsymbol{\omega}}_2 \end{pmatrix} \boldsymbol{H}\,. \tag{14}$$

Equation (14) is the parabolic catadioptric epipolar constraint. It can be verified that

$$\boldsymbol{F} = \boldsymbol{K}_{\widetilde{\omega}_1}^T \begin{pmatrix} \boldsymbol{E} & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{K}_{\widetilde{\omega}_2} \,. \tag{15}$$

A matrix expressed in this way will be called a catadioptric fundamental matrix. The first thing to note about this new $\boldsymbol{F}$ is that since $H\widetilde{\boldsymbol{\omega}} = \breve{\boldsymbol{\omega}}$, we must have

$$\boldsymbol{F}\widetilde{\boldsymbol{\omega}}_2 = 0 \quad \text{and} \quad \boldsymbol{F}^T\widetilde{\boldsymbol{\omega}}_1 = 0 \,.$$

Hence, the lifted left and right images of the absolute conic belong to the left and right nullspace of $\boldsymbol{F}$, respectively. Also since $\boldsymbol{G}$ is rank 2, $\boldsymbol{F}$ will remain rank 2 because $\left(-\breve{\boldsymbol{\omega}}_1^T\boldsymbol{G}, \breve{\boldsymbol{\omega}}_1^T\boldsymbol{G}\breve{\boldsymbol{\omega}}_2\right)$ is linearly dependent on the first three rows.

Note that the expression $\widetilde{\boldsymbol{x}}_1^T\boldsymbol{F}\widetilde{\boldsymbol{x}}_2$ is linear in the entries of the matrix $\boldsymbol{F}$. Hence just like in the perspective case, given a set of correspondences a matrix whose entries are the coefficients in the epipolar equation of each entry of $\boldsymbol{F}$ can be constructed whose nullspace contains the matrix $\boldsymbol{F}$ flattened into a single vector in $\mathbb{R}^{16}$. The nullspace can be calculated using singular value decomposition by selecting the vector with the smallest singular value.

4 The Space of Catadioptric Fundamental Matrices

In the previous section we found that there is a bilinear constraint on the liftings of corresponding image points in the form of a 4×4 matrix analogous to the fundamental matrix for perspective cameras. It would be nice to find the necessary and sufficient conditions that a given matrix be a catadioptric fundamental matrix, that is, of the form (15). We will show that the condition that $\boldsymbol{F}$ be rank 2 is necessary but not sufficient.

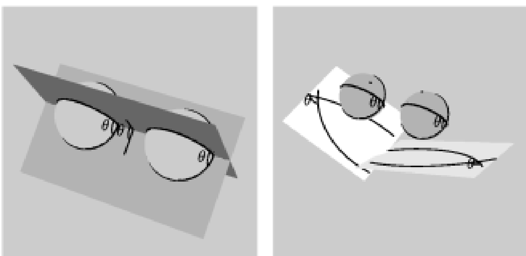

Fig. 3. Left: If two epipolar planes intersect two spheres representing two views at an angle θ, then the angle of intersection of the epipolar great circles is also θ. Right: By the angle preserving property of stereographic projection, the epipolar circles also must intersect at an angle θ.

The condition that we describe is based on the fact that F must preserve angles between epipolar circles. In Figure 3 (left) notice that two epipolar planes with a dihedral angle of θ intersect two spheres, representing two catadioptric views, in two pairs of great circles, both of which pairs have an angle of intersection of θ. Because stereographic projection preserves angles, the projections of the great circles, two pairs of line images must also intersect at an angle θ as shown in Figure 3 (right). The fundamental matrix is a rank 2 space correlation [13] meaning that it maps points to planes, polar planes actually, on which corresponding points must lie.

