Notes on Convex Sets, Polytopes, Polyhedra, Combinatorial Topology, Voronoi Diagrams and Delaunay Triangulations

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Abstract: Some basic mathematical tools such as convex sets, polytopes and combinatorial topology, are used quite heavily in applied fields such as geometric modeling, meshing, computer vision, medical imaging and robotics. This report may be viewed as a tutorial and a set of notes on convex sets, polytopes, polyhedra, combinatorial topology, Voronoi Diagrams and Delaunay Triangulations. It is intended for a broad audience of mathematically inclined readers.

One of my (selfish!) motivations in writing these notes was to understand the concept of *shelling* and how it is used to prove the famous Euler-Poincaré formula (Poincaré, 1899) and the more recent *Upper Bound Theorem* (McMullen, 1970) for polytopes. Another of my motivations was to give a "correct" account of Delaunay triangulations and Voronoi diagrams in terms of (direct and inverse) stereographic projections onto a sphere and prove rigorously that the projective map that sends the (projective) sphere to the (projective) paraboloid works correctly, that is, maps the Delaunay triangulation and Voronoi diagram w.r.t. the lifting onto the sphere to the Delaunay diagram and Voronoi diagrams w.r.t. the traditional lifting onto the paraboloid. Here, the problem is that this map is only well defined (total) in projective space and we are forced to define the notion of convex polyhedron in projective space.

It turns out that in order to achieve (even partially) the above goals, I found that it was necessary to include quite a bit of background material on convex sets, polytopes, polyhedra and projective spaces. I have included a rather thorough treatment of the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes and also of the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra, which is a bit harder. In particular, the *Fourier-Motzkin elimination* method (a version of Gaussian elimination for inequalities) is discussed in some detail. I also had to include some material on projective spaces, projective maps and polar duality w.r.t. a nondegenerate quadric in order to define a suitable notion of "projective polyhedron" based on cones. To the best of our knowledge, this notion of projective polyhedron is new. We also believe that some of our proofs establishing the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra are new.

Key-words: Convex sets, polytopes, polyhedra, shellings, combinatorial topology, Voronoi diagrams, Delaunay triangulations.

Contents

Chapter 1

Introduction

1.1 Motivations and Goals

For the past eight years or so I have been teaching a graduate course whose main goal is to expose students to some fundamental concepts of geometry, keeping in mind their applications to geometric modeling, meshing, computer vision, medical imaging, robotics, *etc.* The audience has been primarily computer science students but a fair number of mathematics students and also students from other engineering disciplines (such as Electrical, Systems, Mechanical and Bioengineering) have been attending my classes. In the past three years, I have been focusing more on convexity, polytopes and combinatorial topology, as concepts and tools from these areas have been used increasingly in meshing and also in computational biology and medical imaging. One of my (selfish!) motivations was to understand the concept of *shelling* and how it is used to prove the famous Euler-Poincaré formula (Poincaré, 1899) and the more recent *Upper Bound Theorem* (McMullen, 1970) for polytopes. Another of my motivations was to give a "correct" account of Delaunay triangulations and Voronoi diagrams in terms of (direct and inverse) stereographic projections onto a sphere and prove rigorously that the projective map that sends the (projective) sphere to the (projective) paraboloid works correctly, that is, maps the Delaunay triangulation and Voronoi diagram w.r.t. the lifting onto the sphere to the Delaunay triangulation and Voronoi diagram w.r.t. the lifting onto the paraboloid. Moreover, the projections of these polyhedra onto the hyperplane $x_{d+1} = 0$, from the sphere or from the paraboloid, are identical. Here, the problem is that this map is only well defined (total) in projective space and we are forced to define the notion of convex polyhedron in projective space.

It turns out that in order to achieve (even partially) the above goals, I found that it was necessary to include quite a bit of background material on convex sets, polytopes, polyhedra and projective spaces. I have included a rather thorough treatment of the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes and also of the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra, which is a bit harder. In particular, the *Fourier-Motzkin elimination* method (a version of Gaussian elimination for inequalities) is discussed in some detail. I also had to include some material on projective spaces, projective maps and polar duality w.r.t. a nondegenerate

quadric, in order to define a suitable notion of "projective polyhedron" based on cones. This notion turned out to be indispensible to give a correct treatment of the Delaunay and Voronoi complexes using inverse stereographic projection onto a sphere and to prove rigorously that the well known projective map between the sphere and the paraboloid maps the Delaunay triangulation and the Voronoi diagram w.r.t. the sphere to the more traditional Delaunay triangulation and Voronoi diagram w.r.t. the paraboloid. To the best of our knowledge, this notion of projective polyhedron is new. We also believe that some of our proofs establishing the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra are new.

Chapter 6 on combinatorial topology is hardly original. However, most texts covering this material are either old fashion or too advanced. Yet, this material is used extensively in meshing and geometric modeling. We tried to give a rather intuitive yet rigorous exposition. We decided to introduce the terminology *combinatorial manifold*, a notion usually referred to as *triangulated manifold*.

A recurring theme in these notes is the process of "conification" (algebraically, "homogenization"), that is, forming a cone from some geometric object. Indeed, "conification" turns an object into a set of lines, and since lines play the role of points in projective geometry, "conification" ("homogenization") is the way to "projectivize" geometric affine objects. Then, these (affine) objects appear as "conic sections" of cones by hyperplanes, just the way the classical conics (ellipse, hyperbola, parabola) appear as conic sections.

It is worth warning our readers that convexity and polytope theory is deceptively simple. This is a subject where most intuitive propositions fail as soon as the dimension of the space is greater than 3 (definitely 4), because our human intuition is not very good in dimension greater than 3. Furthermore, rigorous proofs of seemingly very simple facts are often quite complicated and may require sophisticated tools (for example, shellings, for a correct proof of the Euler-Poincaré formula). Nevertheless, readers are urged to strenghten their geometric intuition; they should just be very vigilant! This is another case where Tate's famous saying is more than pertinent: "Reason geometrically, prove algebraically."

At first, these notes were meant as a complement to Chapter 3 (Properties of Convex Sets: A Glimpse) of my book (*Geometric Methods and Applications*, [20]). However, they turn out to cover much more material. For the reader's convenience, I have included Chapter 3 of my book as part of Chapter 2 of these notes. I also assume some familiarity with affine geometry. The reader may wish to review the basics of affine geometry. These can be found in any standard geometry text (Chapter 2 of Gallier [20] covers more than needed for these notes).

Most of the material on convex sets is taken from Berger [6] (*Geometry II*). Other relevant sources include Ziegler [45], Grünbaum [24] Valentine [43], Barvinok [3], Rockafellar [34], Bourbaki (Topological Vector Spaces) [9] and Lax [26], the last four dealing with affine spaces of infinite dimension. As to polytopes and polyhedra, "the" classic reference is Grünbaum [24]. Other good references include Ziegler [45], Ewald [18], Cromwell [14] and Thomas [40].

The recent book by Thomas contains an excellent and easy going presentation of poly-

tope theory. This book also gives an introduction to the theory of triangulations of point configurations, including the definition of secondary polytopes and state polytopes, which happen to play a role in certain areas of biology. For this, a quick but very efficient presentation of Gröbner bases is provided. We highly recommend Thomas's book [40] as further reading. It is also an excellent preparation for the more advanced book by Sturmfels [39]. However, in our opinion, the "bible" on polytope theory is without any contest, Ziegler [45], a masterly and beautiful piece of mathematics. In fact, our Chapter 7 is heavily inspired by Chapter 8 of Ziegler. However, the pace of Ziegler's book is quite brisk and we hope that our more pedestrian account will inspire readers to go back and read the masters.

In a not too distant future, I would like to write about constrained Delaunay triangulations, a formidable topic, please be patient!

Chapter 2

Basic Properties of Convex Sets

2.1   Convex Sets

Convex sets play a very important role in geometry. In this chapter we state and prove some of the "classics" of convex affine geometry: Carathéodory's theorem, Radon's theorem, and Helly's theorem. These theorems share the property that they are easy to state, but they are deep, and their proof, although rather short, requires a lot of creativity.

Given an affine space E, recall that a subset V of E is *convex* if for any two points $a, b \in V$, we have $c \in V$ for every point $c = (1 - \lambda)a + \lambda b$, with $0 \leq \lambda \leq 1$ ($\lambda \in \mathbb{R}$). Given any two points a, b, the notation $[a, b]$ is often used to denote the line segment between a and b, that is,

$$[a, b] = \{c \in E \mid c = (1 - \lambda)a + \lambda b, \, 0 \leq \lambda \leq 1\},$$

and thus a set V is convex if $[a, b] \subseteq V$ for any two points $a, b \in V$ ($a = b$ is allowed). The empty set is trivially convex, every one-point set $\{a\}$ is convex, and the entire affine space E is, of course, convex.

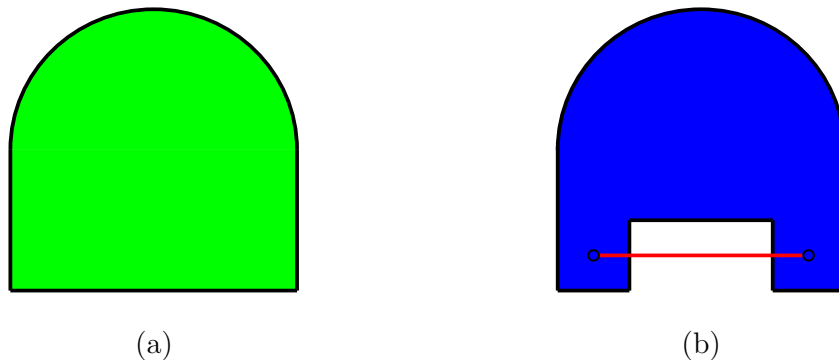(a)   (b)

Figure 2.1: (a) A convex set; (b) A nonconvex set

It is obvious that the intersection of any family (finite or infinite) of convex sets is convex. Then, given any (nonempty) subset S of E, there is a smallest convex set containing S denoted by conv(S) or $\mathcal{C}(S)$, and called the *convex hull of S* (namely, the intersection of all convex sets containing S). The *affine hull* of a subset S of E is the smallest affine set containing S and it will be denoted by $\langle S \rangle$ or aff(S).

Definition 2.1. Given any affine space E, the *dimension* of a nonempty convex subset S of E, denoted by dim S, is the dimension of the smallest affine subset aff(S) containing S.

A good understanding of what conv(S) is, and good methods for computing it, are essential. First we have the following simple but crucial lemma:

Lemma 2.1. *Given an affine space $\langle E, \overrightarrow{E}, + \rangle$, for any family $(a_i)_{i \in I}$ of points in E, the set V of convex combinations $\sum_{i \in I} \lambda_i a_i$ (where $\sum_{i \in I} \lambda_i = 1$ and $\lambda_i \geq 0$) is the convex hull of $(a_i)_{i \in I}$.*

Proof. If $(a_i)_{i \in I}$ is empty, then $V = \emptyset$, because of the condition $\sum_{i \in I} \lambda_i = 1$. As in the case of affine combinations, it is easily shown by induction that any convex combination can be obtained by computing convex combinations of two points at a time. As a consequence, if $(a_i)_{i \in I}$ is nonempty, then the smallest convex subspace containing $(a_i)_{i \in I}$ must contain the set V of all convex combinations $\sum_{i \in I} \lambda_i a_i$. Thus, it is enough to show that V is closed under convex combinations, which is immediately verified. $\square$

In view of Lemma 2.1, it is obvious that any affine subspace of E is convex.

Convex sets also arise in terms of hyperplanes. Given a hyperplane H, if $f \colon E \to \mathbb{R}$ is any nonconstant affine form defining H (i.e., $H = \mathrm{Ker}\, f$, with $f(a + u) = f(a) + \overrightarrow{f}(u)$ for all $a \in E$ and all $u \in \overrightarrow{E}$, where $f(a) \in \mathbb{R}$ and $\overrightarrow{f} \colon \overrightarrow{E} \to \mathbb{R}$ is a nonzero linear form), we can define the two subsets

$$H_+(f) = \{a \in E \mid f(a) \geq 0\} \quad \text{and} \quad H_-(f) = \{a \in E \mid f(a) \leq 0\},$$

called *(closed) half-spaces associated with f*.

Observe that if $\lambda > 0$, then $H_+(\lambda f) = H_+(f)$, but if $\lambda < 0$, then $H_+(\lambda f) = H_-(f)$, and similarly for $H_-(\lambda f)$. However, the set

$$\{H_+(f), H_-(f)\}$$

depends only on the hyperplane H, and the choice of a specific f defining H amounts to the choice of one of the two half-spaces. For this reason, we will also say that $H_+(f)$ and $H_-(f)$ are the *closed half-spaces associated with H*. Clearly, $H_+(f) \cup H_-(f) = E$ and $H_+(f) \cap H_-(f) = H$. It is immediately verified that $H_+(f)$ and $H_-(f)$ are convex. Bounded convex sets arising as the intersection of a finite family of half-spaces associated
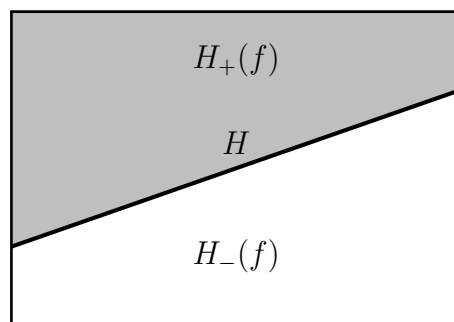
Figure 2.2: The two half-spaces determined by a hyperplane, H

with hyperplanes play a major role in convex geometry and topology (they are called *convex polytopes*).

The convex combinations $\sum_{i \in I} \lambda_i a_i$ arising in computing the convex hull of a family of points $(a_i)_{i \in I}$ have finite support, so they can be written as $\sum_{j \in J} \lambda_j a_j$ for some *finite* subset J of I, but a priori there is no bound on the size such finite sets J. Thus it is natural to wonder whether Lemma 2.1 can be sharpened in two directions:

(1) Is it possible to have a fixed bound on the number of points involved in the convex combinations $\sum_{j \in J} \lambda_j a_j$ (that is, on the size of the index sets J)?

(2) Is it necessary to consider convex combinations of all points, or is it possible to consider only a subset of points with special properties?

The answer is yes in both cases. In Case (1), assuming that the affine space E has dimension m, Carathéodory's theorem asserts that it is enough to consider convex combinations of $m + 1$ points. For example, in the plane $\mathbb{A}^2$, the convex hull of a set S of points is the union of all triangles (interior points included) with vertices in S. In Case (2), the theorem of Krein and Milman asserts that a convex set that is also compact is the convex hull of its extremal points (given a convex set S, a point $a \in S$ is extremal if $S - \{a\}$ is also convex, see Berger [6] or Lang [25]). Next, we prove Carathéodory's theorem.

2.2 Carathéodory's Theorem

The proof of Carathéodory's theorem is really beautiful. It proceeds by contradiction and uses a minimality argument.

Theorem 2.2. *(Carathéodory, 1907) Given any affine space E of dimension m, for any (nonvoid) family $S = (a_i)_{i \in L}$ in E, the convex hull $\mathrm{conv}(S)$ of S is equal to the set of convex combinations of families of $m + 1$ points of S.*

Proof. By Lemma 2.1,

$$\text{conv}(S) = \left\{ \sum_{i \in I} \lambda_i a_i \mid a_i \in S, \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, I \subseteq L, I \text{ finite} \right\}.$$

We would like to prove that

$$\text{conv}(S) = \left\{ \sum_{i \in I} \lambda_i a_i \mid a_i \in S, \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, I \subseteq L, |I| = m + 1 \right\}.$$

We proceed by contradiction. If the theorem is false, there is some point $b \in \text{conv}(S)$ such that b can be expressed as a convex combination $b = \sum_{i \in I} \lambda_i a_i$, where $I \subseteq L$ is a finite set of cardinality $|I| = q$ with $q \geq m + 2$, and b cannot be expressed as any convex combination $b = \sum_{j \in J} \mu_j a_j$ of strictly fewer than q points in S, that is, where $|J| < q$. Such a point $b \in \text{conv}(S)$ is a convex combination

$$b = \lambda_1 a_1 + \cdots + \lambda_q a_q,$$

where $\lambda_1 + \cdots + \lambda_q = 1$ and $\lambda_i > 0$ $(1 \leq i \leq q)$. We shall prove that b can be written as a convex combination of $q - 1$ of the a_i. Pick any origin O in E. Since there are $q > m + 1$ points $a_1, \ldots, a_q$, these points are affinely dependent, and by Lemma 2.6.5 from Gallier [20], there is a family $(\mu_1, \ldots, \mu_q)$ all scalars not all null, such that $\mu_1 + \cdots + \mu_q = 0$ and

$$\sum_{i=1}^{q} \mu_i \mathbf{Oa_i} = 0.$$

Consider the set $T \subseteq \mathbb{R}$ defined by

$$T = \{t \in \mathbb{R} \mid \lambda_i + t\mu_i \geq 0, \mu_i \neq 0, 1 \leq i \leq q\}.$$

The set T is nonempty, since it contains 0. Since $\sum_{i=1}^{q} \mu_i = 0$ and the μ_i are not all null, there are some μ_h, μ_k such that $\mu_h < 0$ and $\mu_k > 0$, which implies that $T = [\alpha, \beta]$, where

$$\alpha = \max_{1 \leq i \leq q} \{-\lambda_i / \mu_i \mid \mu_i > 0\} \quad \text{and} \quad \beta = \min_{1 \leq i \leq q} \{-\lambda_i / \mu_i \mid \mu_i < 0\}$$

(T is the intersection of the closed half-spaces $\{t \in \mathbb{R} \mid \lambda_i + t\mu_i \geq 0, \mu_i \neq 0\}$). Observe that $\alpha < 0 < \beta$, since $\lambda_i > 0$ for all $i = 1, \ldots, q$.

We claim that there is some j $(1 \leq j \leq q)$ such that

$$\lambda_j + \alpha \mu_j = 0.$$

Indeed, since

$$\alpha = \max_{1 \leq i \leq q} \{-\lambda_i / \mu_i \mid \mu_i > 0\},$$

as the set on the right hand side is finite, the maximum is achieved and there is some index j so that $\alpha = -\lambda_j/\mu_j$. If j is some index such that $\lambda_j + \alpha\mu_j = 0$, since $\sum_{i=1}^{q} \mu_i \mathbf{Oa_i} = 0$, we have

$$b = \sum_{i=1}^{q} \lambda_i a_i = O + \sum_{i=1}^{q} \lambda_i \mathbf{Oa_i} + 0,$$

$$= O + \sum_{i=1}^{q} \lambda_i \mathbf{Oa_i} + \alpha \left(\sum_{i=1}^{q} \mu_i \mathbf{Oa_i} \right),$$

$$= O + \sum_{i=1}^{q} (\lambda_i + \alpha\mu_i) \mathbf{Oa_i},$$

$$= \sum_{i=1}^{q} (\lambda_i + \alpha\mu_i) a_i,$$

$$= \sum_{i=1, i \neq j}^{q} (\lambda_i + \alpha\mu_i) a_i,$$

since $\lambda_j + \alpha\mu_j = 0$. Since $\sum_{i=1}^{q} \mu_i = 0$, $\sum_{i=1}^{q} \lambda_i = 1$, and $\lambda_j + \alpha\mu_j = 0$, we have

$$\sum_{i=1, i \neq j}^{q} \lambda_i + \alpha\mu_i = 1,$$

and since $\lambda_i + \alpha\mu_i \geq 0$ for $i = 1, \dots, q$, the above shows that b can be expressed as a convex combination of $q - 1$ points from S. However, this contradicts the assumption that b cannot be expressed as a convex combination of strictly fewer than q points from S, and the theorem is proved. $\qquad\square$

If S is a finite (of infinite) set of points in the affine plane $\mathbb{A}^2$, Theorem 2.2 confirms our intuition that $\mathrm{conv}(S)$ is the union of triangles (including interior points) whose vertices belong to S. Similarly, the convex hull of a set S of points in $\mathbb{A}^3$ is the union of tetrahedra (including interior points) whose vertices belong to S. We get the feeling that triangulations play a crucial role, which is of course true!

An interesting consequence of Carathéodory's theorem is the following result:

Proposition 2.3. *If K is any compact subset of $\mathbb{A}^m$, then the convex hull, $\mathrm{conv}(K)$, of K is also compact.*

Proposition 2.3 can be proved by showing that $\mathrm{conv}(K)$ is the image of some compact subset of $\mathbb{R}^{m+1} \times (\mathbb{A}^m)^{m+1}$ by some well chosen continuous map.

A closer examination of the proof of Theorem 2.2 reveals that the fact that the μ_i's add up to zero ensures that T is a closed interval, but all we need is that T be bounded from

below, and this only requires that some μ_j be strictly positive. As a consequence, we can prove a version of Theorem 2.2 for convex cones. This is a useful result since cones play such an important role in convex optimization. let us recall some basic definitions about cones.

Definition 2.2. Given any vector space E, a subset $C \subseteq E$ is a *convex cone* iff C is closed under *positive linear combinations*, that is, linear combinations of the form

$$\sum_{i \in I} \lambda_i v_i, \quad \text{with} \quad v_i \in C \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for all} \quad i \in I,$$

where I has finite support (all $\lambda_i = 0$ except for finitely many $i \in I$). Given any set of vectors S, the *positive hull* of S, or *cone* spanned by S, denoted cone(S), is the set of all positive linear combinations of vectors in S,

$$\text{cone}(S) = \left\{ \sum_{i \in I} \lambda_i v_i \mid v_i \in S, \ \lambda_i \geq 0 \right\}.$$

Note that a cone always contains 0. When S consists of a finite number of vector, the convex cone cone(S) is called a *polyhedral cone*. We have the following version of Carathéodory's theorem for convex cones:

Theorem 2.4. *Given any vector space E of dimension m, for any (nonvoid) family $S = (v_i)_{i \in L}$ of vectors in E, the cone cone(S) spanned by S is equal to the set of positive combinations of families of m vectors in S.*

The proof of Theorem 2.4 can be easily adapted from the proof of Theorem 2.2 and is left as an exercise.

There is an interesting generalization of Carathéodory's theorem known as the *Colorful Carathéodory theorem*. This theorem due to Bárány and proved in 1982 can be used to give a fairly short proof of a generalization of Helly's theorem known as Tverberg's theorem (see Section 2.4).

Theorem 2.5. *(Colorful Carathéodory theorem) Let E be any affine space of dimension m. For any point $b \in E$, for any sequence of $m + 1$ nonempty subsets $(S_1, \ldots, S_{m+1})$ of E, if $b \in \text{conv}(S_i)$ for $i = 1, \ldots, m+1$, then there exists a sequence of $m+1$ points $(a_1, \ldots, a_{m+1})$, with $a_i \in S_i$, so that $b \in \text{conv}(a_1, \ldots, a_{m+1})$, that is, b is a convex combination of the a_i's.*

Although Theorem 2.5 is not hard to prove, we will not prove it here. Instead, we refer the reader to Matousek [27], Chapter 8, Section 8.2. There is also a stronger version of Theorem 2.5, in which it is enough to assume that $b \in \text{conv}(S_i \cup S_j)$ for all i, j with $1 \leq i < j \leq m+1$.

Now that we have given an answer to the first question posed at the end of Section 2.1 we give an answer to the second question.

2.3 Vertices, Extremal Points and Krein and Milman's Theorem

First, we define the notions of separation and of separating hyperplanes. For this, recall the definition of the closed (or open) half–spaces determined by a hyperplane.

