the property that it maximizes the minimum angle of the triangles involved in any triangulation of P. However, this does not characterize the Delaunay triangulation. Given a connected graph in the plane, it can also be shown that any minimal spanning tree is contained in the Delaunay triangulation of the convex hull of the set of vertices of the graph (O'Rourke [132]).

We will now explore briefly the connection between Delaunay triangulations and convex hulls.

9.4 Delaunay Triangulations and Convex Hulls

In this section we show that there is an intimate relationship between convex hulls and Delaunay triangulations. We will see that given a set P of points in the Euclidean space $\mathbb{E}^m$ of dimension m, we can "lift" these points onto a paraboloid living in the space $\mathbb{E}^{m+1}$ of dimension $m+1$, and that the Delaunay triangulation of P is the projection of the downward-facing faces of the convex hull of the set of lifted points. This remarkable connection was first discovered by Brown [23], and refined by Edelsbrunner and Seidel [55]. For simplicity, we consider the case of a set P of points in the plane $\mathbb{E}^2$, and we assume that they are in general position.

Consider the paraboloid of revolution of equation $z = x^2 + y^2$. A point $p = (x, y)$ in the plane is lifted to the point $l(p) = (X, Y, Z)$ in $\mathbb{E}^3$, where $X = x$, $Y = y$, and $Z = x^2 + y^2$.

The first crucial observation is that a circle in the plane is lifted into a plane curve (an ellipse). Indeed, if such a circle C is defined by the equation

$$x^2 + y^2 + ax + by + c = 0,$$

since $X = x$, $Y = y$, and $Z = x^2 + y^2$, by eliminating $x^2 + y^2$ we get

$$Z = -ax - by - c,$$

and thus X, Y, Z satisfy the linear equation

$$aX + bY + Z + c = 0,$$

which is the equation of a plane. Thus, the intersection of the cylinder of revolution consisting of the lines parallel to the z-axis and passing through a point of the circle C with the paraboloid $z = x^2 + y^2$ is a planar curve (an ellipse).

We can compute the convex hull of the set of lifted points. Let us focus on the downward-facing faces of this convex hull. Let $(l(p_1), l(p_2), l(p_3))$ be such a face. The points p_1, p_2, p_3 belong to the set P. We claim that no other point from P is inside the circle C. Indeed, a point p inside the circle C would lift to a point $l(p)$ on the paraboloid. Since no four points are cocyclic, one of the four points p_1, p_2, p_3, p is further from O than the others; say this point is p_3. Then, the face $(l(p_1), l(p_2), l(p))$ would be below the face $(l(p_1), l(p_2), l(p_3))$, contradicting the fact that $(l(p_1), l(p_2), l(p_3))$

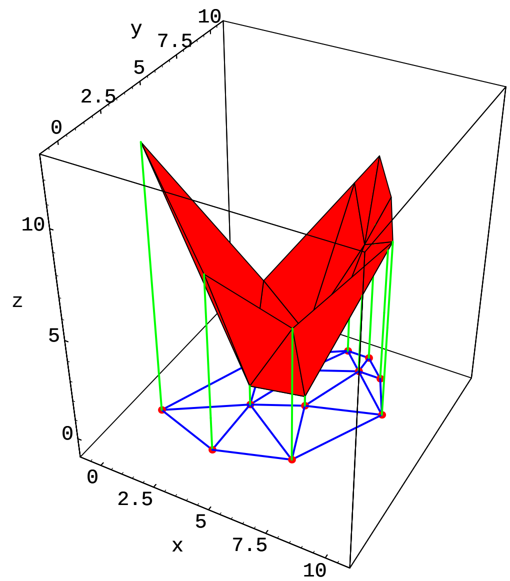

Figure 9.8. A Delaunay triangulation and its lifting to a paraboloid

is one of the downward-facing faces of the convex hull of P. But then, by property (2) of Lemma 9.3.2, the triangle (p_1, p_2, p_3) would belong to the Delaunay triangulation of P.

Therefore, we have shown that *the projection of the part of the convex hull of the lifted set $l(P)$ consisting of the downward-facing faces is the Delaunay triangulation of P*. Figure 9.8 shows the lifting of the Delaunay triangulation shown earlier.

Another example of the lifting of a Delaunay triangulation is shown in Figure 9.9. The fact that a Delaunay triangulation can be obtained by projecting a lower convex hull can be used to find efficient algorithms for computing a Delaunay triangulation. It also holds for higher dimensions.

The Voronoi diagram itself can also be obtained from the lifted set $l(P)$. However, this time, we need to consider tangent planes to the paraboloid at the lifted points. It is fairly obvious that the tangent plane at the lifted point $(a, b, a^2 + b^2)$ is

$$z = 2ax + 2by - (a^2 + b^2).$$

Given two distinct lifted points $(a_1, b_1, a_1^2 + b_1^2)$ and $(a_2, b_2, a_2^2 + b_2^2)$, the intersection of the tangent planes at these points is a line belonging to the plane of equation

$$(b_1 - a_1)x + (b_2 - a_2)y = (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2.$$

This is precisely the equation of the bisector line of the two points (a_1, b_1) and (a_2, b_2). Therefore, *if we look at the paraboloid from $z = +\infty$ (with the*