Assume that points $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively lie on epipolar circles $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ and satisfy $\widetilde{\boldsymbol{p}}^T \boldsymbol{F}\widetilde{\boldsymbol{q}} = 0$. The epipolar circle η can be determined from $\boldsymbol{F}$ since $\widetilde{\boldsymbol{\eta}} = \boldsymbol{Q}^{-1}\boldsymbol{F}^T\widetilde{\boldsymbol{p}}$. Can $\widetilde{\boldsymbol{\gamma}}$ also be determined from $\boldsymbol{F}$? If we knew $\widetilde{\boldsymbol{q}}$ we would have $\widetilde{\gamma} = \boldsymbol{Q}^{-1}\boldsymbol{F}\widetilde{\boldsymbol{q}}$. Let us assume we do not know $\boldsymbol{q}$. But clearly $\widetilde{\boldsymbol{\gamma}}$ is in the range of $\boldsymbol{Q}^{-1}\boldsymbol{F}$. So assume that $F = \lambda_1 \boldsymbol{u}_1\boldsymbol{v}_1^T + \lambda_2\boldsymbol{u}_2\boldsymbol{v}_2^T$ is the singular value decomposition of $\boldsymbol{F}$ which is rank 2. Thus $\widetilde{\boldsymbol{\gamma}} = \boldsymbol{Q}^{-1}(\alpha\boldsymbol{u}_1 + \beta\boldsymbol{u}_2)$ for some α and β. Since $\widetilde{\boldsymbol{\gamma}}^T\boldsymbol{Q}\widetilde{\boldsymbol{p}} = 0$, solutions unique up to scale are $\alpha = \widetilde{\boldsymbol{p}}^T\boldsymbol{u}_2$ and $\beta = -\widetilde{\boldsymbol{p}}^T\boldsymbol{u}_1$. Then $\widetilde{\gamma} = \boldsymbol{Q}^{-1}\boldsymbol{W}\widetilde{\boldsymbol{p}}$ where $\boldsymbol{W} = \boldsymbol{u}_1\boldsymbol{u}_2^T - \boldsymbol{u}_2\boldsymbol{u}_1^T$. In summary, corresponding epipolar circles as a function of the point $\boldsymbol{p}$ in one image are $\widetilde{\gamma} = \boldsymbol{Q}^{-1}\boldsymbol{W}\widetilde{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{\eta}} = \boldsymbol{Q}^{-1}\boldsymbol{F}^T\widetilde{\boldsymbol{p}}$. Note that these two definitions do not depend on any component in $\widetilde{\boldsymbol{p}}$ orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we may therefore rewrite them as

$$\widetilde{\gamma} = \boldsymbol{Q}^{-1}(\beta\boldsymbol{u}_1 - \alpha\boldsymbol{u}_2) \text{ and } \widetilde{\boldsymbol{\eta}} = \boldsymbol{Q}^{-1}(\lambda_1\alpha\boldsymbol{v}_1 + \lambda_2\beta\boldsymbol{v}_2)\,, \tag{16}$$

hence parameterizing all corresponding epipolar circles.

The sets $\{\widetilde{\gamma}\}$ and $\{\widetilde{\boldsymbol{\eta}}\}$ generated by all choices of α and β are two lines in circle space. They therefore represent coaxal circles, whose respective intersections have to be the epipoles. In order for the coaxal circles to have real intersection points the line in circle space ought not to intersect the sphere. For some coaxal system $\boldsymbol{a} + \lambda\boldsymbol{b}$ this is the case if and only if

$$\|\boldsymbol{a} + \lambda\boldsymbol{b}\|_Q^2 = (\boldsymbol{a} + \lambda\boldsymbol{b})^T\,\boldsymbol{Q}\,(\boldsymbol{a} + \lambda\boldsymbol{b}) > 0 \tag{17}$$

for all λ which is the case if and only if the discriminant of the left hand side as a polynomial in λ is negative. The discriminant being negative gives

$$\langle\boldsymbol{a}, \boldsymbol{b}\rangle_Q^2 < \|\boldsymbol{a}\|_Q^2\|\boldsymbol{b}\|_Q^2\,. \tag{18}$$

Lemma 3. If (18) is satisfied then for any two circles $\alpha_1\boldsymbol{a} + \beta_1\boldsymbol{b}$ and $\alpha_2\boldsymbol{a} + \beta_2\boldsymbol{b}$ in the coaxal space,

$$0 \le \frac{\langle\alpha_1\boldsymbol{a} + \beta_1\boldsymbol{b}, \alpha_2\boldsymbol{a} + \beta_2\boldsymbol{b}\rangle_Q^2}{\|\alpha_1\boldsymbol{a} + \beta_1\boldsymbol{b}\|_Q^2\|\alpha_2\boldsymbol{a} + \beta_2\boldsymbol{b}\|_Q^2} \le 1\,,$$

in which case the angle between them is well-defined.