Given a hyperplane H, if $f \colon E \to \mathbb{R}$ is any nonconstant affine form defining H (i.e., $H = \mathrm{Ker}\, f$), we define the *closed half-spaces associated with f* by

$$H_+(f) = \{a \in E \mid f(a) \geq 0\},$$
$$H_-(f) = \{a \in E \mid f(a) \leq 0\}.$$

Observe that if $\lambda > 0$, then $H_+(\lambda f) = H_+(f)$, but if $\lambda < 0$, then $H_+(\lambda f) = H_-(f)$, and similarly for $H_-(\lambda f)$.

Thus, the set $\{H_+(f), H_-(f)\}$ depends only on the hyperplane H, and the choice of a specific f defining H amounts to the choice of one of the two half-spaces.

We also define the *open half–spaces associated with f* as the two sets

$$\overset{\circ}{H}_+(f) = \{a \in E \mid f(a) > 0\},$$
$$\overset{\circ}{H}_-(f) = \{a \in E \mid f(a) < 0\}.$$

The set $\{\overset{\circ}{H}_+(f), \overset{\circ}{H}_-(f)\}$ only depends on the hyperplane H. Clearly, we have $\overset{\circ}{H}_+(f) = H_+(f) - H$ and $\overset{\circ}{H}_-(f) = H_-(f) - H$.

Definition 2.3. Given an affine space E and two nonempty subsets A and B of E, we say that a hyperplane H *separates (resp. strictly separates) A and B* if A is in one and B is in the other of the two half–spaces (resp. open half–spaces) determined by H.

In Figure 2.3 (a), the two closed convex sets A and B are unbounded and B has the hyperplane H for its boundary, while A is asymptotic to H. The hyperplane H is a separating hyperplane for A and B but A and B can't be strictly separated. In Figure 2.3 (b), both A and B are convex and closed, B is unbounded and asymptotic to the hyperplane, H', but A is bounded. Both hyperplanes H and H' strictly separate A and B.

The special case of separation where A is convex and $B = \{a\}$, for some point, a, in A, is of particular importance.

Definition 2.4. Let E be an affine space and let A be any nonempty subset of E. A *supporting hyperplane of A* is any hyperplane H containing some point a of A, and separating $\{a\}$ and A. We say that H is a *supporting hyperplane of A at a*.
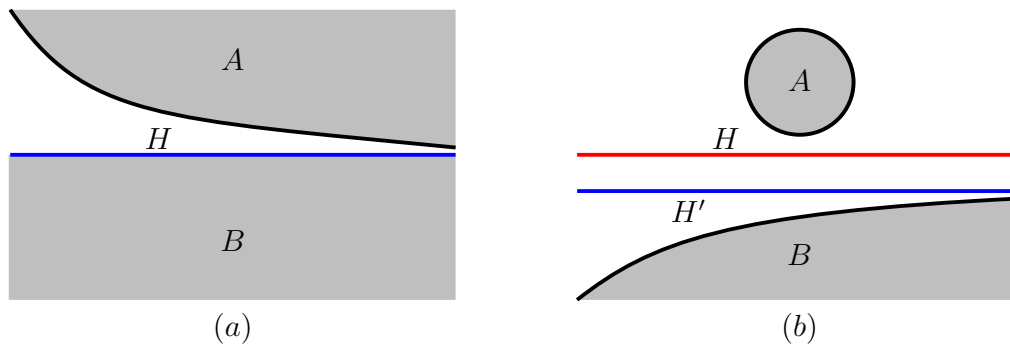
Figure 2.3: (a) A separating hyperplane H. (b) Strictly separating hyperplanes H and H'.
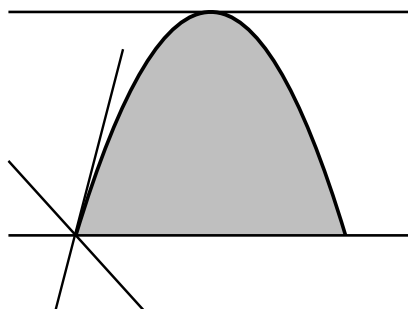
Figure 2.4: Examples of supporting hyperplanes

Observe that if H is a supporting hyperplane of A at a, then we must have $a \in \partial A$. Otherwise, there would be some open ball $B(a, \epsilon)$ of center a contained in A and so there would be points of A (in $B(a, \epsilon)$) in both half-spaces determined by H, contradicting the fact that H is a supporting hyperplane of A at a. Furthermore, $H \cap \overset{\circ}{A} = \emptyset$.

One should experiment with various pictures and realize that supporting hyperplanes at a point may not exist (for example, if A is not convex), may not be unique, and may have several distinct supporting points! (See Figure 2.4).

Next, we need to define various types of boundary points of closed convex sets.

Definition 2.5. Let E be an affine space of dimension d. For any nonempty closed and convex subset A of dimension d, a point $a \in \partial A$ has *order* $k(a)$ if the intersection of all the supporting hyperplanes of A at a is an affine subspace of dimension $k(a)$. We say that $a \in \partial A$ is a *vertex* if $k(a) = 0$; we say that a is *smooth* if $k(a) = d - 1$, i.e., if the supporting hyperplane at a is unique.

A vertex is a boundary point a such that there are d independent supporting hyperplanes
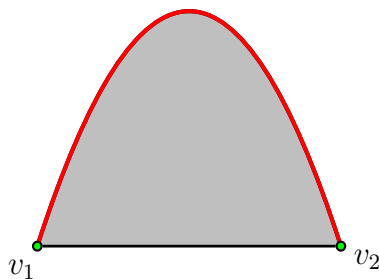
Figure 2.5: Examples of vertices and extreme points

at a. A d-simplex has boundary points of order $0, 1, \ldots, d-1$. The following proposition is shown in Berger [6] (Proposition 11.6.2):

Proposition 2.6. *The set of vertices of a closed and convex subset is countable.*

Another important concept is that of an extremal point.

Definition 2.6. Let E be an affine space. For any nonempty convex subset A, a point $a \in \partial A$ is *extremal* (or *extreme*) if $A - \{a\}$ is still convex.

It is fairly obvious that a point $a \in \partial A$ is extremal if it does not belong to the interior of any closed nontrivial line segment $[x, y] \subseteq A$ ($x \neq y$, $a \neq x$ and $a \neq y$).

Observe that a vertex is extremal, but the converse is false. For example, in Figure 2.5, all the points on the arc of parabola, including v_1 and v_2, are extreme points. However, only v_1 and v_2 are vertices. Also, if $\dim E \geq 3$, the set of extremal points of a compact convex may not be closed. See Berger [6], Chapter 11, Figure 11.6.5.3.

Actually, it is not at all obvious that a nonempty compact convex set possesses extremal points. In fact, a stronger results holds (Krein and Milman's theorem). In preparation for the proof of this important theorem, observe that any compact (nontrivial) interval of $\mathbb{A}^1$ has two extremal points, its two endpoints. We need the following lemma:

Lemma 2.7. *Let E be an affine space of dimension n, and let A be a nonempty compact and convex set. Then, $A = \mathrm{conv}(\partial A)$, i.e., A is equal to the convex hull of its boundary.*

Proof. Pick any a in A, and consider any line D through a. Then, $D \cap A$ is closed and convex. However, since A is compact, it follows that $D \cap A$ is a closed interval $[u, v]$ containing a, and $u, v \in \partial A$. Therefore, $a \in \mathrm{conv}(\partial A)$, as desired. $\qquad \square$

The following important theorem shows that only extremal points matter as far as determining a compact and convex subset from its boundary. The proof of Theorem 2.8 makes use of a proposition due to Minkowski (Proposition 3.19) which will be proved in Section 3.2.

Theorem 2.8. *(Krein and Milman, 1940) Let E be an affine space of dimension n. Every compact and convex nonempty subset A is equal to the convex hull of its set of extremal points.*

Proof. Denote the set of extremal points of A by $\mathrm{Extrem}(A)$. We proceed by induction on $d = \dim E$. When $d = 1$, the convex and compact subset A must be a closed interval $[u, v]$, or a single point. In either cases, the theorem holds trivially. Now, assume $d \geq 2$, and assume that the theorem holds for $d - 1$. It is easily verified that

$$\mathrm{Extrem}(A \cap H) = (\mathrm{Extrem}(A)) \cap H,$$

for every supporting hyperplane H of A (such hyperplanes exist, by Minkowski's proposition (Proposition 3.19)). Observe that Lemma 2.7 implies that if we can prove that

$$\partial A \subseteq \mathrm{conv}(\mathrm{Extrem}(A)),$$

then, since $A = \mathrm{conv}(\partial A)$, we will have established that

$$A = \mathrm{conv}(\mathrm{Extrem}(A)).$$

Let $a \in \partial A$, and let H be a supporting hyperplane of A at a (which exists, by Minkowski's proposition). Now, A and H are convex so $A \cap H$ is convex; H is closed and A is compact, so $H \cap A$ is a closed subset of a compact subset, A, and thus, $A \cap H$ is also compact. Since $A \cap H$ is a compact and convex subset of H and H has dimension $d - 1$, by the induction hypothesis, we have

$$A \cap H = \mathrm{conv}(\mathrm{Extrem}(A \cap H)).$$

However,

$$
\begin{aligned}
\mathrm{conv}(\mathrm{Extrem}(A \cap H)) \;&=\; \mathrm{conv}((\mathrm{Extrem}(A)) \cap H) \\
&=\; \mathrm{conv}(\mathrm{Extrem}(A)) \cap H \subseteq \mathrm{conv}(\mathrm{Extrem}(A)),
\end{aligned}
$$

and so, $a \in A \cap H \subseteq \mathrm{conv}(\mathrm{Extrem}(A))$. Therefore, we proved that

$$\partial A \subseteq \mathrm{conv}(\mathrm{Extrem}(A)),$$

from which we deduce that $A = \mathrm{conv}(\mathrm{Extrem}(A))$, as explained earlier. $\square$

Remark: Observe that Krein and Milman's theorem implies that any nonempty compact and convex set has a nonempty subset of extremal points. This is intuitively obvious, but hard to prove! Krein and Milman's theorem also applies to infinite dimensional affine spaces, provided that they are locally convex, see Valentine [43], Chapter 11, Bourbaki [9], Chapter II, Barvinok [3], Chapter 3, or Lax [26], Chapter 13.

An important consequence of Krein and Millman's theorem is that every convex function on a convex and compact set achieves its maximum at some extremal point.

Definition 2.7. Let A be a nonempty convex subset of $\mathbb{A}^n$. A function $f \colon A \to \mathbb{R}$ is *convex* if

$$f((1 - \lambda)a + \lambda b) \leq (1 - \lambda)f(a) + \lambda f(b)$$

for all $a, b \in A$ and for all $\lambda \in [0, 1]$. The function $f \colon A \to \mathbb{R}$ is *strictly convex* if

$$f((1 - \lambda)a + \lambda b) < (1 - \lambda)f(a) + \lambda f(b)$$

for all $a, b \in A$ with $a \neq b$ and for all λ with $0 < \lambda < 1$. A function $f \colon A \to \mathbb{R}$ is *concave* (resp. *strictly concave*) iff $-f$ is convex (resp. $-f$ is strictly convex).

If f is convex, a simple induction shows that

$$f\left(\sum_{i \in I} \lambda_i a_i\right) \leq \sum_{i \in I} \lambda_i f(a_i)$$

for every finite convex combination in A, *i.e.*, for any finite family $(a_i)_{i \in I}$ of points in A and any family $(\lambda_i)_{i \in I}$ with $\sum_{i \in I} \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i \in I$.

Proposition 2.9. *Let A be a nonempty convex and compact subset of $\mathbb{A}^n$ and let $f \colon A \to \mathbb{R}$ be any function. If f is convex and continuous, then f achieves its maximum at some extreme point of A.*

Proof. Since A is compact and f is continuous, $f(A)$ is a closed interval, $[m, M]$, in $\mathbb{R}$ and so f achieves its minimum m and its maximum M. Say $f(c) = M$, for some $c \in A$. By Krein and Millman's theorem, c is some convex combination of exteme points of A,

$$c = \sum_{i=1}^{k} \lambda_i a_i,$$

with $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_i \geq 0$ and each a_i an extreme point in A. But then, as f is convex,

$$M = f(c) = f\left(\sum_{i=1}^{k} \lambda_i a_i\right) \leq \sum_{i=1}^{k} \lambda_i f(a_i)$$

and if we let

$$f(a_{i_0}) = \max_{1 \leq i \leq k}\{f(a_i)\}$$

for some i_0 such that $1 \leq i_0 \leq k$, then we get

$$M = f(c) \leq \sum_{i=1}^{k} \lambda_i f(a_i) \leq \left(\sum_{i=1}^{k} \lambda_i\right) f(a_{i_0}) = f(a_{i_0}),$$

as $\sum_{i=1}^{k} \lambda_i = 1$. Since M is the maximum value of the function f over A, we have $f(a_{i_0}) \leq M$ and so,

$$M = f(a_{i_0})$$

and f achieves its maximum at the extreme point a_{i_0}, as claimed. $\qquad\square$

Proposition 2.9 plays an important role in convex optimization: It guarantees that the maximum value of a convex objective function on a compact and convex set is achieved at some extreme point. Thus, it is enough to look for a maximum at some extreme point of the domain.

Proposition 2.9 fails for minimal values of a convex function. For example, the function, $x \mapsto f(x) = x^2$, defined on the compact interval $[-1, 1]$ achieves it minimum at $x = 0$, which is not an extreme point of $[-1, 1]$. However, if f is concave, then f achieves its minimum value at some extreme point of A. In particular, if f is affine, it achieves its minimum and its maximum at some extreme points of A.

We conclude this chapter with three other classics of convex geometry.

2.4   Radon's, Tverberg's, Helly's, Theorems and Centerpoints

We begin with *Radon's theorem*.

Theorem 2.10. *(Radon, 1921) Given any affine space E of dimension m, for every subset X of E, if X has at least $m+2$ points, then there is a partition of X into two nonempty disjoint subsets X_1 and X_2 such that the convex hulls of X_1 and X_2 have a nonempty intersection.*

Proof. Pick some origin O in E. Write $X = (x_i)_{i \in L}$ for some index set L (we can let $L = X$). Since by assumption $|X| \geq m + 2$ where $m = \dim(E)$, X is affinely dependent, and by Lemma 2.6.5 from Gallier [20], there is a family $(\mu_k)_{k \in L}$ (of finite support) of scalars, not all null, such that

$$\sum_{k \in L} \mu_k = 0 \quad \text{and} \quad \sum_{k \in L} \mu_k \mathbf{Ox_k} = 0.$$

Since $\sum_{k \in L} \mu_k = 0$, the μ_k are not all null, and $(\mu_k)_{k \in L}$ has finite support, the sets

$$I = \{i \in L \mid \mu_i > 0\} \quad \text{and} \quad J = \{j \in L \mid \mu_j < 0\}$$

are nonempty, finite, and obviously disjoint. Let

$$X_1 = \{x_i \in X \mid \mu_i > 0\} \quad \text{and} \quad X_2 = \{x_i \in X \mid \mu_i \leq 0\}.$$

Again, since the μ_k are not all null and $\sum_{k \in L} \mu_k = 0$, the sets X_1 and X_2 are nonempty, and obviously

$$X_1 \cap X_2 = \emptyset \quad \text{and} \quad X_1 \cup X_2 = X.$$

Furthermore, the definition of I and J implies that $(x_i)_{i \in I} \subseteq X_1$ and $(x_j)_{j \in J} \subseteq X_2$. It remains to prove that $\text{conv}(X_1) \cap \text{conv}(X_2) \neq \emptyset$. The definition of I and J implies that

$$\sum_{k \in L} \mu_k \mathbf{Ox_k} = 0$$
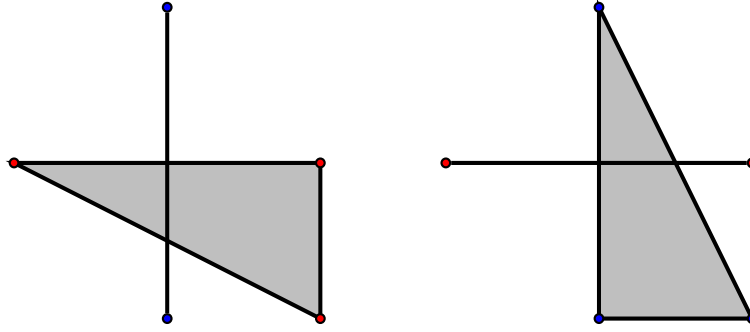
Figure 2.6: Examples of Radon Partitions

can be written as

$$\sum_{i \in I} \mu_i \mathbf{Ox_i} + \sum_{j \in J} \mu_j \mathbf{Ox_j} = 0,$$

that is, as

$$\sum_{i \in I} \mu_i \mathbf{Ox_i} = \sum_{j \in J} -\mu_j \mathbf{Ox_j},$$

where

$$\sum_{i \in I} \mu_i = \sum_{j \in J} -\mu_j = \mu,$$

with $\mu > 0$. Thus, we have

$$\sum_{i \in I} \frac{\mu_i}{\mu} \mathbf{Ox_i} = \sum_{j \in J} -\frac{\mu_j}{\mu} \mathbf{Ox_j},$$

with

$$\sum_{i \in I} \frac{\mu_i}{\mu} = \sum_{j \in J} -\frac{\mu_j}{\mu} = 1,$$

proving that $\sum_{i \in I}(\mu_i/\mu)x_i \in \mathrm{conv}(X_1)$ and $\sum_{j \in J}-(\mu_j/\mu)x_j \in \mathrm{conv}(X_2)$ are identical, and thus that $\mathrm{conv}(X_1) \cap \mathrm{conv}(X_2) \neq \emptyset$. $\qquad\square$

A partition, (X_1, X_2), of X satisfying the conditions of Theorem 2.10 is sometimes called a *Radon partition* of X and any point in $\mathrm{conv}(X_1) \cap \mathrm{conv}(X_2)$ is called a *Radon point* of X. Figure 2.6 shows two Radon partitions of five points in the plane.

It can be shown that a finite set, $X \subseteq E$, has a unique Radon partition iff it has $m + 2$ elements and any $m+1$ points of X are affinely independent. For example, there are exactly two possible cases in the plane as shown in Figure 2.7.

There is also a version of Radon's theorem for the class of cones with an apex. Say that a convex cone, $C \subseteq E$, has an *apex* (or is a *pointed cone*) iff there is some hyperplane, H, such that $C \subseteq H_+$ and $H \cap C = \{0\}$. For example, the cone obtained as the intersection of
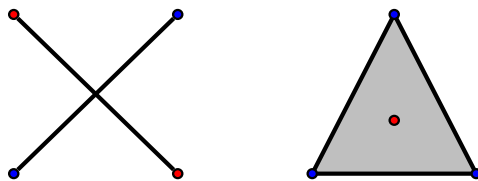
Figure 2.7: The Radon Partitions of four points (in $\mathbb{A}^2$)

two half spaces in $\mathbb{R}^3$ is not pointed since it is a wedge with a line as part of its boundary. Here is the version of Radon's theorem for convex cones:

Theorem 2.11. *Given any vector space E of dimension m, for every subset X of E, if $\mathrm{cone}(X)$ is a pointed cone such that X has at least $m + 1$ nonzero vectors, then there is a partition of X into two nonempty disjoint subsets, X_1 and X_2, such that the cones, $\mathrm{cone}(X_1)$ and $\mathrm{cone}(X_2)$, have a nonempty intersection not reduced to $\{0\}$.*

The proof of Theorem 2.11 is left as an exercise.

There is a beautiful generalization of Radon's theorem known as *Tverberg's Theorem*.

Theorem 2.12. *(Tverberg's Theorem, 1966) Let E be any affine space of dimension m. For any natural number, $r \geq 2$, for every subset, X, of E, if X has at least $(m + 1)(r - 1) + 1$ points, then there is a partition, $(X_1, \ldots, X_r)$, of X into r nonempty pairwise disjoint subsets so that $\bigcap_{i=1}^{r} \mathrm{conv}(X_i) \neq \emptyset$.*

A partition as in Theorem 2.12 is called a *Tverberg partition* and a point in $\bigcap_{i=1}^{r} \mathrm{conv}(X_i)$ is called a *Tverberg point*. Theorem 2.12 was conjectured by Birch and proved by Tverberg in 1966. Tverberg's original proof was technically quite complicated. Tverberg then gave a simpler proof in 1981 and other simpler proofs were later given, notably by Sarkaria (1992) and Onn (1997), using the Colorful Carathéodory theorem. A proof along those lines can be found in Matousek [27], Chapter 8, Section 8.3. A *colored Tverberg theorem* and more can also be found in Matousek [27] (Section 8.3).

Next, we prove a version of *Helly's theorem*.

Theorem 2.13. *(Helly, 1913) Given any affine space E of dimension m, for every family $\{K_1, \ldots, K_n\}$ of n convex subsets of E, if $n \geq m + 2$ and the intersection $\bigcap_{i \in I} K_i$ of any $m + 1$ of the K_i is nonempty (where $I \subseteq \{1, \ldots, n\}$, $|I| = m + 1$), then $\bigcap_{i=1}^{n} K_i$ is nonempty.*

Proof. The proof is by induction on $n \geq m + 1$ and uses Radon's theorem in the induction step. For $n = m + 1$, the assumption of the theorem is that the intersection of any family of $m + 1$ of the K_i's is nonempty, and the theorem holds trivially. Next, let $L = \{1, 2, \ldots, n + 1\}$,

where $n + 1 \geq m + 2$. By the induction hypothesis, $C_i = \bigcap_{j \in (L - \{i\})} K_j$ is nonempty for every $i \in L$.

We claim that $C_i \cap C_j \neq \emptyset$ for some $i \neq j$. If so, as $C_i \cap C_j = \bigcap_{k=1}^{n+1} K_k$, we are done. So, let us assume that the C_i's are pairwise disjoint. Then, we can pick a set $X = \{a_1, \ldots, a_{n+1}\}$ such that $a_i \in C_i$, for every $i \in L$. By Radon's Theorem, there are two nonempty disjoint sets $X_1, X_2 \subseteq X$ such that $X = X_1 \cup X_2$ and $\mathrm{conv}(X_1) \cap \mathrm{conv}(X_2) \neq \emptyset$. However, $X_1 \subseteq K_j$ for every j with $a_j \notin X_1$. This is because $a_j \notin K_j$ for every j, and so, we get

$$X_1 \subseteq \bigcap_{a_j \notin X_1} K_j.$$

Symetrically, we also have

$$X_2 \subseteq \bigcap_{a_j \notin X_2} K_j.$$

Since the K_j's are convex and

$$\left(\bigcap_{a_j \notin X_1} K_j \right) \cap \left(\bigcap_{a_j \notin X_2} K_j \right) = \bigcap_{i=1}^{n+1} K_i,$$

it follows that $\mathrm{conv}(X_1) \cap \mathrm{conv}(X_2) \subseteq \bigcap_{i=1}^{n+1} K_i$, so that $\bigcap_{i=1}^{n+1} K_i$ is nonempty, contradicting the fact that $C_i \cap C_j = \emptyset$ for all $i \neq j$. $\qquad \square$

A more general version of Helly's theorem is proved in Berger [6].

An amusing corollary of Helly's theorem is the following result: Consider $n \geq 4$ line segments in the affine plane $\mathbb{A}^2$ lying on disjoint parallel lines. If every three of these line segments meet a line, then all of these line segments meet a common line.

To prove this fact, pick a coordinate system in which the y-axis is parallel to the common direction of the parallel lines, and for every line segment S, let

$$CS = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \text{the line } y = \alpha x + \beta \text{ meets } S\}.$$

It is not hard to see that CS is convex. Then, by hypothesis the fact that any three line segments S_i, S_j, S_k meet a line means that $CS_i \cap CS_j \cap CS_k \neq \emptyset$, any Helly's Theorem implies that the family of all the convex sets CS_i has a nonempty intersection, which means that there is a line meeting all the line segments S_i.

We conclude this chapter with a nice application of Helly's Theorem to the existence of centerpoints. Centerpoints generalize the notion of median to higher dimensions. Recall that if we have a set of n data points, $S = \{a_1, \ldots, a_n\}$, on the real line, a *median* for S is a point, x, such that both intervals $[x, \infty)$ and $(-\infty, x]$ contain at least $n/2$ of the points in S (by $n/2$, we mean the largest integer greater than or equal to $n/2$).
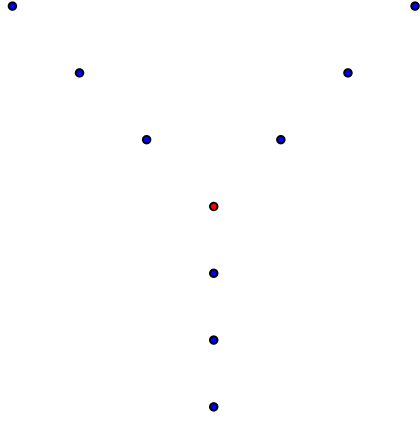
Figure 2.8: Example of a centerpoint

Given any hyperplane, H, recall that the closed half-spaces determined by H are denoted H_+ and H_- and that $H \subseteq H_+$ and $H \subseteq H_-$. We let $\overset{\circ}{H}_+ = H_+ - H$ and $\overset{\circ}{H}_- = H_- - H$ be the *open half-spaces* determined by H.

Definition 2.8. Let $S = \{a_1, \ldots, a_n\}$ be a set of n points in $\mathbb{A}^d$. A point, $c \in \mathbb{A}^d$, is a *centerpoint of S* iff for every hyperplane, H, whenever the closed half-space H_+ (resp. H_-) contains c, then H_+ (resp. H_-) contains at least $\frac{n}{d+1}$ points from S (by $\frac{n}{d+1}$, we mean the largest integer greater than or equal to $\frac{n}{d+1}$, namely the ceiling $\lceil \frac{n}{d+1} \rceil$ of $\frac{n}{d+1}$).

So, for $d = 2$, for each line, D, if the closed half-plane D_+ (resp. D_-) contains c, then D_+ (resp. D_-) contains at least a third of the points from S. For $d = 3$, for each plane, H, if the closed half-space H_+ (resp. H_-) contains c, then H_+ (resp. H_-) contains at least a fourth of the points from S, *etc.* Example 2.8 shows nine points in the plane and one of their centerpoints (in red). This example shows that the bound $\frac{1}{3}$ is tight.

Observe that a point, $c \in \mathbb{A}^d$, is a centerpoint of S iff c belongs to every open half-space, $\overset{\circ}{H}_+$ (resp. $\overset{\circ}{H}_-$) containing at least $\frac{dn}{d+1} + 1$ points from S (again, we mean $\lceil \frac{dn}{d+1} \rceil + 1$).

Indeed, if c is a centerpoint of S and H is any hyperplane such that $\overset{\circ}{H}_+$ (resp. $\overset{\circ}{H}_-$) contains at least $\frac{dn}{d+1} + 1$ points from S, then $\overset{\circ}{H}_+$ (resp. $\overset{\circ}{H}_-$) must contain c as otherwise, the closed half-space, H_- (resp. H_+) would contain c and at most $n - \frac{dn}{d+1} - 1 = \frac{n}{d+1} - 1$ points from S, a contradiction. Conversely, assume that c belongs to every open half-space, $\overset{\circ}{H}_+$ (resp. $\overset{\circ}{H}_-$) containing at least $\frac{dn}{d+1} + 1$ points from S. Then, for any hyperplane, H, if $c \in H_+$ (resp. $c \in H_-$) but H_+ contains at most $\frac{n}{d+1} - 1$ points from S, then the open half-space, $\overset{\circ}{H}_-$ (resp. $\overset{\circ}{H}_+$) would contain at least $n - \frac{n}{d+1} + 1 = \frac{dn}{d+1} + 1$ points from S but not c, a contradiction.

We are now ready to prove the existence of centerpoints.

Theorem 2.14. *(Existence of Centerpoints) Every finite set, $S = \{a_1, \ldots, a_n\}$, of n points in $\mathbb{A}^d$ has some centerpoint.*

Proof. We will use the second characterization of centerpoints involving open half-spaces containing at least $\frac{dn}{d+1} + 1$ points.

Consider the family of sets,

$$
\begin{aligned}
\mathcal{C} \;=\; & \left\{ \operatorname{conv}(S \cap \mathring{H}_+) \mid (\exists H) \left(|S \cap \mathring{H}_+| > \frac{dn}{d+1} \right) \right\} \\
& \cup \left\{ \operatorname{conv}(S \cap \mathring{H}_-) \mid (\exists H) \left(|S \cap \mathring{H}_-| > \frac{dn}{d+1} \right) \right\},
\end{aligned}
$$

where H is a hyperplane.

As S is finite, $\mathcal{C}$ consists of a finite number of convex sets, say $\{C_1, \ldots, C_m\}$. If we prove that $\bigcap_{i=1}^m C_i \neq \emptyset$ we are done, because $\bigcap_{i=1}^m C_i$ is the set of centerpoints of S.

First, we prove by induction on k (with $1 \leq k \leq d+1$), that any intersection of k of the C_i's has at least $\frac{(d+1-k)n}{d+1} + k$ elements from S. For $k = 1$, this holds by definition of the C_i's.

Next, consider the intersection of $k+1 \leq d+1$ of the C_i's, say $C_{i_1} \cap \cdots \cap C_{i_k} \cap C_{i_{k+1}}$. Let

$$
\begin{aligned}
A &= S \cap (C_{i_1} \cap \cdots \cap C_{i_k} \cap C_{i_{k+1}}) \\
B &= S \cap (C_{i_1} \cap \cdots \cap C_{i_k}) \\
C &= S \cap C_{i_{k+1}}.
\end{aligned}
$$

Note that $A = B \cap C$. By the induction hypothesis, B contains at least $\frac{(d+1-k)n}{d+1} + k$ elements from S. As C contains at least $\frac{dn}{d+1} + 1$ points from S, and as

$$
|B \cup C| = |B| + |C| - |B \cap C| = |B| + |C| - |A|
$$

and $|B \cup C| \leq n$, we get $n \geq |B| + |C| - |A|$, that is,

$$
|A| \geq |B| + |C| - n.
$$

It follows that

$$
|A| \geq \frac{(d+1-k)n}{d+1} + k + \frac{dn}{d+1} + 1 - n
$$

that is,

$$
|A| \geq \frac{(d+1-k)n + dn - (d+1)n}{d+1} + k + 1 = \frac{(d+1-(k+1))n}{d+1} + k + 1,
$$

establishing the induction hypothesis.

Now, if $m \leq d+1$, the above claim for $k = m$ shows that $\bigcap_{i=1}^m C_i \neq \emptyset$ and we are done. If $m \geq d+2$, the above claim for $k = d+1$ shows that any intersection of $d+1$ of the C_i's

is nonempty. Consequently, the conditions for applying Helly's Theorem are satisfied and therefore,

$$\bigcap_{i=1}^{m} C_i \neq \emptyset.$$

However, $\bigcap_{i=1}^{m} C_i$ is the set of centerpoints of S and we are done. $\square$

Remark: The above proof actually shows that the set of centerpoints of S is a convex set. In fact, it is a finite intersection of convex hulls of finitely many points, so it is the convex hull of finitely many points, in other words, a polytope. It should also be noted that Theorem 2.14 can be proved easily using Tverberg's theorem (Theorem 2.12). Indeed, for a judicious choice of r, any Tverberg point is a centerpoint!

Jadhav and Mukhopadhyay have given a linear-time algorithm for computing a centerpoint of a finite set of points in the plane. For $d \geq 3$, it appears that the best that can be done (using linear programming) is $O(n^d)$. However, there are good approximation algorithms (Clarkson, Eppstein, Miller, Sturtivant and Teng) and in $\mathbb{E}^3$ there is a near quadratic algorithm (Agarwal, Sharir and Welzl). Recently, Miller and Sheehy (2009) have given an algorithm for finding an approximate centerpoint in sub-exponential time together with a polynomial-checkable proof of the approximation guarantee.

Chapter 3

Separation and Supporting Hyperplanes

3.1 Separation Theorems and Farkas Lemma

It seems intuitively rather obvious that if A and B are two nonempty disjoint convex sets in $\mathbb{A}^2$, then there is a line, H, separating them, in the sense that A and B belong to the two (disjoint) open half–planes determined by H. However, this is not always true! For example, this fails if both A and B are closed and unbounded (find an example). Nevertheless, the result is true if both A and B are open, or if the notion of separation is weakened a little bit. The key result, from which most separation results follow, is a geometric version of the *Hahn-Banach theorem*. In the sequel, we restrict our attention to real affine spaces of finite dimension. Then, if X is an affine space of dimension d, there is an affine bijection f between X and $\mathbb{A}^d$.

Now, $\mathbb{A}^d$ is a topological space, under the usual topology on $\mathbb{R}^d$ (in fact, $\mathbb{A}^d$ is a metric space). Recall that if $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ are any two points in $\mathbb{A}^d$, their **Euclidean distance**, $d(a, b)$, is given by

$$d(a, b) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_d - a_d)^2},$$

which is also the *norm*, $\|\mathbf{ab}\|$, of the vector $\mathbf{ab}$ and that for any $\epsilon > 0$, the *open ball of center a and radius ϵ, $B(a, \epsilon)$*, is given by

$$B(a, \epsilon) = \{b \in \mathbb{A}^d \mid d(a, b) < \epsilon\}.$$

A subset $U \subseteq \mathbb{A}^d$ is *open* (in the *norm topology*) if either U is empty or for every point, $a \in U$, there is some (small) open ball, $B(a, \epsilon)$, contained in U. A subset $C \subseteq \mathbb{A}^d$ is *closed* iff $\mathbb{A}^d - C$ is open. For example, the *closed balls*, $\overline{B(a, \epsilon)}$, where

$$\overline{B(a, \epsilon)} = \{b \in \mathbb{A}^d \mid d(a, b) \leq \epsilon\},$$

are closed. A subset $W \subseteq \mathbb{A}^d$ is *bounded* iff there is some ball (open or closed), B, so that $W \subseteq B$. A subset $W \subseteq \mathbb{A}^d$ is *compact* iff every family, $\{U_i\}_{i\in I}$, that is an open cover of W (which means that $W = \bigcup_{i\in I}(W \cap U_i)$, with each U_i an open set) possesses a finite subcover (which means that there is a finite subset, $F \subseteq I$, so that $W = \bigcup_{i\in F}(W \cap U_i)$). In $\mathbb{A}^d$, it can be shown that a subset W is compact iff W is closed and bounded. Given a function, $f\colon \mathbb{A}^m \to \mathbb{A}^n$, we say that f is *continuous* if $f^{-1}(V)$ is open in $\mathbb{A}^m$ whenever V is open in $\mathbb{A}^n$. If $f\colon \mathbb{A}^m \to \mathbb{A}^n$ is a continuous function, although it is generally **false** that $f(U)$ is open if $U \subseteq \mathbb{A}^m$ is open, it is easily checked that $f(K)$ is compact if $K \subseteq \mathbb{A}^m$ is compact.

An affine space X of dimension d becomes a topological space if we give it the topology for which the open subsets are of the form $f^{-1}(U)$, where U is any open subset of $\mathbb{A}^d$ and $f\colon X \to \mathbb{A}^d$ is an affine bijection.

Given any subset, A, of a topological space, X, the smallest closed set containing A is denoted by $\overline{A}$, and is called the *closure* or *adherence of A*. A subset, A, of X, is *dense in X* if $\overline{A} = X$. The largest open set contained in A is denoted by $\overset{\circ}{A}$, and is called the *interior of A*. The set, $\mathrm{Fr}\, A = \overline{A} \cap \overline{X - A}$, is called the *boundary* (or *frontier*) of A. We also denote the boundary of A by ∂A.

In order to prove the Hahn-Banach theorem, we will need two lemmas. Given any two distinct points $x, y \in X$, we let

$$]x, y[= \{(1 - \lambda)x + \lambda y \in X \mid 0 < \lambda < 1\}.$$

Our first lemma (Lemma 3.1) is intuitively quite obvious so the reader might be puzzled by the length of its proof. However, after proposing several wrong proofs, we realized that its proof is more subtle than it might appear. The proof below is due to Valentine [43]. See if you can find a shorter (and correct) proof!

Lemma 3.1. *Let S be a nonempty convex set and let $x \in \overset{\circ}{S}$ and $y \in \overline{S}$. Then, we have $]x, y[\subseteq \overset{\circ}{S}$.*

Proof. Let $z \in\,]x, y[\,$, that is, $z = (1 - \lambda)x + \lambda y$, with $0 < \lambda < 1$. Since $x \in \overset{\circ}{S}$, we can find some open subset, U, contained in S so that $x \in U$. It is easy to check that the central magnification of center z, $H_{z, \frac{\lambda-1}{\lambda}}$, maps x to y. Then, $V = H_{z, \frac{\lambda-1}{\lambda}}(U)$ is an open subset containing y and as $y \in \overline{S}$, we have $V \cap S \neq \emptyset$. Let $v \in V \cap S$ be a point of S in this intersection. Now, there is a unique point, $u \in U \subseteq S$, such that $H_{z, \frac{\lambda-1}{\lambda}}(u) = v$ and, as S is convex, we deduce that $z = (1 - \lambda)u + \lambda v \in S$. Since U is open, the set

$$W = (1 - \lambda)U + \lambda v = \{(1 - \lambda)w + \lambda v \mid w \in U\} \subseteq S$$

is also open and $z \in W$, which shows that $z \in \overset{\circ}{S}$. $\square$
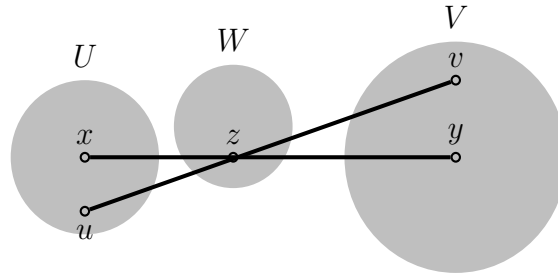
Figure 3.1: Illustration for the proof of Lemma 3.1

Corollary 3.2. *If S is convex, then $\overset{\circ}{S}$ is also convex, and we have $\overset{\circ}{\overline{S}} = \overset{\circ}{S}$. Furthermore, if $\overset{\circ}{S} \neq \emptyset$, then $\overline{S} = \overset{\overset{\circ}{}}{\overline{S}}$.*

Beware that if S is a closed set, then the convex hull conv(S) of S is not necessarily closed!

For example, consider the subset S of $\mathbb{A}^2$ consisting of the points belonging to the right branch of the hyperbola of equation $x^2 - y^2 = 1$, that is,

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 1, \ x \geq 0\}.$$

Then S is convex, but the convex hull of the set $S \cup \{(0,0\}$ is not closed.

However, if S is compact, then conv(S) is also compact, and thus closed (see Proposition 2.3).

There is a simple criterion to test whether a convex set has an empty interior, based on the notion of dimension of a convex set (recall that the dimension of a nonempty convex subset is the dimension of its affine hull).

Proposition 3.3. *A nonempty convex set S has a nonempty interior iff* $\dim S = \dim X$.

Proof. Let $d = \dim X$. First, assume that $\overset{\circ}{S} \neq \emptyset$. Then, S contains some open ball of center a_0, and in it, we can find a frame $(a_0, a_1, \ldots, a_d)$ for X. Thus, $\dim S = \dim X$. Conversely, let $(a_0, a_1, \ldots, a_d)$ be a frame of X, with $a_i \in S$, for $i = 0, \ldots, d$. Then, we have

$$\frac{a_0 + \cdots + a_d}{d + 1} \in \overset{\circ}{S},$$

and $\overset{\circ}{S}$ is nonempty. $\qquad \square$

Proposition 3.3 is false in infinite dimension.
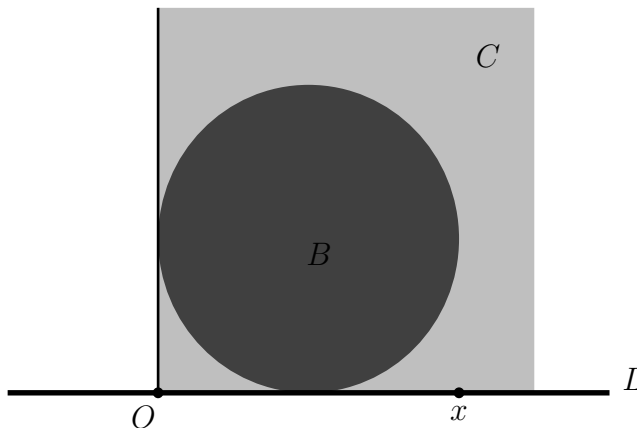
Figure 3.2: Hahn-Banach Theorem in the plane (Lemma 3.5)

We leave the following property as an exercise:

Proposition 3.4. *If S is convex, then $\overline{S}$ is also convex.*

One can also easily prove that convexity is preserved under direct image and inverse image by an affine map.

The next lemma, which seems intuitively obvious, is the core of the proof of the Hahn-Banach theorem. This is the case where the affine space has dimension two. First, we need to define what is a convex cone with vertex x.

Definition 3.1. A convex set, C, is a *convex cone with vertex x* if C is invariant under all central magnifications, $H_{x,\lambda}$, of center x and ratio λ, with $\lambda > 0$ (i.e., $H_{x,\lambda}(C) = C$).

Given a convex set, S, and a point, $x \notin S$, we can define

$$\text{cone}_x(S) = \bigcup_{\lambda > 0} H_{x,\lambda}(S).$$

It is easy to check that this is a convex cone with vertex x.

Lemma 3.5. *Let B be a nonempty open and convex subset of $\mathbb{A}^2$, and let O be a point of $\mathbb{A}^2$ so that $O \notin B$. Then, there is some line, L, through O, so that $L \cap B = \emptyset$.*

Proof. Define the convex cone $C = \text{cone}_O(B)$. As B is open, it is easy to check that each $H_{O,\lambda}(B)$ is open and since C is the union of the $H_{O,\lambda}(B)$ (for $\lambda > 0$), which are open, C itself is open. Also, $O \notin C$. We claim that at least one point, x, of the boundary, ∂C, of C, is distinct from O. Otherwise, $\partial C = \{O\}$ and we claim that $C = \mathbb{A}^2 - \{O\}$, which is not convex, a contradiction. Indeed, as C is convex it is connected, $\mathbb{A}^2 - \{O\}$ itself is connected
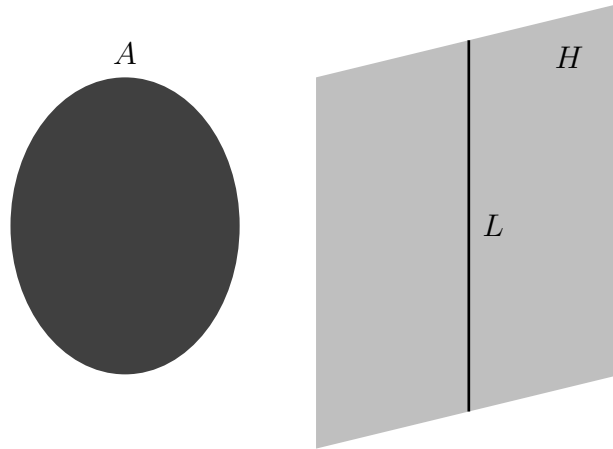
Figure 3.3: Hahn-Banach Theorem, geometric form (Theorem 3.6)

and $C \subseteq \mathbb{A}^2 - \{O\}$. If $C \neq \mathbb{A}^2 - \{O\}$, pick some point $a \neq O$ in $\mathbb{A}^2 - C$ and some point $c \in C$. Now, a basic property of connectivity asserts that every continuous path from a (in the exterior of C) to c (in the interior of C) must intersect the boundary of C, namely, $\{O\}$. However, there are plenty of paths from a to c that avoid O, a contradiction. Therefore, $C = \mathbb{A}^2 - \{O\}$.

Since C is open and $x \in \partial C$, we have $x \notin C$. Furthermore, we claim that $y = 2O - x$ (the symmetric of x w.r.t. O) does not belong to C either. Otherwise, we would have $y \in \overset{\circ}{C} = C$ and $x \in \overline{C}$, and by Lemma 3.1, we would get $O \in C$, a contradiction. Therefore, the line through O and x misses C entirely (since C is a cone), and thus, $B \subseteq C$. $\qquad \square$

Finally, we come to the Hahn-Banach theorem.

Theorem 3.6. *(Hahn-Banach Theorem, geometric form) Let X be a (finite-dimensional) affine space, A be a nonempty open and convex subset of X and L be an affine subspace of X so that $A \cap L = \emptyset$. Then, there is some hyperplane, H, containing L, that is disjoint from A.*

Proof. The case where $\dim X = 1$ is trivial. Thus, we may assume that $\dim X \geq 2$. We reduce the proof to the case where $\dim X = 2$. Let V be an affine subspace of X of maximal dimension containing L and so that $V \cap A = \emptyset$. Pick an origin $O \in L$ in X, and consider the vector space X_O. We would like to prove that V is a hyperplane, i.e., $\dim V = \dim X - 1$. We proceed by contradiction. Thus, assume that $\dim V \leq \dim X - 2$. In this case, the quotient space X/V has dimension at least 2. We also know that X/V is isomorphic to the orthogonal complement, $V^\perp$, of V so we may identify X/V and $V^\perp$. The (orthogonal) projection map, $\pi \colon X \to V^\perp$, is linear, continuous, and we can show that π maps the open subset A to an open subset $\pi(A)$, which is also convex (one way to prove that $\pi(A)$ is open is to observe that

for any point, $a \in A$, a small open ball of center a contained in A is projected by π to an open ball contained in $\pi(A)$ and as π is surjective, $\pi(A)$ is open). Furthermore, $O \notin \pi(A)$. Since $V^{\perp}$ has dimension at least 2, there is some plane P (a subspace of dimension 2) intersecting $\pi(A)$, and thus, we obtain a nonempty open and convex subset $B = \pi(A) \cap P$ in the plane $P \cong \mathbb{A}^2$. So, we can apply Lemma 3.5 to B and the point $O = 0$ in $P \cong \mathbb{A}^2$ to find a line, l, (in P) through O with $l \cap B = \emptyset$. But then, $l \cap \pi(A) = \emptyset$ and $W = \pi^{-1}(l)$ is an affine subspace such that $W \cap A = \emptyset$ and W properly contains V, contradicting the maximality of V. $\square$

Remark: The geometric form of the Hahn-Banach theorem also holds when the dimension of X is infinite but a slightly more sophisticated proof is required. Actually, all that is needed is to prove that a maximal affine subspace containing L and disjoint from A exists. This can be done using Zorn's lemma. For other proofs, see Bourbaki [9], Chapter 2, Valentine [43], Chapter 2, Barvinok [3], Chapter 2, or Lax [26], Chapter 3.

Theorem 3.6 is false if we omit the assumption that A is open.

For a counter-example, let $A \subseteq \mathbb{A}^2$ be the union of the half space $y < 0$ with the closed segment $[0,1]$ on the x-axis and let L be the point $(2,0)$ on the boundary of A. It is also false if A is closed as shown by the following counter-example.

In $\mathbb{E}^3$, consider the closed convex set (cone) A defined by the inequalities

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z^2 \leq xy,$$

and let D be the line given by $x = 0, z = 1$. Then $D \cap A = \emptyset$, both A and D are convex and closed, yet every plane containing D meets A.