Proof: From (17) and from the fact that $(\beta_1\alpha_2 - \alpha_1\beta_2)^2\left(\langle\boldsymbol{a}, \boldsymbol{b}\rangle_Q^2 - \|\boldsymbol{a}\|_Q^2\|\boldsymbol{b}\|_Q^2\right) = \langle\alpha_1\boldsymbol{a} + \beta_1\boldsymbol{b}, \alpha_2\boldsymbol{a} + \beta_2\boldsymbol{b}\rangle_Q^2 - \|\alpha_1\boldsymbol{a} + \beta_1\boldsymbol{b}\|_Q^2\|\alpha_2\boldsymbol{a} + \beta_2\boldsymbol{b}\|_Q^2$. □

Definition. When we say that a rank 2 space correlation $\boldsymbol{F}$ preserves epipolar angles (i.e. angles between epipolar circles) we mean that for all $\widetilde{\boldsymbol{p}}$,

$$\frac{\langle\boldsymbol{W}\widetilde{\boldsymbol{p}}_1, \boldsymbol{W}\widetilde{\boldsymbol{p}}_2\rangle_Q^2}{\|\boldsymbol{W}\widetilde{\boldsymbol{p}}_1\|_Q^2\|\boldsymbol{W}\widetilde{\boldsymbol{p}}_2\|_Q^2} = \frac{\langle\boldsymbol{F}^T\widetilde{\boldsymbol{p}}_1, \boldsymbol{F}^T\widetilde{\boldsymbol{p}}_2\rangle_Q^2}{\|\boldsymbol{F}^T\widetilde{\boldsymbol{p}}_1\|_Q^2\|\boldsymbol{F}^T\widetilde{\boldsymbol{p}}_2\|_Q^2}\,. \tag{19}$$

Equation (19) is obtained by substituting definitions of $\widetilde{\gamma}_i$ and $\widetilde{\boldsymbol{\eta}}_i$ from (16) into (7) while noticing that the $\boldsymbol{Q}^{-1}$'s cancel. This definition skirts the issue of whether this formula actually implies angles are preserved, but if angles are preserved then this formula must be true. Whether the converse is true turns out to be irrelevant.

Proposition 3. If (18) is satisfied by left and right singular vectors of a rank 2 space correlation $\boldsymbol{F}$ having SVD $\lambda_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T$ then the following statement is true. $\boldsymbol{F}$ preserves angles between epipolar circles if and only if

$$\begin{aligned} &\Big(\|\boldsymbol{u}_1\|_Q^2 \|\boldsymbol{u}_2\|_Q^2 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q^2 = \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q^2 \|\boldsymbol{v}_1\|_Q^2 \|\boldsymbol{v}_2\|_Q^2 \\ &\text{and} \quad \lambda_1 \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q \|\boldsymbol{v}_1\|_Q^2 = -\lambda_2 \|\boldsymbol{u}_2\|_Q^2 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q \Big) \\ &\quad \textbf{or} \Big(\lambda_1^2 \|\boldsymbol{u}_1\|_Q^2 \|\boldsymbol{v}_1\|_Q^2 = \lambda_2^2 \|\boldsymbol{u}_2\|_Q^2 \|\boldsymbol{v}_2\|_Q^2,\ \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q = 0 \,\text{and}\, \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q = 0 \Big) \end{aligned} \quad (20)$$

Proof: See appendix for $(\longrightarrow)$.

Corollary 2. A matrix $\boldsymbol{E}$ is an essential matrix if and only if the matrix $\boldsymbol{E}^{(4)}$ satisfies (20), where we define $\boldsymbol{E}^{(4)} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$.