Theorem 3.6 has many important corollaries. For example, we will eventually prove that for any two nonempty disjoint convex sets, A and B, there is a hyperplane separating A and B, but this will take some work (recall the definition of a separating hyperplane given in Definition 2.3). We begin with the following version of the Hahn-Banach theorem:

Theorem 3.7. *(Hahn-Banach, second version) Let X be a (finite-dimensional) affine space, A be a nonempty convex subset of X with nonempty interior and L be an affine subspace of X so that $A \cap L = \emptyset$. Then, there is some hyperplane, H, containing L and separating L and A.*

Proof. Since A is convex, by Corollary 3.2, $\overset{\circ}{A}$ is also convex. By hypothesis, $\overset{\circ}{A}$ is nonempty. So, we can apply Theorem 3.6 to the nonempty open and convex $\overset{\circ}{A}$ and to the affine subspace L. We get a hyperplane H containing L such that $\overset{\circ}{A} \cap H = \emptyset$. However, $A \subseteq \overline{A} = \overline{\overset{\circ}{A}}$ and $\overline{\overset{\circ}{A}}$ is contained in the closed half space (H_+ or H_-) containing $\overset{\circ}{A}$, so H separates A and L. $\square$
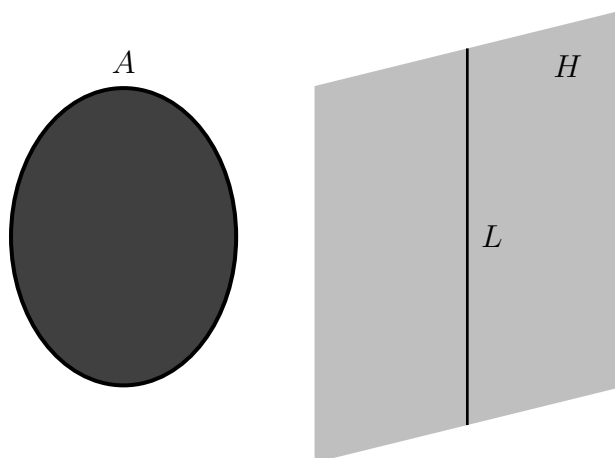
Figure 3.4: Hahn-Banach Theorem, second version (Theorem 3.7)
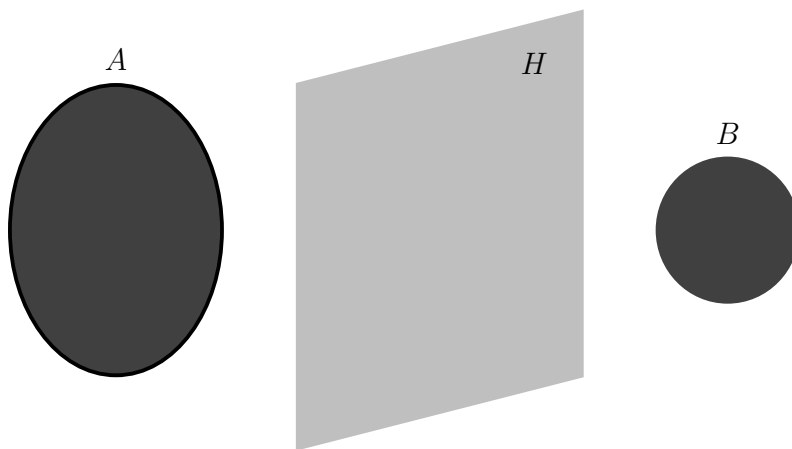
Figure 3.5: Separation Theorem, version 1 (Corollary 3.8)

Corollary 3.8. *Given an affine space, X, let A and B be two nonempty disjoint convex subsets and assume that A has nonempty interior $(\overset{\circ}{A} \neq \emptyset)$. Then, there is a hyperplane separating A and B.*

Proof. Pick some origin O and consider the vector space X_O. Define $C = A - B$ (a special case of the Minkowski sum) as follows:

$$A - B = \{a - b \mid a \in A,\ b \in B\} = \bigcup_{b \in B}(A - b).$$

It is easily verified that $C = A - B$ is convex and has nonempty interior (as a union of subsets having a nonempty interior). Furthermore $O \notin C$, since $A \cap B = \emptyset$.[1] (Note that the definition depends on the choice of O, but this has no effect on the proof.) Since $\overset{\circ}{C}$ is nonempty, we can apply Theorem 3.7 to C and to the affine subspace $\{O\}$ and we get a hyperplane, H, separating C and $\{O\}$. Let f be any linear form defining the hyperplane H. We may assume that $f(a - b) \leq 0$, for all $a \in A$ and all $b \in B$, i.e., $f(a) \leq f(b)$. Consequently, if we let $\alpha = \sup\{f(a) \mid a \in A\}$ (which makes sense, since the set $\{f(a) \mid a \in A\}$ is bounded), we have $f(a) \leq \alpha$ for all $a \in A$ and $f(b) \geq \alpha$ for all $b \in B$, which shows that the affine hyperplane defined by $f - \alpha$ separates A and B. $\qquad\square$

Remark: Theorem 3.7 and Corollary 3.8 also hold in the infinite dimensional case, see Lax [26], Chapter 3, or Barvinok, Chapter 3.

Since a hyperplane, H, separating A and B as in Corollary 3.8 is the boundary of each of the two half–spaces that it determines, we also obtain the following corollary:

Corollary 3.9. *Given an affine space, X, let A and B be two nonempty disjoint open and convex subsets. Then, there is a hyperplane strictly separating A and B.*

Beware that Corollary 3.9 *fails* for *closed* convex sets.

However, Corollary 3.9 holds if we also assume that A (or B) is compact, as shown in Corollary 3.10.

We need to review the notion of distance from a point to a subset. Let X be a metric space with distance function, d. Given any point, $a \in X$, and any nonempty subset, B, of X, we let

$$d(a, B) = \inf_{b \in B} d(a, b)$$

---

[1]Readers who prefer a purely affine argument may define $C = A - B$ as the *affine* subset

$$A - B = \{O + a - b \mid a \in A,\ b \in B\}.$$

Again, $O \notin C$ and C is convex. We can pick the affine form, f, defining a separating hyperplane, H, of C and $\{O\}$, so that $f(O + a - b) \leq f(O)$, for all $a \in A$ and all $b \in B$, i.e., $f(a) \leq f(b)$.

(where inf is the notation for least upper bound).

Now, if X is an affine space of dimension d, it can be given a metric structure by giving the corresponding vector space a metric structure, for instance, the metric induced by a Euclidean structure. We have the following important property: For any nonempty closed subset, $S \subseteq X$ (not necessarily convex), and any point, $a \in X$, there is some point $s \in S$ "achieving the distance from a to S," i.e., so that

$$d(a, S) = d(a, s).$$

The proof uses the fact that the distance function is continuous and that a continuous function attains its minimum on a compact set, and is left as an exercise.

Corollary 3.10. *Given an affine space, X, let A and B be two nonempty disjoint closed and convex subsets, with A compact. Then, there is a hyperplane strictly separating A and B.*

Proof sketch. First, we pick an origin O and we give $X_O \cong \mathbb{A}^n$ a Euclidean structure. Let d denote the associated distance. Given any subsets A of X, let

$$A + B(O, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\},$$

where $B(a, \epsilon)$ denotes the open ball, $B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\}$, of center a and radius $\epsilon > 0$. Note that

$$A + B(O, \epsilon) = \bigcup_{a \in A} B(a, \epsilon),$$

which shows that $A + B(O, \epsilon)$ is open; furthermore it is easy to see that if A is convex, then $A + B(O, \epsilon)$ is also convex. Now, the function $a \mapsto d(a, B)$ (where $a \in A$) is continuous and since A is compact, it achieves its minimum, $d(A, B) = \min_{a \in A} d(a, B)$, at some point, a, of A. Say, $d(A, B) = \delta$. Since B is closed, there is some $b \in B$ so that $d(A, B) = d(a, B) = d(a, b)$ and since $A \cap B = \emptyset$, we must have $\delta > 0$. Thus, if we pick $\epsilon < \delta/2$, we see that

$$(A + B(O, \epsilon)) \cap (B + B(O, \epsilon)) = \emptyset.$$

Now, $A + B(O, \epsilon)$ and $B + B(O, \epsilon)$ are open, convex and disjoint and we conclude by applying Corollary 3.9. $\square$

Finally, we have the separation theorem announced earlier for arbitrary nonempty convex subsets.

Theorem 3.11. *(Separation of disjoint convex sets) Given an affine space, X, let A and B be two nonempty disjoint convex subsets. Then, there is a hyperplane separating A and B.*
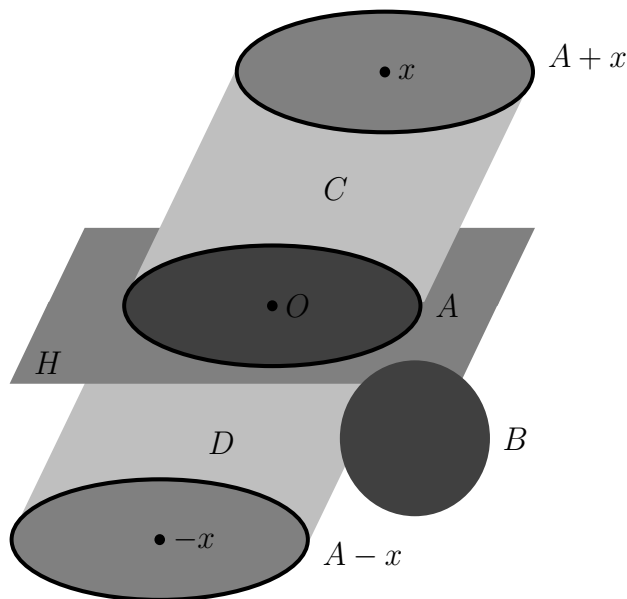
Figure 3.6: Separation Theorem, final version (Theorem 3.11)

Proof. The proof is by descending induction on dim A. If dim $A = $ dim X, we know from Proposition 3.3 that A has nonempty interior and we conclude using Corollary 3.8. Next, asssume that the induction hypothesis holds if dim $A \geq n$ and assume dim $A = n - 1$. Pick an origin $O \in A$ and let H be a hyperplane containing A. Pick $x \in X$ outside H and define $C = \text{conv}(A \cup \{A + x\})$ where $A + x = \{a + x \mid a \in A\}$ and $D = \text{conv}(A \cup \{A - x\})$ where $A - x = \{a - x \mid a \in A\}$. Note that $C \cup D$ is convex. If $B \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$, then the convexity of B and $C \cup D$ implies that $A \cap B \neq \emptyset$, a contradiction. Without loss of generality, assume that $B \cap C = \emptyset$. Since x is outside H, we have dim $C = n$ and by the induction hypothesis, there is a hyperplane, H_1 separating C and B. As $A \subseteq C$, we see that H_1 also separates A and B. $\square$

Remarks:

(1) The reader should compare this proof (from Valentine [43], Chapter II) with Berger's proof using compactness of the projective space $\mathbb{P}^d$, see Berger [6] (Corollary 11.4.7).

(2) Rather than using the Hahn-Banach theorem to deduce separation results, one may proceed differently and use the following intuitively obvious lemma, as in Valentine [43] (Theorem 2.4):

   Lemma 3.12. *If A and B are two nonempty convex sets such that $A \cup B = X$ and $A \cap B = \emptyset$, then $V = \overline{A} \cap \overline{B}$ is a hyperplane.*

One can then deduce Corollaries 3.8 and Theorem 3.11. Yet another approach is followed in Barvinok [3].

(3) How can some of the above results be generalized to infinite dimensional affine spaces, especially Theorem 3.6 and Corollary 3.8? One approach is to simultaneously relax the notion of interior and tighten a little the notion of closure, in a more "linear and less topological" fashion, as in Valentine [43].

Given any subset $A \subseteq X$ (where X may be infinite dimensional, but is a Hausdorff topological vector space), say that a point $x \in X$ is *linearly accessible from* A iff there is some $a \in A$ with $a \neq x$ and $]a, x[\subseteq A$. We let lina A be the set of all points linearly accessible from A and lin $A = A \cup$ lina A.

A point $a \in A$ is a *core point of* A iff for every $y \in X$, with $y \neq a$, there is some $z \in]a, y[$, such that $[a, z] \subseteq A$. The set of all core points is denoted core A.

It is not difficult to prove that lin $A \subseteq \overline{A}$ and $\overset{\circ}{A} \subseteq$ core A. If A has nonempty interior, then lin $A = \overline{A}$ and $\overset{\circ}{A} =$ core A. Also, if A is convex, then core A and lin A are convex. Then, Lemma 3.12 still holds (where X is not necessarily finite dimensional) if we redefine V as $V = $ lin $A \cap$ lin B and allow the possibility that V could be X itself. Corollary 3.8 also holds in the general case if we assume that core A is nonempty. For details, see Valentine [43], Chapter I and II.

(4) Yet another approach is to define the notion of an algebraically open convex set, as in Barvinok [3]. A convex set, A, is *algebraically open* iff the intersection of A with every line, L, is an open interval, possibly empty or infinite at either end (or all of L). An open convex set is algebraically open. Then, the Hahn-Banach theorem holds provided that A is an algebraically open convex set and similarly, Corollary 3.8 also holds provided A is algebraically open. For details, see Barvinok [3], Chapter 2 and 3. We do not know how the notion "algebraically open" relates to the concept of core.

(5) Theorems 3.6, 3.7 and Corollary 3.8 are proved in Lax [26] using the notion of *gauge function* in the more general case where A has some core point (but beware that Lax uses the terminology *interior point* instead of core point!).

An important special case of separation is the case where A is convex and $B = \{a\}$, for some point, a, in A.

A "cute" application of Corollary 3.10 is one of the many versions of "Farkas Lemma" (1893-1894, 1902), a basic result in the theory of linear programming. For any vector, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and any real, $\alpha \in \mathbb{R}$, write $x \geq \alpha$ iff $x_i \geq \alpha$, for $i = 1, \ldots, n$.

The proof of Farkas Lemma Version I (Proposition 3.14) relies on the fact that a polyhedral cone cone$(a_1, \ldots, a_m)$ is closed. Although it seems obvious that a polyhedral cone should be closed, a rigorous proof is not entirely trivial.

Indeed, the fact that a polyhedral cone is closed relies crucially on the fact that C is spanned by a finite number of vectors, because the cone generated by an infinite set may not be closed. For example, consider the closed disk $D \subseteq \mathbb{R}^2$ of center $(0, 1)$ and radius 1, which is tangent to the x-axis at the origin. Then the cone(D) consists of the open upper half-plane *plus* the origin $(0, 0)$, but this set is not closed.

Proposition 3.13. *Every polyhedral cone C is closed.*

Proof. This is proved by showing that

1. Every primitive cone is closed.

2. A polyhedral cone C is the union of finitely many primitive cones, where a *primitive cone* is a polyhedral cone spanned by linearly independent vectors.

Assume that $(a_1, \ldots, a_m)$ are linearly independent vectors in $\mathbb{R}^n$, and consider any sequence $(x^{(k)})_{k \geq 0}$

$$x^{(k)} = \sum_{i=1}^{m} \lambda_i^{(k)} a_i$$

of vectors in the primitive cone cone($\{a_1, \ldots, a_m\}$), which means that $\lambda_j^{(k)} \geq 0$ for $i = 1, \ldots, m$ and all $k \geq 0$. The vectors $x^{(k)}$ belong to the subspace U spanned by $(a_1, \ldots, a_m)$, and U is closed. Assume that the sequence $(x^{(k)})_{k \geq 0}$ converges to a limit $x \in \mathbb{R}^n$. Since U is closed and $x^{(k)} \in U$ for all $k \geq 0$, we have $x \in U$. If we write $x = x_1 a_1 + \cdots + x_m a_m$, we would like to prove that $x_i \geq 0$ for $i = 1, \ldots, m$. The sequence the $(x^{(k)})_{k \geq 0}$ converges to x iff

$$\lim_{k \mapsto \infty} \left\| x^{(k)} - x \right\| = 0,$$

iff

$$\lim_{k \mapsto \infty} \left(\sum_{i=1}^{m} |\lambda_i^{(k)} - x_i|^2 \right)^{1/2} = 0$$

iff

$$\lim_{k \mapsto \infty} \lambda_i^{(k)} = x_i, \quad i = 1, \ldots, m.$$

Since $\lambda_i^{(k)} \geq 0$ for $i = 1, \ldots, m$ and all $k \geq 0$, we have $x_i \geq 0$ for $i = 1, \ldots, m$, so $x \in$ cone($\{a_1, \ldots, a_m\}$).

Next, assume that x belongs to the polyhedral cone C. Consider a positive combination

$$x = \lambda_1 a_1 + \cdots + \lambda_k a_k, \tag{$*_1$}$$

for some nonzero $a_1, \ldots, a_k \in C$, with $\lambda_i \geq 0$ and with k *minimal*. Since k is minimal, we must have $\lambda_i > 0$ for $i = 1, \ldots, k$. We claim that $(a_1, \ldots, a_k)$ are linearly independent.

If not, there is some nontrivial linear combination

$$\mu_1 a_1 + \cdots + \mu_k a_k = 0, \tag{$*_2$}$$

and since the a_i are nonzero, $\mu_j \neq 0$ for some at least some j. We may assume that $\mu_j < 0$ for some j (otherwise, we consider the family $(-\mu_i)_{1 \leq i \leq k}$), so let

$$J = \{j \in \{1, \ldots, k\} \mid \mu_j < 0\}.$$

For any $t \in \mathbb{R}$, since $x = \lambda_1 a_1 + \cdots + \lambda_k a_k$, using $(*_2)$ we get

$$x = (\lambda_1 + t\mu_1)a_1 + \cdots + (\lambda_k + t\mu_k)a_k, \tag{$*_3$}$$

and if we pick

$$t = \min_{j \in J} \left(-\frac{\lambda_j}{\mu_j} \right) \geq 0,$$

we have $(\lambda_i + t\mu_i) \geq 0$ for $i = 1, \ldots, k$, but $\lambda_j + t\mu_j = 0$ for some $j \in J$, so $(*_3)$ is an expression of x with less that k nonzero coefficients, contadicting the minimality of k in $(*_1)$. Therefore, $(a_1, \ldots, a_k)$ are linearly independent.

Since a polyhedral cone C is spanned by finitely many vectors, there are finitely many primitive cones (corresponding to linearly independent subfamilies), and since every $x \in C$, belongs to some primitive cone, C is the union of a finite number of primitive cones. Since every primitive cone is closed, as a union of finitely many closed sets, C itself is closed. $\quad\square$

Lemma 3.14. *(Farkas Lemma, Version I) Given any $d \times n$ real matrix, A, and any vector, $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:*

    (a) The linear system, $Ax = z$, has a solution, $x = (x_1, \ldots, x_n)$, such that $x \geq 0$ and $x_1 + \cdots + x_n = 1$, or

    (b) There is some $c \in \mathbb{R}^d$ and some $\alpha \in \mathbb{R}$ such that $c^\top z < \alpha$ and $c^\top A \geq \alpha$.

Proof. Let $A_1, \ldots, A_n \in \mathbb{R}^d$ be the n points corresponding to the columns of A. Then, either $z \in \mathrm{conv}(\{A_1, \ldots, A_n\})$ or $z \notin \mathrm{conv}(\{A_1, \ldots, A_n\})$. In the first case, we have a convex combination

$$z = x_1 A_1 + \cdots + x_n A_n$$

where $x_i \geq 0$ and $x_1 + \cdots + x_n = 1$, so $x = (x_1, \ldots, x_n)$ is a solution satisfying (a).

In the second case, by Corollary 3.10, there is a hyperplane, H, strictly separating $\{z\}$ and $\mathrm{conv}(\{A_1, \ldots, A_n\})$, which is closed by Proposition 3.13. In fact, observe that $z \notin \mathrm{conv}(\{A_1, \ldots, A_n\})$ iff there is a hyperplane, H, such that $z \in \overset{\circ}{H}_-$ and $A_i \in H_+$, or $z \in \overset{\circ}{H}_+$ and $A_i \in H_-$, for $i = 1, \ldots, n$. As the affine hyperplane, H, is the zero locus of an equation of the form

$$c_1 y_1 + \cdots + c_d y_d = \alpha,$$

either $c^\top z < \alpha$ and $c^\top A_i \geq \alpha$ for $i = 1, \ldots, n$, that is, $c^\top A \geq \alpha$, or $c^\top z > \alpha$ and $c^\top A \leq \alpha$. In the second case, $(-c)^\top z < -\alpha$ and $(-c)^\top A \geq -\alpha$, so (b) is satisfied by either c and α or by $-c$ and $-\alpha$. $\quad\square$

Remark: If we relax the requirements on solutions of $Ax = z$ and only require $x \geq 0$ ($x_1 + \cdots + x_n = 1$ is no longer required) then, in condition (b), we can take $\alpha = 0$. This is another version of Farkas Lemma. In this case, instead of considering the convex hull of $\{A_1, \ldots, A_n\}$ we are considering the convex cone,

$$\mathrm{cone}(A_1, \ldots, A_n) = \{\lambda A_1 + \cdots + \lambda_n A_n \mid \lambda_i \geq 0,\ 1 \leq i \leq n\},$$

that is, we are dropping the condition $\lambda_1 + \cdots + \lambda_n = 1$. For this version of Farkas Lemma we need the following separation lemma:

Proposition 3.15. *Let $C \subseteq \mathbb{E}^d$ be any closed convex cone with vertex O. Then, for every point, a, not in C, there is a hyperplane, H, passing through O separating a and C with $a \notin H$.*

Proof. Since C is closed and convex and $\{a\}$ is compact and convex, by Corollary 3.10, there is a hyperplane, H', strictly separating a and C. Let H be the hyperplane through O parallel to H'. Since C and a lie in the two disjoint open half-spaces determined by H', the point a cannot belong to H. Suppose that some point, $b \in C$, lies in the open half-space determined by H and a. Then, the line, L, through O and b intersects H' in some point, c, and as C is a cone, the half line determined by O and b is contained in C. So, $c \in C$ would belong to H', a contradiction. Therefore, C is contained in the closed half-space determined by H that does not contain a, as claimed. $\qquad\square$
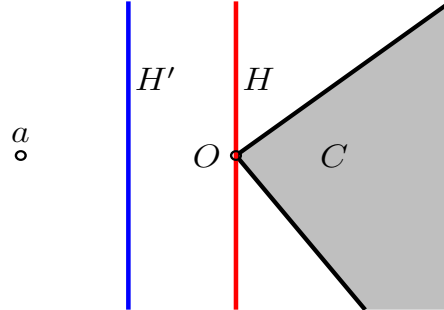
Figure 3.7: Illustration for the proof of Proposition 3.15

Lemma 3.16. *(Farkas Lemma, Version II) Given any $d \times n$ real matrix, A, and any vector, $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:*

(a) *The linear system, $Ax = z$, has a solution, x, such that $x \geq 0$, or*

(b) *There is some $c \in \mathbb{R}^d$ such that $c^\top z < 0$ and $c^\top A \geq 0$.*

Proof. The proof is analogous to the proof of Lemma 3.14 except that it uses Proposition 3.15 instead of Corollary 3.10 and either $z \in \mathrm{cone}(A_1, \ldots, A_n)$ or $z \notin \mathrm{cone}(A_1, \ldots, A_n)$. $\qquad\square$

One can show that Farkas II implies Farkas I. Here is another version of Farkas Lemma having to do with a system of inequalities, $Ax \leq z$. Although, this version may seem weaker that Farkas II, it is actually equivalent to it!

Lemma 3.17. *(Farkas Lemma, Version III) Given any $d \times n$ real matrix, A, and any vector, $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:*

(a) The system of inequalities, $Ax \leq z$, has a solution, x, or

(b) There is some $c \in \mathbb{R}^d$ such that $c \geq 0$, $c^\top z < 0$ and $c^\top A = 0$.

Proof. We use two tricks from linear programming:

1. We convert the system of inequalities, $Ax \leq z$, into a system of equations by introducing a vector of "slack variables", $\gamma = (\gamma_1, \ldots, \gamma_d)$, where the system of equations is

$$(A, I)\begin{pmatrix} x \\ \gamma \end{pmatrix} = z,$$

    with $\gamma \geq 0$.

2. We replace each "unconstrained variable", x_i, by $x_i = X_i - Y_i$, with $X_i, Y_i \geq 0$.

Then, the original system $Ax \leq z$ has a solution, x (unconstrained), iff the system of equations

$$(A, -A, I)\begin{pmatrix} X \\ Y \\ \gamma \end{pmatrix} = z$$

has a solution with $X, Y, \gamma \geq 0$. By Farkas II, this system has no solution iff there exists some $c \in \mathbb{R}^d$ with $c^\top z < 0$ and

$$c^\top (A, -A, I) \geq 0,$$

that is, $c^\top A \geq 0$, $-c^\top A \geq 0$, and $c \geq 0$. However, these four conditions reduce to $c^\top z < 0$, $c^\top A = 0$ and $c \geq 0$. $\qquad \square$

These versions of Farkas lemma are statements of the form $(P \vee Q) \wedge \neg(P \wedge Q)$, which is easily seen to be equivalent to $\neg P \equiv Q$, namely, the logical equivalence of $\neg P$ and Q. Therefore, Farkas-type lemmas can be interpreted as criteria for the unsolvablity of various kinds of systems of linear equations or systems of linear inequalities, in the form of a separation property.

For example, Farkas II (Lemma 3.16) says that a system of linear equations, $Ax = z$, does not have any solution, $x \geq 0$, iff there is some $c \in \mathbb{R}^d$ such that $c^\top z < 0$ and $c^\top A \geq 0$. This means that there is a hyperplane, H, of equation $c^\top y = 0$, such that the columns vectors, A_j, forming the matrix A all lie in the positive closed half space, H_+, but z lies in the interior of the other half space, H_-, determined by H. Therefore, z can't be in the cone spanned by the A_j's.

Farkas III says that a system of linear inequalities, $Ax \leq z$, does not have any solution (at all) iff there is some $c \in \mathbb{R}^d$ such that $c \geq 0$, $c^\top z < 0$ and $c^\top A = 0$. This time, there is also a hyperplane of equation $c^\top y = 0$, with $c \geq 0$, such that the columns vectors, A_j, forming the matrix A all lie in H but z lies in the interior of the half space, H_-, determined by H. In the "easy" direction, if there is such a vector c and some x satisfying $Ax \leq z$, since $c \geq 0$, we get $c^\top Ax \leq c^\top z$, but $c^\top Ax = 0$ and $c^\top z < 0$, a contradiction.

What is the crirerion for the insolvability of a system of inequalities $Ax \leq z$ with $x \geq 0$? This problem is equivalent to the insolvability of the set of inequalities

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \leq \begin{pmatrix} z \\ 0 \end{pmatrix}$$

and by Farkas III, this system has no solution iff there is some vector, (c_1, c_2), with $(c_1, c_2) \geq 0$,

$$(c_1^\top, c_2^\top) \begin{pmatrix} A \\ -I \end{pmatrix} = 0 \quad \text{and} \quad (c_1^\top, c_2^\top) \begin{pmatrix} z \\ 0 \end{pmatrix} < 0.$$

The above conditions are equivalent to $c_1 \geq 0$, $c_2 \geq 0$, $c_1^\top A - c_2^\top = 0$ and $c_1^\top z < 0$, which reduce to $c_1 \geq 0$, $c_1^\top A \geq 0$ and $c_1^\top z < 0$.

We can put all these versions together to prove the following version of Farkas lemma:

Lemma 3.18. *(Farkas Lemma, Version IIIb) For any $d \times n$ real matrix, A, and any vector, $z \in \mathbb{R}^d$, the following statements are equivalent:*

(1) *The system, $Ax = z$, has no solution $x \geq 0$ iff there is some $c \in \mathbb{R}^d$ such that $c^\top A \geq 0$ and $c^\top z < 0$.*

(2) *The system, $Ax \leq z$, has no solution iff there is some $c \in \mathbb{R}^d$ such that $c \geq 0$, $c^\top A = 0$ and $c^\top z < 0$.*

(3) *The system, $Ax \leq z$, has no solution $x \geq 0$ iff there is some $c \in \mathbb{R}^d$ such that $c \geq 0$, $c^\top A \geq 0$ and $c^\top z < 0$.*

Proof. We already proved that (1) implies (2) and that (2) implies (3). The proof that (3) implies (1) is left as an easy exercise. $\square$

The reader might wonder what is the criterion for the unsolvability of a system $Ax = z$, without any condition on x. However, since the unsolvability of the system $Ax = b$ is equivalent to the unsolvability of the system

$$\begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} z \\ -z \end{pmatrix},$$

using (2), the above system is unsolvable iff there is some $(c_1, c_2) \geq (0, 0)$ such that

$$(c_1^\top, c_2^\top) \begin{pmatrix} A \\ -A \end{pmatrix} = 0 \quad \text{and} \quad (c_1^\top, c_2^\top) \begin{pmatrix} z \\ -z \end{pmatrix} < 0,$$

and these are equivalent to $c_1^\top A - c_2^\top A = 0$ and $c_1^\top z - c_2^\top z < 0$, namely, $c^\top A = 0$ and $c^\top z < 0$ where $c = c_1 - c_2 \in \mathbb{R}^d$. However, this simply says that c is orthogonal to the columns $A^1, \ldots, A^n$ of A and that z is not orthogonal to c, so z cannot belong to the column space of A, a criterion that we already knew from linear algebra.

As in Matousek and Gartner [28], we can summarize these various criteria in the following table:

	The system $Ax \leq z$	The system $Ax = z$
has no solution $x \geq 0$ iff	$\exists c \in \mathbb{R}^d$, such that $c \geq 0$, $c^\top A \geq 0$ and $c^\top z < 0$	$\exists c \in \mathbb{R}^d$, such that $c^\top A \geq 0$ and $c^\top z < 0$
has no solution $x \in \mathbb{R}^n$ iff	$\exists c \in \mathbb{R}^d$, such that, $c \geq 0$, $c^\top A = 0$ and $c^\top z < 0$	$\exists c \in \mathbb{R}^d$, such that $c^\top A = 0$ and $c^\top z < 0$

Remark: The strong duality theorem in linear programming can be proved using Lemma 3.18(c).

3.2 Supporting Hyperplanes and Minkowski's Proposition

Recall the definition of a supporting hyperplane given in Definition 2.4. We have the following important proposition first proved by Minkowski (1896):

Proposition 3.19. *(Minkowski) Let A be a nonempty, closed, and convex subset. Then, for every point $a \in \partial A$, there is a supporting hyperplane to A through a.*

Proof. Let $d = \dim A$. If $d < \dim X$ (i.e., A has empty interior), then A is contained in some affine subspace V of dimension $d < \dim X$, and any hyperplane containing V is a supporting hyperplane for every $a \in A$. Now, assume $d = \dim X$, so that $\overset{\circ}{A} \neq \emptyset$. If $a \in \partial A$, then $\{a\} \cap \overset{\circ}{A} = \emptyset$. By Theorem 3.6, there is a hyperplane H separating $\overset{\circ}{A}$ and $L = \{a\}$. However, by Corollary 3.2, since $\overset{\circ}{A} \neq \emptyset$ and A is closed, we have

$$A = \overline{A} = \overline{\overset{\circ}{A}}.$$

Now, the half–space containing $\overset{\circ}{A}$ is closed, and thus, it contains $\overline{\overset{\circ}{A}} = A$. Therefore, H separates A and $\{a\}$. $\qquad\square$

Remark: The assumption that A is closed is convenient but unnecessary. Indeed, the proof of Proposition 3.19 shows that the proposition holds for every boundary point, $a \in \partial A$ (assuming $\partial A \neq \emptyset$).

Beware that Proposition 3.19 is false when the dimension of X is infinite and when $\overset{\circ}{A} = \emptyset$.

The proposition below gives a sufficient condition for a closed subset to be convex.

Proposition 3.20. *Let A be a closed subset with nonempty interior. If there is a supporting hyperplane for every point $a \in \partial A$, then A is convex.*

Proof. We leave it as an exercise (see Berger [6], Proposition 11.5.4). $\qquad\square$

The condition that A has nonempty interior is crucial!

The proposition below characterizes closed convex sets in terms of (closed) half–spaces. It is another intuitive fact whose rigorous proof is nontrivial.

Proposition 3.21. *Let A be a nonempty closed and convex subset. Then, A is the intersection of all the closed half–spaces containing it.*

Proof. Let A' be the intersection of all the closed half–spaces containing A. It is immediately checked that A' is closed and convex and that $A \subseteq A'$. Assume that $A' \neq A$, and pick $a \in A' - A$. Then, we can apply Corollary 3.10 to $\{a\}$ and A and we find a hyperplane, H, strictly separating A and $\{a\}$; this shows that A belongs to one of the two half-spaces determined by H, yet a does not belong to the same half-space, contradicting the definition of A'. $\qquad\square$

3.3   Polarity and Duality

Let $E = \mathbb{E}^n$ be a Euclidean space of dimension n. Pick any origin, O, in $\mathbb{E}^n$ (we may assume $O = (0, \ldots, 0)$). We know that the inner product on $E = \mathbb{E}^n$ induces a duality between E and its dual E^* (for example, see Chapter 6, Section 2 of Gallier [20]), namely, $u \mapsto \varphi_u$, where φ_u is the linear form defined by $\varphi_u(v) = u \cdot v$, for all $v \in E$. For geometric purposes, it is more convenient to recast this duality as a correspondence between points and hyperplanes, using the notion of polarity with respect to the unit sphere, $S^{n-1} = \{a \in \mathbb{E}^n \mid \|\mathbf{Oa}\| = 1\}$.

First, we need the following simple fact: For every hyperplane, H, not passing through O, there is a *unique* point, h, so that

$$H = \{a \in \mathbb{E}^n \mid \mathbf{Oh} \cdot \mathbf{Oa} = 1\}.$$

Indeed, any hyperplane, H, in $\mathbb{E}^n$ is the null set of some equation of the form

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = \beta,$$

and if $O \notin H$, then $\beta \neq 0$. Thus, any hyperplane, H, not passing through O is defined by an equation of the form

$$h_1 x_1 + \cdots + h_n x_n = 1,$$

if we set $h_i = \alpha_i/\beta$. So, if we let $h = (h_1, \ldots, h_n)$, we see that

$$H = \{a \in \mathbb{E}^n \mid \mathbf{Oh} \cdot \mathbf{Oa} = 1\},$$

as claimed. Now, assume that

$$H = \{a \in \mathbb{E}^n \mid \mathbf{Oh_1} \cdot \mathbf{Oa} = 1\} = \{a \in \mathbb{E}^n \mid \mathbf{Oh_2} \cdot \mathbf{Oa} = 1\}.$$

The functions $a \mapsto \mathbf{Oh_1} \cdot \mathbf{Oa} - 1$ and $a \mapsto \mathbf{Oh_2} \cdot \mathbf{Oa} - 1$ are two affine forms defining the same hyperplane, so there is a nonzero scalar, λ, so that

$$\mathbf{Oh_1} \cdot \mathbf{Oa} - 1 = \lambda(\mathbf{Oh_2} \cdot \mathbf{Oa} - 1) \quad \text{for all } a \in \mathbb{E}^n$$

(see Gallier [20], Chapter 2, Section 2.10). In particular, for $a = O$, we find that $\lambda = 1$, and so,

$$\mathbf{Oh_1} \cdot \mathbf{Oa} = \mathbf{Oh_2} \cdot \mathbf{Oa} \quad \text{for all } a,$$

which implies $h_1 = h_2$. This proves the uniqueness of h.

Using the above, we make the following definition:

Definition 3.2. Given any point, $a \neq O$, the *polar hyperplane of a (w.r.t. S^{n-1})* or *dual of* a is the hyperplane, $a^\dagger$, given by

$$a^\dagger = \{b \in \mathbb{E}^n \mid \mathbf{Oa} \cdot \mathbf{Ob} = 1\}.$$

Given a hyperplane, H, not containing O, the *pole of H (w.r.t S^{n-1})* or *dual of H* is the (unique) point, $H^\dagger$, so that

$$H = \{a \in \mathbb{E}^n \mid \mathbf{OH}^\dagger \cdot \mathbf{Oa} = 1\}.$$

We often abbreviate polar hyperplane to polar. We immediately check that $a^{\dagger\dagger} = a$ and $H^{\dagger\dagger} = H$, so, we obtain a bijective correspondence between $\mathbb{E}^n - \{O\}$ and the set of hyperplanes not passing through O.

When a is outside the sphere S^{n-1}, there is a nice geometric interpetation for the polar hyperplane, $H = a^\dagger$. Indeed, in this case, since

$$H = a^\dagger = \{b \in \mathbb{E}^n \mid \mathbf{Oa} \cdot \mathbf{Ob} = 1\}$$

and $\|\mathbf{Oa}\| > 1$, the hyperplane H intersects S^{n-1} (along an $(n-2)$-dimensional sphere) and if b is any point on $H \cap S^{n-1}$, we claim that $\mathbf{Ob}$ and $\mathbf{ba}$ are orthogonal. This means that $H \cap S^{n-1}$ is the set of points on S^{n-1} where the lines through a and tangent to S^{n-1} touch S^{n-1} (they form a cone tangent to S^{n-1} with apex a). Indeed, as $\mathbf{Oa} = \mathbf{Ob} + \mathbf{ba}$ and $b \in H \cap S^{n-1}$ i.e., $\mathbf{Oa} \cdot \mathbf{Ob} = 1$ and $\|\mathbf{Ob}\|^2 = 1$, we get

$$1 = \mathbf{Oa} \cdot \mathbf{Ob} = (\mathbf{Ob} + \mathbf{ba}) \cdot \mathbf{Ob} = \|\mathbf{Ob}\|^2 + \mathbf{ba} \cdot \mathbf{Ob} = 1 + \mathbf{ba} \cdot \mathbf{Ob},$$
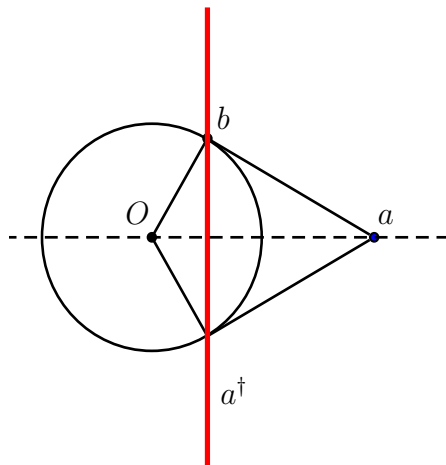
Figure 3.8: The polar, $a^\dagger$, of a point, a, outside the sphere S^{n-1}

which implies $\mathbf{ba} \cdot \mathbf{Ob} = 0$. When $a \in S^{n-1}$, the hyperplane $a^\dagger$ is tangent to S^{n-1} at a.

Also, observe that for any point, $a \neq O$, and any hyperplane, H, not passing through O, if $a \in H$, then, $H^\dagger \in a^\dagger$, i.e, the pole, $H^\dagger$, of H belongs to the polar, $a^\dagger$, of a. Indeed, $H^\dagger$ is the unique point so that

$$H = \{b \in \mathbb{E}^n \mid \mathbf{OH}^\dagger \cdot \mathbf{Ob} = 1\}$$

and

$$a^\dagger = \{b \in \mathbb{E}^n \mid \mathbf{Oa} \cdot \mathbf{Ob} = 1\};$$

since $a \in H$, we have $\mathbf{OH}^\dagger \cdot \mathbf{Oa} = 1$, which shows that $H^\dagger \in a^\dagger$.

If $a = (a_1, \ldots, a_n)$, the equation of the polar hyperplane, $a^\dagger$, is

$$a_1 X_1 + \cdots + a_n X_n = 1.$$

Remark: As we noted, polarity in a Euclidean space suffers from the minor defect that the polar of the origin is undefined and, similarly, the pole of a hyperplane through the origin does not make sense. If we embed $\mathbb{E}^n$ into the projective space, $\mathbb{P}^n$, by adding a "hyperplane at infinity" (a copy of $\mathbb{P}^{n-1}$), thereby viewing $\mathbb{P}^n$ as the disjoint union $\mathbb{P}^n = \mathbb{E}^n \cup \mathbb{P}^{n-1}$, then the polarity correspondence can be defined everywhere. Indeed, the polar of the origin is the hyperplane at infinity ($\mathbb{P}^{n-1}$) and since $\mathbb{P}^{n-1}$ can be viewed as the set of hyperplanes through the origin in $\mathbb{E}^n$, the pole of a hyperplane through the origin is the corresponding "point at infinity" in $\mathbb{P}^{n-1}$.

Now, we would like to extend this correspondence to subsets of $\mathbb{E}^n$, in particular, to convex sets. Given a hyperplane, H, not containing O, we denote by H_- the closed half-space containing O.

Definition 3.3. Given any subset, A, of $\mathbb{E}^n$, the set

$$A^* = \{b \in \mathbb{E}^n \mid \mathbf{Oa} \cdot \mathbf{Ob} \leq 1, \quad \text{for all } a \in A\} = \bigcap_{\substack{a \in A \\ a \neq O}} (a^\dagger)_-,$$

is called the *polar dual* or *reciprocal* of A.

For simplicity of notation, we write $a^\dagger_-$ for $(a^\dagger)_-$. Observe that $\{O\}^* = \mathbb{E}^n$, so it is convenient to set $O^\dagger_- = \mathbb{E}^n$, even though $O^\dagger$ is undefined. By definition, A^* is convex even if A is not. Furthermore, note that

(1) $A \subseteq A^{**}$.

(2) If $A \subseteq B$, then $B^* \subseteq A^*$.

(3) If A is convex and closed, then $A^* = (\partial A)^*$.

It follows immediately from (1) and (2) that $A^{***} = A^*$. Also, if $B^n(r)$ is the (closed) ball of radius $r > 0$ and center O, it is obvious by definition that $B^n(r)^* = B^n(1/r)$.

In Figure 3.9, the polar dual of the polygon $(v_1, v_2, v_3, v_4, v_5)$ is the polygon shown in green. This polygon is cut out by the half-planes determined by the polars of the vertices $(v_1, v_2, v_3, v_4, v_5)$ and containing the center of the circle. These polar lines are all easy to determine by drawing for each vertex, v_i, the tangent lines to the circle and joining the contact points. The construction of the polar of v_3 is shown in detail.

Remark: We chose a different notation for polar hyperplanes and polars ($a^\dagger$ and $H^\dagger$) and polar duals (A^*), to avoid the potential confusion between $H^\dagger$ and H^*, where H is a hyperplane (or $a^\dagger$ and $\{a\}^*$, where a is a point). Indeed, they are completely different! For example, the polar dual of a hyperplane is either a line orthogonal to H through O, if $O \in H$, or a semi-infinite line through O and orthogonal to H whose endpoint is the pole, $H^\dagger$, of H, whereas, $H^\dagger$ is a single point! Ziegler ([45], Chapter 2) use the notation $A^\triangle$ instead of A^* for the polar dual of A.

We would like to investigate the duality induced by the operation $A \mapsto A^*$. Unfortunately, it is not always the case that $A^{**} = A$, but this is true when A is closed and convex, as shown in the following proposition:

Proposition 3.22. *Let A be any subset of $\mathbb{E}^n$ (with origin O).*

(i) *If A is bounded, then $O \in \overset{\circ}{A}^*$; if $O \in \overset{\circ}{A}$, then A^* is bounded.*

(ii) *If A is a closed and convex subset containing O, then $A^{**} = A$.*
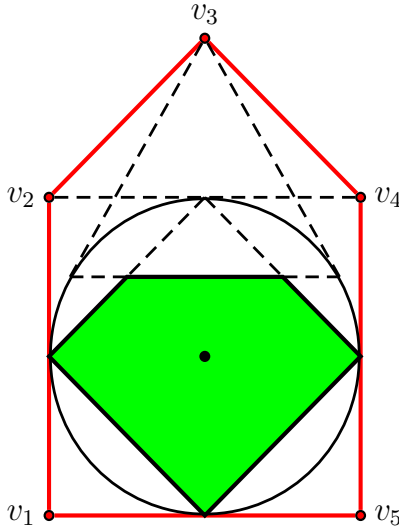
Figure 3.9: The polar dual of a polygon

Proof. (i) If A is bounded, then $A \subseteq B^n(r)$ for some $r > 0$ large enough. Then, $B^n(r)^* = B^n(1/r) \subseteq A^*$, so that $O \in \mathring{A}^*$. If $O \in \mathring{A}$, then $B^n(r) \subseteq A$ for some r small enough, so $A^* \subseteq B^n(r)^* = B^r(1/r)$ and A^* is bounded.

(ii) We always have $A \subseteq A^{**}$. We prove that if $b \notin A$, then $b \notin A^{**}$; this shows that $A^{**} \subseteq A$ and thus, $A = A^{**}$. Since A is closed and convex and $\{b\}$ is compact (and convex!), by Corollary 3.10, there is a hyperplane, H, strictly separating A and b and, in particular, $O \notin H$, as $O \in A$. If $h = H^\dagger$ is the pole of H, we have

$$\mathbf{Oh} \cdot \mathbf{Ob} > 1 \quad \text{and} \quad \mathbf{Oh} \cdot \mathbf{Oa} < 1, \quad \text{for all } a \in A$$

since $H_- = \{a \in \mathbb{E}^n \mid \mathbf{Oh} \cdot \mathbf{Oa} \leq 1\}$. This shows that $b \notin A^{**}$, since

$$
\begin{aligned}
A^{**} &= \{c \in \mathbb{E}^n \mid \mathbf{Od} \cdot \mathbf{Oc} \leq 1 \quad \text{for all } d \in A^*\} \\
&= \{c \in \mathbb{E}^n \mid (\forall d \in \mathbb{E}^n)(\text{if} \quad \mathbf{Od} \cdot \mathbf{Oa} \leq 1 \quad \text{for all } a \in A, \quad \text{then} \quad \mathbf{Od} \cdot \mathbf{Oc} \leq 1)\},
\end{aligned}
$$

just let $c = b$ and $d = h$. $\square$

Remark: For an arbitrary subset, $A \subseteq \mathbb{E}^n$, it can be shown that $A^{**} = \overline{\text{conv}(A \cup \{O\})}$, the topological closure of the convex hull of $A \cup \{O\}$.

Proposition 3.22 will play a key role in studying polytopes, but before doing this, we need one more proposition.

Proposition 3.23. *Let A be any closed convex subset of $\mathbb{E}^n$ such that $O \in \overset{\circ}{A}$. The polar hyperplanes of the points of the boundary of A constitute the set of supporting hyperplanes of A^*. Furthermore, for any $a \in \partial A$, the points of A^* where $H = a^\dagger$ is a supporting hyperplane of A^* are the poles of supporting hyperplanes of A at a.*

Proof. Since $O \in \overset{\circ}{A}$, we have $O \notin \partial A$, and so, for every $a \in \partial A$, the polar hyperplane $a^\dagger$ is well-defined. Pick any $a \in \partial A$ and let $H = a^\dagger$ be its polar hyperplane. By definition, $A^* \subseteq H_-$, the closed half-space determined by H and containing O. If T is any supporting hyperplane to A at a, as $a \in T$, we have $t = T^\dagger \in a^\dagger = H$. Furthermore, it is a simple exercise to prove that $t \in (T_-)^*$ (in fact, $(T_-)^*$ is the interval with endpoints O and t). Since $A \subseteq T_-$ (because T is a supporting hyperplane to A at a), we deduce that $t \in A^*$, and thus, H is a supporting hyperplane to A^* at t. By Proposition 3.22, as A is closed and convex, $A^{**} = A$; it follows that all supporting hyperplanes to A^* are indeed obtained this way. $\square$

Chapter 4

Polyhedra and Polytopes

4.1 Polyhedra, $\mathcal{H}$-Polytopes and $\mathcal{V}$-Polytopes

There are two natural ways to define a convex polyhedron, A:

(1) As the convex hull of a finite set of points.