Proof: A matrix $\boldsymbol{E}^{(4)}$ has SVD $\lambda_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T$ where $\boldsymbol{u}_i, \boldsymbol{v}_i \in \boldsymbol{\pi}_\infty$. Because they lie on $\boldsymbol{\pi}_\infty$, the dot product reduces to the Euclidean dot product and therefore (18) is just the Schwartz inequality satisfied by any vectors, and also by the properties of the SVD, $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q = 0$, $\|\boldsymbol{u}_i\|_Q = \|\boldsymbol{v}_i\|_Q = 1$.

If $\boldsymbol{E}$ is an essential matrix then $\lambda_1 = \lambda_2$ and then the second clause of (20) is satisfied. Therefore $\boldsymbol{E}^{(4)}$ is angle preserving.

If $\boldsymbol{E}^{(4)}$ is angle preserving then since $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q = 0$, the second clause applies and $\lambda_1^2 = \lambda_2^2$, thus $\boldsymbol{E}$ is an essential matrix. □

Lemma 4. If a rank 2 space correlation $\boldsymbol{F}$ preserves angles and the transformation $\boldsymbol{K} \in \mathcal{L}$ then $\boldsymbol{F}\boldsymbol{K}$ preserves angles between epipolar circles. Similarly for $\boldsymbol{K}^T \boldsymbol{F}$.

Proof: $\langle \boldsymbol{K}^T \boldsymbol{F}^T \widetilde{\boldsymbol{p}}_1, \boldsymbol{K}^T \boldsymbol{F}^T \widetilde{\boldsymbol{p}}_2 \rangle_Q = \langle \boldsymbol{F}^T \widetilde{\boldsymbol{p}}_1, \boldsymbol{F}^T \widetilde{\boldsymbol{q}}_2 \rangle_Q$ since $\boldsymbol{K}\boldsymbol{Q}\boldsymbol{K}^T = \lambda \boldsymbol{Q}$. For the other notice by relabeling the SVD, Proposition 3 implies that if $\boldsymbol{F}$ is angle preserving then $\boldsymbol{F}^T$ is too. □

Lemma 5. If $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy (18) then the nullspace of $(\boldsymbol{a}^T, \boldsymbol{b}^T)$ intersects $\boldsymbol{Q}$.

Proof: $\boldsymbol{Q}^{-1}\boldsymbol{a}$ and $\boldsymbol{Q}^{-1}\boldsymbol{b}$ also satisfy (18) and their span is a line not intersecting the sphere. Let $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ be two lines through the span and tangent to the sphere at points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Both $\boldsymbol{p}_i$ are orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$ because they lie on the polar planes of $\boldsymbol{Q}^{-1}\boldsymbol{a}$ and $\boldsymbol{Q}^{-1}\boldsymbol{b}$ and therefore satisfy $\boldsymbol{p}_i^T \boldsymbol{Q}\boldsymbol{Q}^{-1}\boldsymbol{a} = \boldsymbol{p}_i^T \boldsymbol{Q}\boldsymbol{Q}^{-1}\boldsymbol{b} = 0$. The $\boldsymbol{p}_i$'s are therefore a basis of the nullspace which obviously intersects the sphere. □

Theorem. A rank 2 space correlation $\boldsymbol{F}$ can be decomposed as $\boldsymbol{K}_1^T \boldsymbol{E}^{(4)} \boldsymbol{K}_2$ where $\boldsymbol{K}_i \in \mathcal{L}_N$ and $\boldsymbol{E}$ is an essential matrix if and only if (20) and (18) are satisfied by the vectors of its singular value decomposition.