(2) As a subset of $\mathbb{E}^n$ cut out by a finite number of hyperplanes, more precisely, as the intersection of a finite number of (closed) half-spaces.

As stated, these two definitions are not equivalent because (1) implies that a polyhedron is bounded, whereas (2) allows unbounded subsets. Now, if we require in (2) that the convex set A is bounded, it is quite clear for $n = 2$ that the two definitions (1) and (2) are equivalent; for $n = 3$, it is intuitively clear that definitions (1) and (2) are still equivalent, but proving this equivalence rigorously does not appear to be that easy. What about the equivalence when $n \geq 4$?

It turns out that definitions (1) and (2) are equivalent for all n, but this is a nontrivial theorem and a rigorous proof does not come by so cheaply. Fortunately, since we have Krein and Milman's theorem at our disposal and polar duality, we can give a rather short proof. The hard direction of the equivalence consists in proving that definition (1) implies definition (2). This is where the duality induced by polarity becomes handy, especially, the fact that $A^{**} = A$! (under the right hypotheses). First, we give precise definitions (following Ziegler [45]).

Definition 4.1. Let $\mathcal{E}$ be any affine Euclidean space of finite dimension, n.[1] An $\mathcal{H}$-*polyhedron* in $\mathcal{E}$, for short, a *polyhedron*, is any subset, $P = \bigcap_{i=1}^{p} C_i$, of $\mathcal{E}$ defined as the intersection of a finite number, $p \geq 1$, of closed half-spaces, C_i; an $\mathcal{H}$-*polytope* in $\mathcal{E}$ is a bounded polyhedron and a $\mathcal{V}$-*polytope* is the convex hull, $P = \mathrm{conv}(S)$, of a finite set of points, $S \subseteq \mathcal{E}$.

---

[1]This means that the vector space, $\overrightarrow{\mathcal{E}}$, associated with $\mathcal{E}$ is a Euclidean space.
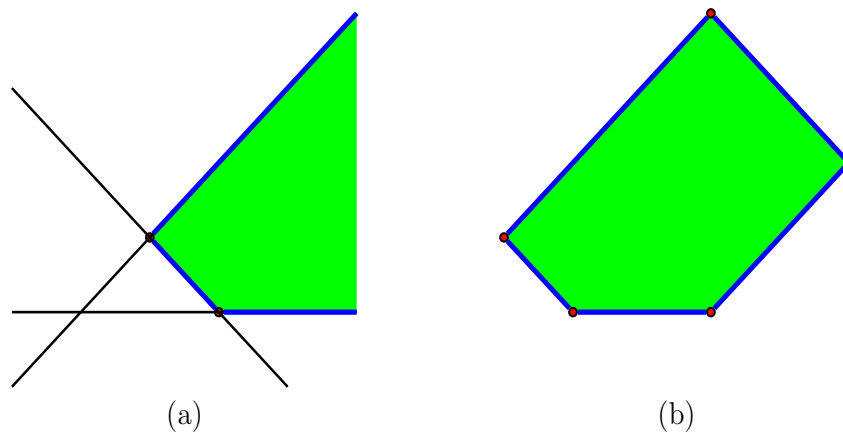
Figure 4.1: (a) An $\mathcal{H}$-polyhedron. (b) A $\mathcal{V}$-polytope

Obviously, polyhedra and polytopes are convex and closed (in $\mathcal{E}$). Since the notions of $\mathcal{H}$-polytope and $\mathcal{V}$-polytope are equivalent (see Theorem 4.7), we often use the simpler locution polytope. Examples of an $\mathcal{H}$-polyhedron and of a $\mathcal{V}$-polytope are shown in Figure 4.1.

Note that Definition 4.1 allows $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes to have an empty interior, which is somewhat of an inconvenience. This is not a problem, since we may always restrict ourselves to the affine hull of P (some affine space, E, of dimension $d \leq n$, where $d = \dim(P)$, as in Definition 2.1) as we now show.

Proposition 4.1. *Let $A \subseteq \mathcal{E}$ be a $\mathcal{V}$-polytope or an $\mathcal{H}$-polyhedron, let $E = \mathrm{aff}(A)$ be the affine hull of A in $\mathcal{E}$ (with the Euclidean structure on E induced by the Euclidean structure on $\mathcal{E}$) and write $d = \dim(E)$. Then, the following assertions hold:*

  (1) The set, A, is a $\mathcal{V}$-polytope in E (i.e., viewed as a subset of E) iff A is a $\mathcal{V}$-polytope in $\mathcal{E}$.

  (2) The set, A, is an $\mathcal{H}$-polyhedron in E (i.e., viewed as a subset of E) iff A is an $\mathcal{H}$-polyhedron in $\mathcal{E}$.

Proof. (1) This follows immediately because E is an affine subspace of $\mathcal{E}$ and every affine subspace of $\mathcal{E}$ is closed under affine combinations and so, *a fortiori*, under convex combinations. We leave the details as an easy exercise.

(2) Assume A is an $\mathcal{H}$-polyhedron in $\mathcal{E}$ and that $d < n$. By definition, $A = \bigcap_{i=1}^{p} C_i$, where the C_i are closed half-spaces determined by some hyperplanes, $H_1, \ldots, H_p$, in $\mathcal{E}$. (Observe that the hyperplanes, H_i's, associated with the closed half-spaces, C_i, may not be distinct. For example, we may have $C_i = (H_i)_+$ and $C_j = (H_i)_-$, for the two closed half-spaces

determined by H_i.) As $A \subseteq E$, we have

$$A = A \cap E = \bigcap_{i=1}^{p}(C_i \cap E),$$

where $C_i \cap E$ is one of the closed half-spaces determined by the hyperplane, $H_i' = H_i \cap E$, in E. Thus, A is also an $\mathcal{H}$-polyhedron in E.

Conversely, assume that A is an $\mathcal{H}$-polyhedron in E and that $d < n$. As any hyperplane, H, in $\mathcal{E}$ can be written as the intersection, $H = H_- \cap H_+$, of the two closed half-spaces that it bounds, E itself can be written as the intersection,

$$E = \bigcap_{i=1}^{p} E_i = \bigcap_{i=1}^{p}(E_i)_+ \cap (E_i)_-,$$

of finitely many half-spaces in $\mathcal{E}$. Now, as A is an $\mathcal{H}$-polyhedron in E, we have

$$A = \bigcap_{j=1}^{q} C_j,$$

where the C_j are closed half-spaces in E determined by some hyperplanes, H_j, in E. However, each H_j can be extended to a hyperplane, H_j', in $\mathcal{E}$, and so, each C_j can be extended to a closed half-space, C_j', in $\mathcal{E}$ and we still have

$$A = \bigcap_{j=1}^{q} C_j'.$$

Consequently, we get

$$A = A \cap E = \bigcap_{i=1}^{p}((E_i)_+ \cap (E_i)_-) \cap \bigcap_{j=1}^{q} C_j',$$

which proves that A is also an $\mathcal{H}$-polyhedron in $\mathcal{E}$. $\square$

The following simple proposition shows that we may assume that $\mathcal{E} = \mathbb{E}^n$:

Proposition 4.2. *Given any two affine Euclidean spaces, E and F, if $h\colon E \to F$ is any affine map then:*

(1) *If A is any $\mathcal{V}$-polytope in E, then $h(E)$ is a $\mathcal{V}$-polytope in F.*

(2) *If h is bijective and A is any $\mathcal{H}$-polyhedron in E, then $h(E)$ is an $\mathcal{H}$-polyhedron in F.*

Proof. (1) As any affine map preserves affine combinations it also preserves convex combination. Thus, $h(\mathrm{conv}(S)) = \mathrm{conv}(h(S))$, for any $S \subseteq E$.

(2) Say $A = \bigcap_{i=1}^{p} C_i$ in E. Consider any half-space, C, in E and assume that

$$C = \{x \in E \mid \varphi(x) \leq 0\},$$

for some affine form, φ, defining the hyperplane, $H = \{x \in E \mid \varphi(x) = 0\}$. Then, as h is bijective, we get

$$
\begin{aligned}
h(C) &= \{h(x) \in F \mid \varphi(x) \leq 0\} \\
&= \{y \in F \mid \varphi(h^{-1}(y)) \leq 0\} \\
&= \{y \in F \mid (\varphi \circ h^{-1})(y) \leq 0\}.
\end{aligned}
$$

This shows that $h(C)$ is one of the closed half-spaces in F determined by the hyperplane, $H' = \{y \in F \mid (\varphi \circ h^{-1})(y) = 0\}$. Furthermore, as h is bijective, it preserves intersections so

$$h(A) = h\left(\bigcap_{i=1}^{p} C_i\right) = \bigcap_{i=1}^{p} h(C_i),$$

a finite intersection of closed half-spaces. Therefore, $h(A)$ is an $\mathcal{H}$-polyhedron in F.   □

By Proposition 4.2 we may assume that $\mathcal{E} = \mathbb{E}^d$ and by Proposition 4.1 we may assume that $\dim(A) = d$. These propositions justify the type of argument beginning with: "We may assume that $A \subseteq \mathbb{E}^d$ has dimension d, that is, that A has nonempty interior". This kind of reasonning will occur many times.

Since the boundary of a closed half-space, C_i, is a hyperplane, H_i, and since hyperplanes are defined by affine forms, a closed half-space is defined by the locus of points satisfying a "linear" inequality of the form $a_i \cdot x \leq b_i$ or $a_i \cdot x \geq b_i$, for some vector $a_i \in \mathbb{R}^n$ and some $b_i \in \mathbb{R}$. Since $a_i \cdot x \geq b_i$ is equivalent to $(-a_i) \cdot x \leq -b_i$, we may restrict our attention to inequalities with a $\leq$ sign. Thus, if A is the $p \times n$ matrix whose i^{th} row is a_i, we see that the $\mathcal{H}$-polyhedron, P, is defined by the system of linear inequalities, $Ax \leq b$, where $b = (b_1, \ldots, b_p) \in \mathbb{R}^p$. We write

$$P = P(A, b), \quad with \quad P(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

An equation, $a_i \cdot x = b_i$, may be handled as the conjunction of the two inequalities $a_i \cdot x \leq b_i$ and $(-a_i) \cdot x \leq -b_i$. Also, if $0 \in P$, observe that we must have $b_i \geq 0$ for $i = 1, \ldots, p$. In this case, every inequality for which $b_i > 0$ can be normalized by dividing both sides by b_i, so we may assume that $b_i = 1$ or $b_i = 0$. This observation will be useful to show that the polar dual of an $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.

Remark: Some authors call "convex" polyhedra and "convex" polytopes what we have simply called polyhedra and polytopes. Since Definition 4.1 implies that these objects are
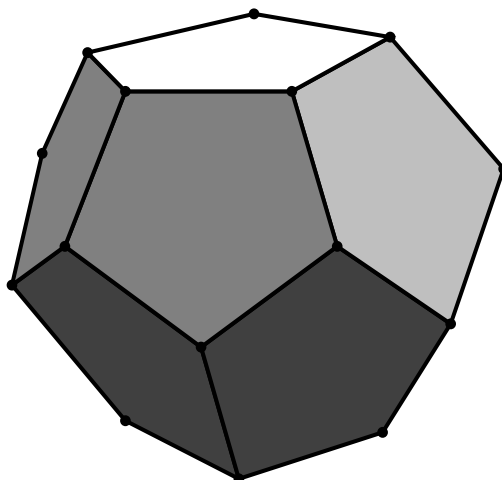
Figure 4.2: Example of a polytope (a dodecahedron)

convex and since we are not going to consider non-convex polyhedra in this chapter, we stick to the simpler terminology.

One should consult Ziegler [45], Berger [6], Grunbaum [24] and especially Cromwell [14], for pictures of polyhedra and polytopes. Figure 4.2 shows the picture a polytope whose faces are all pentagons. This polytope is called a *dodecahedron*. The dodecahedron has 12 faces, 30 edges and 20 vertices.

Even better and a lot more entertaining, take a look at the spectacular web sites of George Hart,

Virtual Polyedra: http://www.georgehart.com/virtual-polyhedra/vp.html,

George Hart's web site: http://www.georgehart.com/

and also

Zvi Har'El's web site: http://www.math.technion.ac.il/ rl/

The *Uniform Polyhedra* web site: http://www.mathconsult.ch/showroom/unipoly/

Paper Models of Polyhedra: http://www.korthalsaltes.com/

Bulatov's *Polyhedra Collection*: http://www.physics.orst.edu/ bulatov/polyhedra/

Paul Getty's *Polyhedral Solids*: http://home.teleport.com/ tpgettys/poly.shtml

Jill Britton's *Polyhedra Pastimes*: http://ccins.camosun.bc.ca/ jbritton/jbpolyhedra.htm

and many other web sites dealing with polyhedra in one way or another by searching for "polyhedra" on *Google*!

Obviously, an n-simplex is a $\mathcal{V}$-polytope. The *standard n-cube* is the set

$$\{(x_1,\ldots,x_n) \in \mathbb{E}^n \mid |x_i| \leq 1, \quad 1 \leq i \leq n\}.$$

The standard cube is a $\mathcal{V}$-polytope. The *standard n-cross-polytope* (or *n-co-cube*) is the set

$$\{(x_1,\ldots,x_n) \in \mathbb{E}^n \mid \sum_{i=1}^{n} |x_i| \leq 1\}.$$

It is also a $\mathcal{V}$-polytope.

What happens if we take the dual of a $\mathcal{V}$-polytope (resp. an $\mathcal{H}$-polytope)? The following proposition, although very simple, is an important step in answering the above question:

Proposition 4.3. *Let $S = \{a_i\}_{i=1}^p$ be a finite set of points in $\mathbb{E}^n$ and let $A = \mathrm{conv}(S)$ be its convex hull. If $S \neq \{O\}$, then, the dual, A^*, of A w.r.t. the center O is the $\mathcal{H}$-polyhedron given by*

$$A^* = \bigcap_{i=1}^{p} (a_i^\dagger)_-.$$

Furthermore, if $O \in \overset{\circ}{A}$, then A^* is an $\mathcal{H}$-polytope, i.e., the dual of a $\mathcal{V}$-polytope with nonempty interior is an $\mathcal{H}$-polytope. If $A = S = \{O\}$, then $A^* = \mathbb{E}^d$.

Proof. By definition, we have

$$A^* = \{b \in \mathbb{E}^n \mid \mathbf{Ob} \cdot (\sum_{j=1}^{p} \lambda_j \mathbf{Oa_j}) \leq 1, \quad \lambda_j \geq 0, \sum_{j=1}^{p} \lambda_j = 1\},$$

and the right hand side is clearly equal to $\bigcap_{i=1}^{p}\{b \in \mathbb{E}^n \mid \mathbf{Ob} \cdot \mathbf{Oa_i} \leq 1\} = \bigcap_{i=1}^{p} (a_i^\dagger)_-$, which is a polyhedron. (Recall that $(a_i^\dagger)_- = \mathbb{E}^n$ if $a_i = O$.) If $O \in \overset{\circ}{A}$, then A^* is bounded (by Proposition 3.22) and so, A^* is an $\mathcal{H}$-polytope. $\qquad\square$

Thus, the dual of the convex hull of a finite set of points, $\{a_1,\ldots,a_p\}$, is the intersection of the half-spaces containing O determined by the polar hyperplanes of the points a_i.

It is convenient to restate Proposition 4.3 using matrices. First, observe that the proof of Proposition 4.3 shows that

$$\mathrm{conv}(\{a_1,\ldots,a_p\})^* = \mathrm{conv}(\{a_1,\ldots,a_p\} \cup \{O\})^*.$$

Therefore, we may assume that not all $a_i = O$ ($1 \leq i \leq p$). If we pick O as an origin, then every point a_j can be identified with a vector in $\mathbb{E}^n$ and O corresponds to the zero vector, 0. Observe that any set of p points, $a_j \in \mathbb{E}^n$, corresponds to the $n \times p$ matrix, A, whose j^{th}

column is a_j. Then, the equation of the the polar hyperplane, $a_j^\dagger$, of any $a_j (\neq 0)$ is $a_j \cdot x = 1$, that is

$$a_j^\top x = 1.$$

Consequently, the system of inequalities defining $\mathrm{conv}(\{a_1, \ldots, a_p\})^*$ can be written in matrix form as

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\},$$

where $\mathbf{1}$ denotes the vector of $\mathbb{R}^p$ with all coordinates equal to 1. We write $P(A^\top, \mathbf{1}) = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\}$. There is a useful converse of this property as proved in the next proposition.

Proposition 4.4. *Given any set of p points, $\{a_1, \ldots, a_p\}$, in $\mathbb{R}^n$ with $\{a_1, \ldots, a_p\} \neq \{0\}$, if A is the $n \times p$ matrix whose j^{th} column is a_j, then*

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = P(A^\top, \mathbf{1}),$$

with $P(A^\top, \mathbf{1}) = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\}$.

Conversely, given any $p \times n$ matrix, A, not equal to the zero matrix, we have

$$P(A, \mathbf{1})^* = \mathrm{conv}(\{a_1, \ldots, a_p\} \cup \{0\}),$$

where $a_i \in \mathbb{R}^n$ is the ith row of A or, equivalently,

$$P(A, \mathbf{1})^* = \{x \in \mathbb{R}^n \mid x = A^\top t, \ t \in \mathbb{R}^p, \ t \geq 0, \ \mathbb{I}t = 1\},$$

where $\mathbb{I}$ is the row vector of length p whose coordinates are all equal to 1.

Proof. Only the second part needs a proof. Let $B = \mathrm{conv}(\{a_1, \ldots, a_p\} \cup \{0\})$, where $a_i \in \mathbb{R}^n$ is the ith *row* of A. Then, by the first part,

$$B^* = P(A, \mathbf{1}).$$

As $0 \in B$, by Proposition 3.22, we have $B = B^{**} = P(A, \mathbf{1})^*$, as claimed. □

Remark: Proposition 4.4 still holds if A is the zero matrix because then, the inequalities $A^\top x \leq \mathbf{1}$ (or $Ax \leq \mathbf{1}$) are trivially satisfied. In the first case, $P(A^\top, \mathbf{1}) = \mathbb{E}^d$ and in the second case, $P(A, \mathbf{1}) = \mathbb{E}^d$.

Using the above, the reader should check that the dual of a simplex is a simplex and that the dual of an n-cube is an n-cross polytope.

Observe that not every $\mathcal{H}$-polyhedron is of the form $P(A, \mathbf{1})$. Firstly, 0 belongs to the interior of $P(A, \mathbf{1})$ and, secondly cones with apex 0 can't be described in this form. However, we will see in Section 4.3 that the full class of polyhedra can be captured is we allow inequalities of the form $a^\top x \leq 0$. In order to find the corresponding "$\mathcal{V}$-definition" we will

need to add positive combinations of vectors to convex combinations of points. Intuitively, these vectors correspond to "points at infinity".

We will see shortly that if A is an $\mathcal{H}$-polytope and if $O \in \overset{\circ}{A}$, then A^* is also an $\mathcal{H}$-polytope. For this, we will prove first that an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. This requires taking a closer look at polyhedra.

Note that some of the hyperplanes cutting out a polyhedron may be redundant. If $A = \bigcap_{i=1}^{t} C_i$ is a polyhedron (where each closed half-space, C_i, is associated with a hyperplane, H_i, so that $\partial C_i = H_i$), we say that $\bigcap_{i=1}^{t} C_i$ is an *irredundant decomposition of A* if A cannot be expressed as $A = \bigcap_{i=1}^{m} C_i'$ with $m < t$ (for some closed half-spaces, C_i'). The following proposition shows that the C_i in an irredundant decomposition of A are uniquely determined by A.

Proposition 4.5. *Let A be a polyhedron with nonempty interior and assume that $A = \bigcap_{i=1}^{t} C_i$ is an irredundant decomposition of A. Then,*

(i) *Up to order, the C_i's are uniquely determined by A.*

(ii) *If $H_i = \partial C_i$ is the boundary of C_i, then $H_i \cap A$ is a polyhedron with nonempty interior in H_i, denoted $\mathrm{Facet}_i\, A$, and called a* **facet** *of A.*

(iii) *We have $\partial A = \bigcup_{i=1}^{t} \mathrm{Facet}_i\, A$, where the union is irredundant, i.e., $\mathrm{Facet}_i\, A$ is not a subset of $\mathrm{Facet}_j\, A$, for all $i \neq j$.*

Proof. (ii) Fix any i and consider $A_i = \bigcap_{j \neq i} C_j$. As $A = \bigcap_{i=1}^{t} C_i$ is an irredundant decomposition, there is some $x \in A_i - C_i$. Pick any $a \in \overset{\circ}{A}$. By Lemma 3.1, we get $b = [a, x] \cap H_i \in \overset{\circ}{A}_i$, so b belongs to the interior of $H_i \cap A_i$ in H_i.

(iii) As $\partial A = A - \overset{\circ}{A} = A \cap (\overset{\circ}{A})^c$ (where B^c denotes the complement of a subset B of $\mathbb{E}^n$)

and $\partial C_i = H_i$, we get

$$
\begin{aligned}
\partial A &= \left(\bigcap_{i=1}^{t} C_i \right) - \left(\bigcap_{j=1}^{t} C_j \right)^{\circ} \\
&= \left(\bigcap_{i=1}^{t} C_i \right) - \left(\bigcap_{j=1}^{t} \mathring{C}_j \right) \\
&= \left(\bigcap_{i=1}^{t} C_i \right) \cap \left(\bigcap_{j=1}^{t} \mathring{C}_j \right)^{c} \\
&= \left(\bigcap_{i=1}^{t} C_i \right) \cap \left(\bigcup_{j=1}^{t} (\mathring{C}_j)^{c} \right) \\
&= \bigcup_{j=1}^{t} \left(\left(\bigcap_{i=1}^{t} C_i \right) \cap (\mathring{C}_j)^{c} \right) \\
&= \bigcup_{j=1}^{t} \left(\partial C_j \cap \left(\bigcap_{i \neq j} C_i \right) \right) \\
&= \bigcup_{j=1}^{t} (H_j \cap A) = \bigcup_{j=1}^{t} \text{Facet}_j\, A.
\end{aligned}
$$

If we had $\text{Facet}_i\, A \subseteq \text{Facet}_j\, A$, for some $i \neq j$, then, by (ii), as the affine hull of $\text{Facet}_i\, A$ is H_i and the affine hull of $\text{Facet}_j\, A$ is H_j, we would have $H_i \subseteq H_j$, a contradiction.

(i) As the decomposition is irredundant, the H_i are pairwise distinct. Also, by (ii), each facet, $\text{Facet}_i\, A$, has dimension $d-1$ (where $d = \dim A$). Then, in (iii), we can show that the decomposition of ∂A as a union of polytopes of dimension $d-1$ whose pairwise nonempty intersections have dimension at most $d-2$ (since they are contained in pairwise distinct hyperplanes) is unique up to permutation. Indeed, assume that

$$\partial A = F_1 \cup \cdots \cup F_m = G_1 \cup \cdots \cup G_n,$$

where the F_i's and G'_j are polyhedra of dimension $d-1$ and each of the unions is irredundant. Then, we claim that for each F_i, there is some $G_{\varphi(i)}$ such that $F_i \subseteq G_{\varphi(i)}$. If not, F_i would be expressed as a union

$$F_i = (F_i \cap G_{i_1}) \cup \cdots \cup (F_i \cap G_{i_k})$$

where $\dim(F_i \cap G_{i_j}) \leq d-2$, since the hyperplanes containing F_i and the G_j's are pairwise distinct, which is absurd, since $\dim(F_i) = d-1$. By symmetry, for each G_j, there is some $F_{\psi(j)}$ such that $G_j \subseteq F_{\psi(j)}$. But then, $F_i \subseteq F_{\psi(\varphi(i))}$ for all i and $G_j \subseteq G_{\varphi(\psi(j))}$ for all j which implies $\psi(\varphi(i)) = i$ for all i and $\varphi(\psi(j)) = j$ for all j since the unions are irredundant. Thus, φ and ψ are mutual inverses and the B_j's are just a permutation of the A_i's, as claimed.

Therefore, the facets, $\text{Facet}_i\, A$, are uniquely determined by A and so are the hyperplanes, $H_i = \text{aff}(\text{Facet}_i\, A)$, and the half-spaces, C_i, that they determine. $\qquad\qquad\square$

As a consequence, if A is a polyhedron, then so are its facets and the same holds for $\mathcal{H}$-polytopes. If A is an $\mathcal{H}$-polytope and H is a hyperplane with $H \cap \overset{\circ}{A} \neq \emptyset$, then $H \cap A$ is an $\mathcal{H}$-polytope whose facets are of the form $H \cap F$, where F is a facet of A.

We can use induction and define k-faces, for $0 \leq k \leq n - 1$.

Definition 4.2. Let $A \subseteq \mathbb{E}^n$ be a polyhedron with nonempty interior. We define a k-face of A to be a facet of a $(k + 1)$-face of A, for $k = 0, \ldots, n - 2$, where an $(n - 1)$-face is just a facet of A. The 1-faces are called *edges*. Two k-faces are *adjacent* if their intersection is a $(k - 1)$-face.

The polyhedron A itself is also called a *face* (of itself) or *n-face* and the k-faces of A with $k \leq n - 1$ are called *proper faces* of A. If $A = \bigcap_{i=1}^{t} C_i$ is an irredundant decomposition of A and H_i is the boundary of C_i, then the hyperplane, H_i, is called the *supporting hyperplane* of the facet $H_i \cap A$. We suspect that the 0-faces of a polyhedron are vertices in the sense of Definition 2.5. This is true and, in fact, the vertices of a polyhedron coincide with its extreme points (see Definition 2.6).

Proposition 4.6. *Let $A \subseteq \mathbb{E}^n$ be a polyhedron with nonempty interior.*

(1) For any point, $a \in \partial A$, on the boundary of A, the intersection of all the supporting hyperplanes to A at a coincides with the intersection of all the faces that contain a. In particular, points of order k of A are those points in the relative interior of the k-faces of A[2]; thus, 0-faces coincide with the vertices of A.

(2) The vertices of A coincide with the extreme points of A.

Proof. (1) If H is a supporting hyperplane to A at a, then, one of the half-spaces, C, determined by H, satisfies $A = A \cap C$. It follows from Proposition 4.5 that if $H \neq H_i$ (where the hyperplanes H_i are the supporting hyperplanes of the facets of A), then C is redundant, from which (1) follows.

(2) If $a \in \partial A$ is not extreme, then $a \in [y, z]$, where $y, z \in \partial A$. However, this implies that a has order $k \geq 1$, i.e, a is not a vertex. $\qquad\qquad\square$

4.2 The Equivalence of $\mathcal{H}$-Polytopes and $\mathcal{V}$-Polytopes

We are now ready for the theorem showing the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes. This is a nontrivial theorem usually attributed to Weyl and Minkowski (for example, see Barvinok [3]).

---

[2]Given a convex set, S, in $\mathbb{A}^n$, its *relative interior* is its interior in the affine hull of S (which might be of dimension strictly less than n).

Theorem 4.7. *(Weyl-Minkowski) If A is an $\mathcal{H}$-polytope, then A has a finite number of extreme points (equal to its vertices) and A is the convex hull of its set of vertices; thus, an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. Moreover, A has a finite number of k-faces (for $k = 0, \ldots, d-2$, where $d = \dim(A)$). Conversely, the convex hull of a finite set of points is an $\mathcal{H}$-polytope. As a consequence, a $\mathcal{V}$-polytope is an $\mathcal{H}$-polytope.*

Proof. By restricting ourselves to the affine hull of A (some $\mathbb{E}^d$, with $d \leq n$) we may assume that A has nonempty interior. Since an $\mathcal{H}$-polytope has finitely many facets, we deduce by induction that an $\mathcal{H}$-polytope has a finite number of k-faces, for $k = 0, \ldots, d-2$. In particular, an $\mathcal{H}$-polytope has finitely many vertices. By proposition 4.6, these vertices are the extreme points of A and since an $\mathcal{H}$-polytope is compact and convex, by the theorem of Krein and Milman (Theorem 2.8), A is the convex hull of its set of vertices.

Conversely, again, we may assume that A has nonempty interior by restricting ourselves to the affine hull of A. Then, pick an origin, O, in the interior of A and consider the dual, A^*, of A. By Proposition 4.3, the convex set A^* is an $\mathcal{H}$-polytope. By the first part of the proof of Theorem 4.7, the $\mathcal{H}$-polytope, A^*, is the convex hull of its vertices. Finally, as the hypotheses of Proposition 3.22 and Proposition 4.3 (again) hold, we deduce that $A = A^{**}$ is an $\mathcal{H}$-polytope. $\qquad\square$

In view of Theorem 4.7, we are justified in dropping the $\mathcal{V}$ or $\mathcal{H}$ in front of polytope, and will do so from now on. Theorem 4.7 has some interesting corollaries regarding the dual of a polytope.

Corollary 4.8. *If A is any polytope in $\mathbb{E}^n$ such that the interior of A contains the origin, O, then the dual, A^*, of A is also a polytope whose interior contains O and $A^{**} = A$.*

Corollary 4.9. *If A is any polytope in $\mathbb{E}^n$ whose interior contains the origin, O, then the k-faces of A are in bijection with the $(n-k-1)$-faces of the dual polytope, A^*. This correspondence is as follows: If $Y = \mathrm{aff}(F)$ is the k-dimensional subspace determining the k-face, F, of A then the subspace, $Y^* = \mathrm{aff}(F^*)$, determining the corresponding face, F^*, of A^*, is the intersection of the polar hyperplanes of points in Y.*

Proof. Immediate from Proposition 4.6 and Proposition 3.23. $\qquad\square$

We also have the following proposition whose proof would not be that simple if we only had the notion of an $\mathcal{H}$-polytope (as a matter of fact, there is a way of proving Theorem 4.7 using Proposition 4.10)

Proposition 4.10. *If $A \subseteq \mathbb{E}^n$ is a polytope and $f \colon \mathbb{E}^n \to \mathbb{E}^m$ is an affine map, then $f(A)$ is a polytope in $\mathbb{E}^m$.*

Proof. Immediate, since an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope and since affine maps send convex sets to convex sets. $\qquad\square$

The reader should check that the Minkowski sum of polytopes is a polytope.

We were able to give a short proof of Theorem 4.7 because we relied on a powerful theorem, namely, Krein and Milman. A drawback of this approach is that it bypasses the interesting and important problem of designing algorithms for finding the vertices of an $\mathcal{H}$-polyhedron from the sets of inequalities defining it. A method for doing this is Fourier-Motzkin elimination, see Ziegler [45] (Chapter 1) and Section 4.3. This is also a special case of *linear programming*.

It is also possible to generalize the notion of $\mathcal{V}$-polytope to polyhedra using the notion of cone and to generalize the equivalence theorem to $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra.

4.3   The Equivalence of $\mathcal{H}$-Polyhedra and $\mathcal{V}$-Polyhedra

The equivalence of $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes can be generalized to polyhedral sets, *i.e.*, finite intersections of closed half-spaces that are not necessarily bounded. This equivalence was first proved by Motzkin in the early 1930's. It can be proved in several ways, some involving cones.

Definition 4.3. Let $\mathcal{E}$ be any affine Euclidean space of finite dimension, n (with associated vector space, $\overrightarrow{\mathcal{E}}$). A subset, $C \subseteq \overrightarrow{\mathcal{E}}$, is a *cone* if C is closed under linear combinations involving only nonnegative scalars called *positive combinations*. Given a subset, $V \subseteq \overrightarrow{\mathcal{E}}$, the *conical hull* or *positive hull* of V is the set

$$\mathrm{cone}(V) = \Big\{ \sum_I \lambda_i v_i \mid \{v_i\}_{i \in I} \subseteq V,\ \lambda_i \geq 0 \quad \text{for all } i \in I \Big\}.$$

A $\mathcal{V}$-*polyhedron* or *polyhedral set* is a subset, $A \subseteq \mathcal{E}$, such that

$$A = \mathrm{conv}(Y) + \mathrm{cone}(V) = \{a + v \mid a \in \mathrm{conv}(Y),\ v \in \mathrm{cone}(V)\},$$

where $V \subseteq \overrightarrow{\mathcal{E}}$ is a finite set of vectors and $Y \subseteq \mathcal{E}$ is a finite set of points.

A set, $C \subseteq \overrightarrow{\mathcal{E}}$, is a $\mathcal{V}$-*cone* or *polyhedral cone* if C is the positive hull of a finite set of vectors, that is,

$$C = \mathrm{cone}(\{u_1, \ldots, u_p\}),$$

for some vectors, $u_1, \ldots, u_p \in \overrightarrow{\mathcal{E}}$. An $\mathcal{H}$-*cone* is any subset of $\overrightarrow{\mathcal{E}}$ given by a finite intersection of closed half-spaces cut out by hyperplanes through 0.

The positive hull, $\mathrm{cone}(V)$, of V is also denoted $\mathrm{pos}(V)$. Observe that a $\mathcal{V}$-cone can be viewed as a polyhedral set for which $Y = \{O\}$, a single point. However, if we take the point O as the origin, we may view a $\mathcal{V}$-polyhedron, A, for which $Y = \{O\}$, as a $\mathcal{V}$-cone. We will switch back and forth between these two views of cones as we find it convenient, this should

not cause any confusion. In this section, we favor the view that $\mathcal{V}$-cones are special kinds of $\mathcal{V}$-polyhedra. As a consequence, a ($\mathcal{V}$ or $\mathcal{H}$)-cone always contains 0, sometimes called an *apex* of the cone.

A set of the form $\{a + tu \mid t \geq 0\}$, where $a \in \mathcal{E}$ is a point and $u \in \overrightarrow{\mathcal{E}}$ is a nonzero vector, is called a *half-line* or *ray*. Then, we see that a $\mathcal{V}$-polyhedron, $A = \mathrm{conv}(Y) + \mathrm{cone}(V)$, is the convex hull of the union of a finite set of points with a finite set of rays. In the case of a $\mathcal{V}$-cone, all these rays meet in a common point, an apex of the cone.

Propositions 4.1 and 4.2 generalize easily to $\mathcal{V}$-polyhedra and cones.

Proposition 4.11. *Let $A \subseteq \mathcal{E}$ be a $\mathcal{V}$-polyhedron or an $\mathcal{H}$-polyhedron, let $E = \mathrm{aff}(A)$ be the affine hull of A in $\mathcal{E}$ (with the Euclidean structure on E induced by the Euclidean structure on $\mathcal{E}$) and write $d = \dim(E)$. Then, the following assertions hold:*

(1) *The set, A, is a $\mathcal{V}$-polyhedron in E (i.e., viewed as a subset of E) iff A is a $\mathcal{V}$-polyhedron in $\mathcal{E}$.*

(2) *The set, A, is an $\mathcal{H}$-polyhedron in E (i.e., viewed as a subset of E) iff A is an $\mathcal{H}$-polyhedron in $\mathcal{E}$.*

Proof. We already proved (2) in Proposition 4.1. For (1), observe that the direction, $\overrightarrow{E}$, of E is a linear subspace of $\overrightarrow{\mathcal{E}}$. Consequently, E is closed under affine combinations and $\overrightarrow{E}$ is closed under linear combinations and the result follows immediately. $\square$

Proposition 4.12. *Given any two affine Euclidean spaces, E and F, if $h \colon E \to F$ is any affine map then:*

(1) *If A is any $\mathcal{V}$-polyhedron in E, then $h(E)$ is a $\mathcal{V}$-polyhedron in F.*

(2) *If $g \colon \overrightarrow{E} \to \overrightarrow{F}$ is any linear map and if C is any $\mathcal{V}$-cone in $\overrightarrow{E}$, then $g(C)$ is a $\mathcal{V}$-cone in $\overrightarrow{F}$.*

(3) *If h is bijective and A is any $\mathcal{H}$-polyhedron in E, then $h(E)$ is an $\mathcal{H}$-polyhedron in F.*

Proof. We already proved (3) in Proposition 4.2. For (1), using the fact that $h(a + u) = h(a) + \overrightarrow{h}(u)$ for any affine map, h, where $\overrightarrow{h}$ is the linear map associated with h, we get

$$h(\mathrm{conv}(Y) + \mathrm{cone}(V)) = \mathrm{conv}(h(Y)) + \mathrm{cone}(\overrightarrow{h}(V)).$$

For (2), as g is linear, we get

$$g(\mathrm{cone}(V)) = \mathrm{cone}(g(V)),$$

establishing the proposition. $\square$

Propositions 4.11 and 4.12 allow us to assume that $\mathcal{E} = \mathbb{E}^d$ and that our ($\mathcal{V}$ or $\mathcal{H}$)-polyhedra, $A \subseteq \mathbb{E}^d$, have nonempty interior (*i.e.* $\dim(A) = d$).

The generalization of Theorem 4.7 is that every $\mathcal{V}$-polyhedron, A, is an $\mathcal{H}$-polyhedron and conversely. At first glance, it may seem that there is a small problem when $A = \mathbb{E}^d$. Indeed, Definition 4.3 allows the possibility that $\mathrm{cone}(V) = \mathbb{E}^d$ for some finite subset, $V \subseteq \mathbb{R}^d$. This is because it is possible to generate a basis of $\mathbb{R}^d$ using finitely many positive combinations. On the other hand the definition of an $\mathcal{H}$-polyhedron, A, (Definition 4.1) assumes that $A \subseteq \mathbb{E}^n$ is cut out by *at least one* hyperplane. So, A is always contained in some half-space of $\mathbb{E}^n$ and strictly speaking, $\mathbb{E}^n$ is not an $\mathcal{H}$-polyhedron! The simplest way to circumvent this difficulty is to decree that $\mathbb{E}^n$ itself is a polyhedron, which we do.

Yet another solution is to assume that, unless stated otherwise, every finite set of vectors, V, that we consider when defining a polyhedron has the property that there is some hyperplane, H, through the origin so that all the vectors in V lie in only one of the two closed half-spaces determined by H. But then, the polar dual of a polyhedron can't be a single point! Therefore, we stick to our decision that $\mathbb{E}^n$ itself is a polyhedron.

To prove the equivalence of $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra, Ziegler proceeds as follows: First, he shows that the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra reduces to the equivalence of $\mathcal{V}$-cones and $\mathcal{H}$-cones using an "old trick" of projective geometry, namely, "homogenizing" [45] (Chapter 1). Then, he uses two dual versions of Fourier-Motzkin elimination to pass from $\mathcal{V}$-cones to $\mathcal{H}$-cones and conversely.

Since the homogenization method is an important technique we will describe it in some detail. However, it turns out that the double dualization technique used in the proof of Theorem 4.7 can be easily adapted to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron. Moreover, it can also be used to prove that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron! So, we will not describe the version of Fourier-Motzkin elimination used to go from $\mathcal{V}$-cones to $\mathcal{H}$-cones. However, we will present the Fourier-Motzkin elimination method used to go from $\mathcal{H}$-cones to $\mathcal{V}$-cones.

Here is the generalization of Proposition 4.3 to polyhedral sets. In order to avoid confusion between the origin of $\mathbb{E}^d$ and the center of polar duality we will denote the origin by O and the center of our polar duality by Ω. Given any nonzero vector, $u \in \mathbb{R}^d$, let $u_-^\dagger$ be the closed half-space

$$u_-^\dagger = \{x \in \mathbb{R}^d \mid x \cdot u \leq 0\}.$$

In other words, $u_-^\dagger$ is the closed half-space bounded by the hyperplane through Ω normal to u and on the "opposite side" of u.

Proposition 4.13. *Let $A = \mathrm{conv}(Y) + \mathrm{cone}(V) \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron with $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Then, for any point, Ω, if $A \neq \{\Omega\}$, then the polar dual, A^*, of A w.r.t. Ω is the $\mathcal{H}$-polyhedron given by*

$$A^* = \bigcap_{i=1}^{p} (y_i^\dagger)_- \cap \bigcap_{j=1}^{q} (v_j^\dagger)_-.$$

Furthermore, if A has nonempty interior and Ω belongs to the interior of A, then A^* is bounded, that is, A^* is an $\mathcal{H}$-polytope. If $A = \{\Omega\}$, then A^* is the special polyhedron, $A^* = \mathbb{E}^d$.

Proof. By definition of A^* w.r.t. Ω, we have

$$
\begin{aligned}
A^* &= \left\{ x \in \mathbb{E}^d \;\middle|\; \mathbf{\Omega x} \cdot \mathbf{\Omega} \left(\sum_{i=1}^{p} \lambda_i \mathbf{y_i} + \sum_{j=1}^{q} \mu_j \mathbf{v_j} \right) \leq 1,\; \lambda_i \geq 0,\; \sum_{i=1}^{p} \lambda_i = 1,\; \mu_j \geq 0 \right\} \\
&= \left\{ x \in \mathbb{E}^d \;\middle|\; \sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} + \sum_{j=1}^{q} \mu_j \mathbf{\Omega x} \cdot v_j \leq 1,\; \lambda_i \geq 0,\; \sum_{i=1}^{p} \lambda_i = 1,\; \mu_j \geq 0 \right\}.
\end{aligned}
$$

When $\mu_j = 0$ for $j = 1, \ldots, q$, we get

$$
\sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} \leq 1, \quad \lambda_i \geq 0,\; \sum_{i=1}^{p} \lambda_i = 1
$$

and we check that

$$
\begin{aligned}
\left\{ x \in \mathbb{E}^d \;\middle|\; \sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} \leq 1,\; \lambda_i \geq 0,\; \sum_{i=1}^{p} \lambda_i = 1 \right\} &= \bigcap_{i=1}^{p} \{x \in \mathbb{E}^d \mid \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} \leq 1\} \\
&= \bigcap_{i=1}^{p} (y_i^\dagger)_-.
\end{aligned}
$$

The points in A^* must also satisfy the conditions

$$
\sum_{j=1}^{q} \mu_j \mathbf{\Omega x} \cdot v_j \leq 1 - \alpha, \quad \mu_j \geq 0,\; \mu_j > 0 \text{ for some } j,\; 1 \leq j \leq q,
$$

with $\alpha \leq 1$ (here $\alpha = \sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i}$). In particular, for every $j \in \{1, \ldots, q\}$, if we set $\mu_k = 0$ for $k \in \{1, \ldots, q\} - \{j\}$, we should have

$$
\mu_j \mathbf{\Omega x} \cdot v_j \leq 1 - \alpha \quad \text{for all} \quad \mu_j > 0,
$$

that is,

$$
\mathbf{\Omega x} \cdot v_j \leq \frac{1 - \alpha}{\mu_j} \quad \text{for all} \quad \mu_j > 0,
$$

which is equivalent to

$$
\mathbf{\Omega x} \cdot v_j \leq 0.
$$

Consequently, if $x \in A^*$, we must also have

$$
x \in \bigcap_{j=1}^{q} \{x \in \mathbb{E}^d \mid \mathbf{\Omega x} \cdot v_j \leq 0\} = \bigcap_{j=1}^{q} (v_j^\dagger)_-.
$$

Therefore,

$$A^* \subseteq \bigcap_{i=1}^{p}(y_i^\dagger)_- \cap \bigcap_{j=1}^{q}(v_j^\dagger)_-.$$

However, the reverse inclusion is obvious and thus, we have equality. If Ω belongs to the interior of A, we know from Proposition 3.22 that A^* is bounded. Therefore, A^* is indeed an $\mathcal{H}$-polytope of the above form. $\qquad\square$

It is fruitful to restate Proposition 4.13 in terms of matrices (as we did for Proposition 4.3). First, observe that

$$(\text{conv}(Y) + \text{cone}(V))^* = (\text{conv}(Y \cup \{\Omega\}) + \text{cone}(V))^*.$$

If we pick Ω as an origin then we can represent the points in Y as vectors. The old origin is still denoted O and Ω is now denoted 0. The zero vector is denoted $\mathbf{0}$.

If $A = \text{conv}(Y) + \text{cone}(V) \neq \{0\}$, let Y be the $d \times p$ matrix whose i^{th} column is y_i and let V is the $d \times q$ matrix whose j^{th} column is v_j. Then Proposition 4.13 says that

$$(\text{conv}(Y) + \text{cone}(V))^* = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}.$$

We write $P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}) = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}$.

If $A = \text{conv}(Y) + \text{cone}(V) = \{0\}$, then both Y and V must be zero matrices but then, the inequalities $Y^\top x \leq \mathbf{1}$ and $V^\top x \leq \mathbf{0}$ are trivially satisfied by all $x \in \mathbb{E}^d$, so even in this case we have

$$\mathbb{E}^d = (\text{conv}(Y) + \text{cone}(V))^* = P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}).$$

The converse of Proposition 4.13 also holds as shown below.

Proposition 4.14. *Let $\{y_1, \ldots, y_p\}$ be any set of points in $\mathbb{E}^d$ and let $\{v_1, \ldots, v_q\}$ be any set of nonzero vectors in $\mathbb{R}^d$. If Y is the $d \times p$ matrix whose i^{th} column is y_i and V is the $d \times q$ matrix whose j^{th} column is v_j, then*

$$(\text{conv}(\{y_1, \ldots, y_p\}) + \text{cone}(\{v_1, \ldots, v_q\}))^* = P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}),$$

with $P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}) = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}$.

Conversely, given any $p \times d$ matrix, Y, and any $q \times d$ matrix, V, we have

$$P(Y, \mathbf{1}; V, \mathbf{0})^* = \text{conv}(\{y_1, \ldots, y_p\} \cup \{0\}) + \text{cone}(\{v_1, \ldots, v_q\}),$$

where $y_i \in \mathbb{R}^n$ is the ith row of Y and $v_j \in \mathbb{R}^n$ is the jth row of V or, equivalently,

$$P(Y, \mathbf{1}; V, \mathbf{0})^* = \{x \in \mathbb{R}^d \mid x = Y^\top u + V^\top t,\ u \in \mathbb{R}^p,\ t \in \mathbb{R}^q,\ u, t \geq 0,\ \mathbb{I}u = 1\},$$

where $\mathbb{I}$ is the row vector of length p whose coordinates are all equal to 1.

Proof. Only the second part needs a proof. Let

$$B = \mathrm{conv}(\{y_1, \ldots, y_p\} \cup \{0\}) + \mathrm{cone}(\{v_1, \ldots, v_q\}),$$

where $y_i \in \mathbb{R}^p$ is the ith *row* of Y and $v_j \in \mathbb{R}^q$ is the jth *row* of V. Then, by the first part,

$$B^* = P(Y, \mathbf{1}; V, \mathbf{0}).$$

As $0 \in B$, by Proposition 3.22, we have $B = B^{**} = P(Y, \mathbf{1}; V, \mathbf{0})^*$, as claimed. $\square$

Proposition 4.14 has the following important Corollary:

Proposition 4.15. *The following assertions hold:*

(1) *The polar dual, A^*, of every $\mathcal{H}$-polyhedron, is a $\mathcal{V}$-polyhedron.*

(2) *The polar dual, A^*, of every $\mathcal{V}$-polyhedron, is an $\mathcal{H}$-polyhedron.*

Proof. (1) We may assume that $0 \in A$, in which case, A is of the form $A = P(Y, \mathbf{1}; V, \mathbf{0})$. By the second part of Proposition 4.14, A^* is a $\mathcal{V}$-polyhedron.