Proof: Assume $\boldsymbol{F}$ is a rank 2 space correlation for which there exists $\boldsymbol{K}_{i=1,2} \in \mathcal{L}_N$ and an essential matrix $\boldsymbol{E}$ such that $\boldsymbol{F} = \boldsymbol{K}_1^T \boldsymbol{E}^{(4)} \boldsymbol{K}_2$. First, in Corollary 2 we saw that $\boldsymbol{E}^{(4)}$'s singular vectors satisfy (18), since $\boldsymbol{K}_{i=1,2} \in \mathcal{L}$, the inequality is preserved by the pre- and post-multiplication of these matrices, implying (18) is satisfied by $\boldsymbol{F}$ as well (even though the singular vectors change the spans are equal). By Corollary 2, $\boldsymbol{E}^{(4)}$ preserves angles between epipolar circles, and therefore by Lemma 4, $\boldsymbol{E}\boldsymbol{K}_2$ and then $\boldsymbol{K}_1^T\boldsymbol{E}^{(4)}\boldsymbol{K}_2$ also preserve angles between epipolar circles. By Proposition 3, $\boldsymbol{K}_1^T\boldsymbol{E}^{(4)}\boldsymbol{K}_2$, a rank 2 space correlation preserving epipolar angles and satisfying (18) must satisfy condition (20).

Now assume that $\boldsymbol{F}$ is an angle preserving, rank 2 space correlation satisfying (18), show that it is decomposable. Since it is rank 2 and satisfies (17), by Lemma 5 there is some $\widetilde{\boldsymbol{\omega}}_1$ inside the sphere such that $\boldsymbol{F}^T\widetilde{\boldsymbol{\omega}} = 0$ and some $\widetilde{\boldsymbol{\omega}}_2$ inside the sphere such that $\boldsymbol{F}\widetilde{\boldsymbol{\omega}} = 0$. If we calculate $\boldsymbol{K}_{\widetilde{\omega}_1}^{-1}$ we find that for some $\boldsymbol{a}, \boldsymbol{b}$ that $\boldsymbol{K}_{\widetilde{\omega}_1}^{-1} = \left(\boldsymbol{a}, \boldsymbol{b}, \alpha\widetilde{\boldsymbol{\omega}}', \beta\widetilde{\boldsymbol{\omega}}\right)$. The important point is that if the singular value decomposition of $\boldsymbol{F} = \lambda_1\boldsymbol{u}_1\boldsymbol{v}_1^T + \lambda_2\boldsymbol{u}_2\boldsymbol{v}_2^T$, then because $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are orthogonal to $\widetilde{\boldsymbol{\omega}}_1$ which is the last column of $\boldsymbol{K}_{\widetilde{\omega}_1}$, $\boldsymbol{K}_{\widetilde{\omega}_1}^{-T}\boldsymbol{u}_i \in \boldsymbol{\pi}_\infty$. $\boldsymbol{K}_{\widetilde{\omega}_2}$ has the same effect on $\boldsymbol{v}_i$. Therefore

$$\boldsymbol{K}_{\widetilde{\omega}_1}^{-T}\boldsymbol{F}\boldsymbol{K}_{\widetilde{\omega}_2}^{-1} = \begin{pmatrix} \boldsymbol{E} & 0 \\ 0 & 0 \end{pmatrix} .$$

We now show that $\boldsymbol{E}$ is an essential matrix. Since $\boldsymbol{F}$ preserves angles between epipolar circles, so does $\boldsymbol{K}_{\widetilde{\omega}_1}^{-T}\boldsymbol{F}\boldsymbol{K}_{\widetilde{\omega}_2}^{-1}$. Since it preserves angles, by Corollary 2, it must be an essential matrix with equal non-null singular values. Thus $\boldsymbol{F} = \boldsymbol{K}_{\widetilde{\omega}_1}^T\boldsymbol{E}\boldsymbol{K}_{\widetilde{\omega}_2}$ for some $\boldsymbol{K}_{\widetilde{\omega}_i} \in \mathcal{L}_N$ and some essential matrix $\boldsymbol{E}$. □

5 Conclusion

In this paper we introduced the spherical circle space to describe points and line images in parabolic catadioptric views. We described the class of linear transformations in that space which turned out to be the Lorentz group. We derived the catadioptric fundamental matrix and proved that the lifted image of the absolute conic belongs to its nullspace. Based on the fact that angles between epipolar circles are preserved we proved necessary and sufficient conditions for a matrix to be a catadioptric fundamental matrix.

Appendix (Proof of Proposition 3)

($\Longrightarrow$) Since (18) is true, (19) is well-defined for all $\widetilde{\boldsymbol{p}}_1$ and $\widetilde{\boldsymbol{p}}_2$. It is therefore true when $\widetilde{\boldsymbol{p}}_1 = \boldsymbol{u}_1$ and $\widetilde{\boldsymbol{p}}_2 = \alpha\boldsymbol{u}_1 + \boldsymbol{u}_2$. Substitute these definitions into (19) and cross-multiply the denominators. If the both sides are equal for all α then for all α the polynomial

$$\begin{aligned} f(\alpha) \equiv \langle -\boldsymbol{u}_2, \boldsymbol{u}_1 - \alpha\boldsymbol{u}_2\rangle_Q^2 \|\alpha\lambda_1\boldsymbol{v}_1\|_Q^2 \|\alpha\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2\|_Q^2 - \\ \langle\lambda_1\boldsymbol{v}_1, \alpha\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2\rangle_Q^2 \| - \boldsymbol{u}_2\|_Q^2 \|\boldsymbol{u}_1 - \alpha\boldsymbol{u}_2\|_Q^2 = 0 . \end{aligned} \tag{21}$$

In order that this polynomial be zero everywhere all its coefficients must be zero. Then the coefficients of α^0, α^1, and α^2 generate the three equations below which have been divided by $\lambda_1^i\lambda_2^j$ where appropriate ($\lambda_i > 0$ by assumption):

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle_Q \langle \boldsymbol{u}_2, \boldsymbol{u}_2 \rangle_Q \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q^2 = \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q^2 \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle_Q \langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle_Q \,, \tag{22}$$

$$\begin{aligned} &\lambda_1 \|\boldsymbol{v}_1\|_Q^2 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q \left(\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q^2 - \|\boldsymbol{u}_1\|_Q^2 \|\boldsymbol{u}_2\|_Q^2 \right) \\ &\quad = -\lambda_2 \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q \|\boldsymbol{u}_2\|_Q^2 \left(\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q^2 - \|\boldsymbol{v}_1\|_Q^2 \|\boldsymbol{v}_2\|_Q^2 \right) \,, \end{aligned} \tag{23}$$

$$\begin{aligned} &\lambda_1^2 \|\boldsymbol{v}_1\|_Q^4 \left(\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q^2 - \|\boldsymbol{u}_1\|_Q^2 \|\boldsymbol{u}_2\|_Q^2 \right) \\ &\quad = -\lambda_2^2 \|\boldsymbol{u}_2\|_Q^4 \left(\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q^2 - \|\boldsymbol{v}_1\|_Q^2 \|\boldsymbol{v}_2\|_Q^2 \right) \,. \end{aligned} \tag{24}$$

Condition (18) and hence (17) implies that $\|\boldsymbol{u}_i\|_Q^2 > 0$ and $\|\boldsymbol{v}_i\|_Q^2 > 0$. Thus if neither $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q = 0$ nor $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q = 0$, then we can solve for $\|\boldsymbol{v}_1\|_Q^2$ in equation (22), and substitute into (23) and (24); then both reduce to $\lambda_1 \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q \|\boldsymbol{v}_1\|_Q^2 = -\lambda_2 \|\boldsymbol{u}_2\|_Q^2 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q$. This satisfies the first clause of (20).

Otherwise if $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q = 0$, then (22) implies that $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q = 0$; the converse is true as well by (22). Substituting $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle_Q = 0$ and $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle_Q = 0$ into equation (23) gives no constraint and into equation (24) yields $\lambda_1^2 \|\boldsymbol{u}_1\|_Q^2 \|\boldsymbol{v}_1\|_Q^2 = \lambda_2^2 \|\boldsymbol{u}_2\|_Q^2 \|\boldsymbol{v}_2\|_Q^2$. Then the second clause of (20) is satisfied. Therefore if $\boldsymbol{F}$ preserves the angles between epipolar circles according to the definition given above and has left and right singular vectors satisfying (18), then one of the conditions in (20) is true.

□

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