(2) This is the first part of Proposition 4.14. $\square$

We can now use Proposition 4.13, Proposition 3.22 and Krein and Milman's Theorem to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron.

Proposition 4.16. *Every $\mathcal{V}$-polyhedron, A, is an $\mathcal{H}$-polyhedron.*

Proof. Let A be a $\mathcal{V}$-polyhedron of dimension d. Thus, $A \subseteq \mathbb{E}^d$ has nonempty interior so we can pick some point, Ω, in the interior of A. If $d = 0$, then $A = \{0\} = \mathbb{E}^0$ and we are done. Otherwise, by Proposition 4.13, the polar dual, A^*, of A w.r.t. Ω is an $\mathcal{H}$-polytope. Then, as in the proof of Theorem 4.7, using Krein and Milman's Theorem we deduce that A^* is a $\mathcal{V}$-polytope. Now, as Ω belongs to A, by Proposition 3.22 (even if A is not bounded) we have $A = A^{**}$ and by Proposition 4.13, we conclude that $A = A^{**}$ is an $\mathcal{H}$-polyhedron. $\square$

Interestingly, we can now prove easily that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.

Proposition 4.17. *Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.*

Proof. Let A be an $\mathcal{H}$-polyhedron of dimension d. By Proposition 4.15, the polar dual, A^*, of A is a $\mathcal{V}$-polyhedron. By Proposition 4.16, A^* is an $\mathcal{H}$-polyhedron and again, by Proposition 4.15, we deduce that $A^{**} = A$ is a $\mathcal{V}$-polyhedron ($A = A^{**}$ because $0 \in A$). $\square$

Putting together Propositions 4.16 and 4.17 we obtain our main theorem:

Theorem 4.18. *(Equivalence of $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra) Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron and conversely.*

Even though we proved the main result of this section, it is instructive to consider a more computational proof making use of cones and an elimination method known as *Fourier-Motzkin elimination*.

4.4   Fourier-Motzkin Elimination and the Polyhedron-Cone Correspondence

The problem with the converse of Proposition 4.16 when A is unbounded (*i.e.*, not compact) is that Krein and Milman's Theorem does not apply. We need to take into account "points at infinity" corresponding to certain vectors. The trick we used in Proposition 4.16 is that the polar dual of a $\mathcal{V}$-polyhedron with nonempty interior is an $\mathcal{H}$-*polytope*. This reduction to polytopes allowed us to use Krein and Milman to convert an $\mathcal{H}$-polytope to a $\mathcal{V}$-polytope and then again we took the polar dual.

Another trick is to switch to cones by "homogenizing." Given any subset, $S \subseteq \mathbb{E}^d$, we can form the cone, $C(S) \subseteq \mathbb{E}^{d+1}$, by "placing" a copy of S in the hyperplane, $H_{d+1} \subseteq \mathbb{E}^{d+1}$, of equation $x_{d+1} = 1$, and drawing all the half-lines from the origin through any point of S. If S is given by m polynomial inequalities of the form

$$P_i(x_1, \ldots, x_d) \leq b_i,$$

where $P_i(x_1, \ldots, x_d)$ is a polynomial of total degree n_i, this amounts to forming the new homogeneous inequalities

$$x_{d+1}^{n_i} P_i \left(\frac{x_1}{x_{d+1}}, \ldots, \frac{x_d}{x_{d+1}} \right) - b_i x_{d+1}^{n_i} \leq 0$$

together with $x_{d+1} \geq 0$. In particular, if the P_i's are linear forms (which means that $n_i = 1$), then our inequalities are of the form

$$a_i \cdot x \leq b_i,$$

where $a_i \in \mathbb{R}^d$ is some vector and the homogenized inequalities are

$$a_i \cdot x - b_i x_{d+1} \leq 0.$$

It will be convenient to formalize the process of putting a copy of a subset, $S \subseteq \mathbb{E}^d$, in the hyperplane, $H_{d+1} \subseteq \mathbb{E}^{d+1}$, of equation $x_{d+1} = 1$, as follows: For every point, $a \in \mathbb{E}^d$, let

$$\widehat{a} = \begin{pmatrix} a \\ 1 \end{pmatrix} \in \mathbb{E}^{d+1}$$

and let $\widehat{S} = \{\widehat{a} \mid a \in S\}$. Obviously, the map $S \mapsto \widehat{S}$ is a bijection between the subsets of $\mathbb{E}^d$ and the subsets of H_{d+1}. We will use this bijection to identify S and $\widehat{S}$ and use the simpler notation, S, unless confusion arises. Let's apply this to polyhedra.

Let $P \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-polyhedron. Then, P is cut out by m hyperplanes, H_i, and for each H_i, there is a nonzero vector, a_i, and some $b_i \in \mathbb{R}$ so that

$$H_i = \{x \in \mathbb{E}^d \mid a_i \cdot x = b_i\}$$

and P is given by

$$P = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid a_i \cdot x \leq b_i\}.$$

If A denotes the $m \times d$ matrix whose i-th row is a_i and b is the vector $b = (b_1, \ldots, b_m)$, then we can write

$$P = P(A, b) = \{x \in \mathbb{E}^d \mid Ax \leq b\}.$$

We "homogenize" $P(A, b)$ as follows: Let $C(P)$ be the subset of $\mathbb{E}^{d+1}$ defined by

$$\begin{aligned} C(P) &= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax \leq x_{d+1}b, \; x_{d+1} \geq 0 \right\} \\ &= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax - x_{d+1}b \leq 0, \; -x_{d+1} \leq 0 \right\}. \end{aligned}$$

Thus, we see that $C(P)$ is the $\mathcal{H}$-cone given by the system of inequalities

$$\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and that

$$\widehat{P} = C(P) \cap H_{d+1}.$$

Conversely, if Q is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), it is clear that $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $H_{d+1} \approx \mathbb{E}^d$.

Let us now assume that $P \subseteq \mathbb{E}^d$ is a $\mathcal{V}$-polyhedron, $P = \operatorname{conv}(Y) + \operatorname{cone}(V)$, where $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Define $\widehat{Y} = \{\widehat{y}_1, \ldots, \widehat{y}_p\} \subseteq \mathbb{E}^{d+1}$, and $\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_q\} \subseteq \mathbb{E}^{d+1}$, by

$$\widehat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \qquad \widehat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

We check immediately that

$$C(P) = \operatorname{cone}(\{\widehat{Y} \cup \widehat{V}\})$$

is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$ such that

$$\widehat{P} = C(P) \cap H_{d+1},$$

where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$.

Conversely, if $C = \operatorname{cone}(W)$ is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$, with $w_{i\,d+1} \geq 0$ for every $w_i \in W$, we prove next that $P = C \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron.

Proposition 4.19. *(Polyhedron–Cone Correspondence) We have the following correspondence between polyhedra in $\mathbb{E}^d$ and cones in $\mathbb{E}^{d+1}$:*

(1) *For any $\mathcal{H}$-polyhedron, $P \subseteq \mathbb{E}^d$, if $P = P(A, b) = \{x \in \mathbb{E}^d \mid Ax \leq b\}$, where A is an $m \times d$-matrix and $b \in \mathbb{R}^m$, then $C(P)$ given by*

$$\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ and $\widehat{P} = C(P) \cap H_{d+1}$, where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$. Conversely, if Q is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), then $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $H_{d+1} \approx \mathbb{E}^d$.

(2) *Let $P \subseteq \mathbb{E}^d$ be any $\mathcal{V}$-polyhedron, where $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$ with $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Define $\widehat{Y} = \{\widehat{y}_1, \ldots, \widehat{y}_p\} \subseteq \mathbb{E}^{d+1}$, and $\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_q\} \subseteq \mathbb{E}^{d+1}$, by*

$$\widehat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \qquad \widehat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

Then,

$$C(P) = \mathrm{cone}(\{\widehat{Y} \cup \widehat{V}\})$$

is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$ such that

$$\widehat{P} = C(P) \cap H_{d+1},$$

Conversely, if $C = \mathrm{cone}(W)$ is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$, with $w_{i\,d+1} \geq 0$ for every $w_i \in W$, then $P = C \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron in $H_{d+1} \approx \mathbb{E}^d$.

In both (1) and (2), $\widehat{P} = \{\widehat{p} \mid p \in P\}$, with

$$\widehat{p} = \begin{pmatrix} p \\ 1 \end{pmatrix} \in \mathbb{E}^{d+1}.$$

Proof. We already proved everything except the last part of the proposition. Let

$$\widehat{Y} = \left\{ \frac{w_i}{w_{i\,d+1}} \,\middle|\, w_i \in W,\ w_{i\,d+1} > 0 \right\}$$

and

$$\widehat{V} = \{w_j \in W \mid w_{j\,d+1} = 0\}.$$

We claim that

$$P = C \cap H_{d+1} = \mathrm{conv}(\widehat{Y}) + \mathrm{cone}(\widehat{V}),$$

and thus, P is a $\mathcal{V}$-polyhedron.

Recall that any element, $z \in C$, can be written as

$$z = \sum_{k=1}^{s} \mu_k w_k, \quad \mu_k \geq 0.$$

Thus, we have

$$
\begin{aligned}
z &= \sum_{k=1}^{s} \mu_k w_k \quad \mu_k \geq 0 \\
&= \sum_{w_{i\,d+1}>0} \mu_i w_i + \sum_{w_{j\,d+1}=0} \mu_j w_j \quad \mu_i, \mu_j \geq 0 \\
&= \sum_{w_{i\,d+1}>0} w_{i\,d+1}\mu_i \frac{w_i}{w_{i\,d+1}} + \sum_{w_{j\,d+1}=0} \mu_j w_j, \quad \mu_i, \mu_j \geq 0 \\
&= \sum_{w_{i\,d+1}>0} \lambda_i \frac{w_i}{w_{i\,d+1}} + \sum_{w_{j\,d+1}=0} \mu_j w_j, \quad \lambda_i, \mu_j \geq 0.
\end{aligned}
$$

Now, $z \in C \cap H_{d+1}$ iff $z_{d+1} = 1$ iff $\sum_{i=1}^{p} \lambda_i = 1$ (where p is the number of elements in $\widehat{Y}$), since the $(d+1)^{\text{th}}$ coordinate of $\frac{w_i}{w_{i\,d+1}}$ is equal to 1, and the above shows that

$$
P = C \cap H_{d+1} = \operatorname{conv}(\widehat{Y}) + \operatorname{cone}(\widehat{V}),
$$

as claimed. $\qquad \square$

By Proposition 4.19, if P is an $\mathcal{H}$-polyhedron, then $C(P)$ is an $\mathcal{H}$-cone. If we can prove that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone, then again, Proposition 4.19 shows that $\widehat{P} = C(P) \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron and so, P is a $\mathcal{V}$-polyhedron. Therefore, in order to prove that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron it suffices to show that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone.

By a similar argument, Proposition 4.19 shows that in order to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron it suffices to show that every $\mathcal{V}$-cone is an $\mathcal{H}$-cone. We will not prove this direction again since we already have it by Proposition 4.16.

It remains to prove that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone. Let $C \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-cone. Then, C is cut out by m hyperplanes, H_i, through 0. For each H_i, there is a nonzero vector, u_i, so that

$$
H_i = \{x \in \mathbb{E}^d \mid u_i \cdot x = 0\}
$$

and C is given by

$$
C = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid u_i \cdot x \leq 0\}.
$$

If A denotes the $m \times d$ matrix whose i-th row is u_i, then we can write

$$
C = P(A, 0) = \{x \in \mathbb{E}^d \mid Ax \leq 0\}.
$$

Observe that $C = C_0(A) \cap H_w$, where

$$
C_0(A) = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid Ax \leq w \right\}
$$

is an $\mathcal{H}$-cone in $\mathbb{E}^{d+m}$ and

$$H_w = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid w = 0 \right\}$$

is an affine subspace in $\mathbb{E}^{d+m}$.

We claim that $C_0(A)$ is a $\mathcal{V}$-cone. This follows by observing that for every $\begin{pmatrix} x \\ w \end{pmatrix}$ satisfying $Ax \leq w$, we can write

$$\begin{pmatrix} x \\ w \end{pmatrix} = \sum_{i=1}^d |x_i|(\mathrm{sign}(x_i)) \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} + \sum_{j=1}^m (w_j - (Ax)_j) \begin{pmatrix} 0 \\ e_j \end{pmatrix},$$

and then

$$C_0(A) = \mathrm{cone}\left(\left\{ \pm \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} \mid 1 \leq i \leq d \right\} \cup \left\{ \begin{pmatrix} 0 \\ e_j \end{pmatrix} \mid 1 \leq j \leq m \right\} \right).$$

Since $C = C_0(A) \cap H_w$ is now the intersection of a $\mathcal{V}$-cone with an affine subspace, to prove that C is a $\mathcal{V}$-cone it is enough to prove that the intersection of a $\mathcal{V}$-cone with a hyperplane is also a $\mathcal{V}$-cone. For this, we use *Fourier-Motzkin elimination*. It suffices to prove the result for a hyperplane, H_k, in $\mathbb{E}^{d+m}$ of equation $y_k = 0$ ($1 \leq k \leq d + m$).

Proposition 4.20. *(Fourier-Motzkin Elimination) Say $C = \mathrm{cone}(Y) \subseteq \mathbb{E}^d$ is a $\mathcal{V}$-cone. Then, the intersection $C \cap H_k$ (where H_k is the hyperplane of equation $y_k = 0$) is a $\mathcal{V}$-cone, $C \cap H_k = \mathrm{cone}(Y^{/k})$, with*

$$Y^{/k} = \{y_i \mid y_{ik} = 0\} \cup \{y_{ik}y_j - y_{jk}y_i \mid y_{ik} > 0, \ y_{jk} < 0\},$$

the set of vectors obtained from Y by "eliminating the k-th coordinate". Here, each y_i is a vector in $\mathbb{R}^d$.

Proof. The only nontrivial direction is to prove that $C \cap H_k \subseteq \mathrm{cone}(Y^{/k})$. For this, consider any $v = \sum_{i=1}^d t_i y_i \in C \cap H_k$, with $t_i \geq 0$ and $v_k = 0$. Such a v can be written

$$v = \sum_{i\mid y_{ik}=0} t_i y_i + \sum_{i\mid y_{ik}>0} t_i y_i + \sum_{j\mid y_{jk}<0} t_j y_j$$

and as $v_k = 0$, we have

$$\sum_{i\mid y_{ik}>0} t_i y_{ik} + \sum_{j\mid y_{jk}<0} t_j y_{jk} = 0.$$

If $t_i y_{ik} = 0$ for $i = 1, \ldots, d$, we are done. Otherwise, we can write

$$\Lambda = \sum_{i\mid y_{ik}>0} t_i y_{ik} = \sum_{j\mid y_{jk}<0} -t_j y_{jk} > 0.$$

Then,

$$
\begin{aligned}
v &= \sum_{i\mid y_{ik}=0} t_i y_i + \frac{1}{\Lambda} \sum_{i\mid y_{ik}>0} \left(\sum_{j\mid y_{jk}<0} -t_j y_{jk} \right) t_i y_i + \frac{1}{\Lambda} \sum_{j\mid y_{jk}<0} \left(\sum_{i\mid y_{ik}>0} t_i y_{ik} \right) t_j y_j \\
&= \sum_{i\mid y_{ik}=0} t_i y_i + \sum_{\substack{i\mid y_{ik}>0 \\ j\mid y_{jk}<0}} \frac{t_i t_j}{\Lambda} \left(y_{ik} y_j - y_{jk} y_i \right).
\end{aligned}
$$

Since the kth coordinate of $y_{ik} y_j - y_{jk} y_i$ is 0, the above shows that any $v \in C \cap H_k$ can be written as a positive combination of vectors in $Y^{/k}$. $\qquad\square$

As discussed above, Proposition 4.20 implies (again!)

Corollary 4.21. *Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.*

Another way of proving that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron is to use cones. This method is interesting in its own right so we discuss it briefly.

Let $P = \text{conv}(Y) + \text{cone}(V) \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron. We can view Y as a $d \times p$ matrix whose ith column is the ith vector in Y and V as $d \times q$ matrix whose jth column is the jth vector in V. Then, we can write

$$
P = \{ x \in \mathbb{R}^d \mid (\exists u \in \mathbb{R}^p)(\exists t \in \mathbb{R}^d)(x = Yu + Vt,\ u \geq 0,\ \mathbb{I}u = 1,\ t \geq 0) \},
$$

where $\mathbb{I}$ is the row vector

$$
\mathbb{I} = \underbrace{(1, \ldots, 1)}_{p}.
$$

Now, observe that P can be interpreted as the projection of the $\mathcal{H}$-polyhedron, $\widetilde{P} \subseteq \mathbb{E}^{d+p+q}$, given by

$$
\widetilde{P} = \{ (x, u, t) \in \mathbb{R}^{d+p+q} \mid x = Yu + Vt,\ u \geq 0,\ \mathbb{I}u = 1,\ t \geq 0 \}
$$

onto $\mathbb{R}^d$. Consequently, if we can prove that the projection of an $\mathcal{H}$-polyhedron is also an $\mathcal{H}$-polyhedron, then we will have proved that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron. Here again, it is possible that $P = \mathbb{E}^d$, but that's fine since $\mathbb{E}^d$ has been declared to be an $\mathcal{H}$-polyhedron.

In view of Proposition 4.19 and the discussion that followed, it is enough to prove that the projection of any $\mathcal{H}$-cone is an $\mathcal{H}$-cone. This can be done by using a type of Fourier-Motzkin elimination dual to the method used in Proposition 4.20. We state the result without proof and refer the interested reader to Ziegler [45], Section 1.2–1.3, for full details.

Proposition 4.22. *If $C = P(A, 0) \subseteq \mathbb{E}^d$ is an $\mathcal{H}$-cone, then the projection, $\text{proj}_k(C)$, onto the hyperplane, H_k, of equation $y_k = 0$ is given by $\text{proj}_k(C) = \text{elim}_k(C) \cap H_k$, with $\text{elim}_k(C) = \{ x \in \mathbb{R}^d \mid (\exists t \in \mathbb{R})(x + te_k \in P) \} = \{ z - te_k \mid z \in P,\ t \in \mathbb{R} \} = P(A^{/k}, 0)$ and where the rows of $A^{/k}$ are given by*

$$
A^{/k} = \{ a_i \mid a_{i\,k} = 0 \} \cup \{ a_{i\,k} a_j - a_{j\,k} a_i \mid a_{i\,k} > 0,\ a_{j\,k} < 0 \}.
$$

It should be noted that both Fourier-Motzkin elimination methods generate a quadratic number of new vectors or inequalities at each step and thus they lead to a combinatorial explosion. Therefore, these methods become intractable rather quickly. The problem is that many of the new vectors or inequalities are redundant. Therefore, it is important to find ways of detecting redundancies and there are various methods for doing so. Again, the interested reader should consult Ziegler [45], Chapter 1.

There is yet another way of proving that an $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron without using Fourier-Motzkin elimination. As we already observed, Krein and Milman's theorem does not apply if our polyhedron is unbounded. Actually, the full power of Krein and Milman's theorem is not needed to show that an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. The crucial point is that if P is an $\mathcal{H}$-polytope with nonempty interior, then *every* line, ℓ, through any point, a, in the interior of P intersects P in a line segment. This is because P is compact and ℓ is closed, so $P \cap \ell$ is a compact subset of a line thus, a closed interval $[b, c]$ with $b < a < c$, as a is in the interior of P. Therefore, we can use induction on the dimension of P to show that every point in P is a convex combination of vertices of the facets of P. Now, if P is unbounded and cut out by at least two half-spaces (so, P is not a half-space), then we claim that for every point, a, in the interior of P, there is *some* line through a that intersects two facets of P. This is because if we pick the origin in the interior of P, we may assume that P is given by an irredundant intersection, $P = \bigcap_{i=1}^{t}(H_i)_-$, and for any point, a, in the interior of P, there is a line, ℓ, through P *in general position w.r.t. P*, which means that ℓ is not parallel to any of the hyperplanes H_i and intersects all of them in distinct points (see Definition 7.2). Fortunately, lines in general position always exist, as shown in Proposition 7.3. Using this fact, we can prove the following result:

Proposition 4.23. *Let $P \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-polyhedron, $P = \bigcap_{i=1}^{t}(H_i)_-$ (an irredundant decomposition), with nonempty interior. If $t = 1$, that is, $P = (H_1)_-$ is a half-space, then*

$$P = a + \mathrm{cone}(u_1, \ldots, u_{d-1}, -u_1, \ldots, -u_{d-1}, u_d),$$

where a is any point in H_1, the vectors $u_1, \ldots, u_{d-1}$ form a basis of the direction of H_1 and u_d is normal to (the direction of) H_1. (When $d = 1$, P is the half-line, $P = \{a + tu_1 \mid t \geq 0\}$.) If $t \geq 2$, then every point, $a \in P$, can be written as a convex combination, $a = (1 - \alpha)b + \alpha c$ ($0 \leq \alpha \leq 1$), where b and c belong to two distinct facets, F and G, of P and where

$$F = \mathrm{conv}(Y_F) + \mathrm{cone}(V_F) \quad and \quad G = \mathrm{conv}(Y_G) + \mathrm{cone}(V_G),$$

for some finite sets of points, Y_F and Y_G and some finite sets of vectors, V_F and V_G. (Note: $\alpha = 0$ or $\alpha = 1$ is allowed.) Consequently, every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.

Proof. We proceed by induction on the dimension, d, of P. If $d = 1$, then P is either a closed interval, $[b, c]$, or a half-line, $\{a + tu \mid t \geq 0\}$, where $u \neq 0$. In either case, the proposition is clear.

For the induction step, assume $d > 1$. Since every facet, F, of P has dimension $d - 1$, the induction hypothesis holds for F, that is, there exist a finite set of points, Y_F, and a finite

set of vectors, V_F, so that

$$F = \text{conv}(Y_F) + \text{cone}(V_F).$$

Thus, every point on the boundary of P is of the desired form. Next, pick any point, a, in the interior of P. Then, from our previous discussion, there is a line, ℓ, through a in general position w.r.t. P. Consequently, the intersection points, $z_i = \ell \cap H_i$, of the line ℓ with the hyperplanes supporting the facets of P exist and are all distinct. If we give ℓ an orientation, the z_i's can be sorted and there is a unique k such that a lies between $b = z_k$ and $c = z_{k+1}$. But then, $b \in F_k = F$ and $c \in F_{k+1} = G$, where F and G the facets of P supported by H_k and H_{k+1}, and $a = (1 - \alpha)b + \alpha c$, a convex combination. Consequently, every point in P is a mixed convex + positive combination of finitely many points associated with the facets of P and finitely many vectors associated with the directions of the supporting hyperplanes of the facets P. Conversely, it is easy to see that any such mixed combination must belong to P and therefore, P is a $\mathcal{V}$-polyhedron. $\qquad\square$

We conclude this section with a version of Farkas Lemma for polyhedral sets.

Lemma 4.24. *(Farkas Lemma, Version IV) Let Y be any $d \times p$ matrix and V be any $d \times q$ matrix. For every $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:*

(a) *There exist $u \in \mathbb{R}^p$ and $t \in \mathbb{R}^q$, with $u \geq 0$, $t \geq 0$, $\mathbb{I}u = 1$ and $z = Yu + Vt$.*

(b) *There is some vector, $(\alpha, c) \in \mathbb{R}^{d+1}$, such that $c^\top y_i \geq \alpha$ for all i with $1 \leq i \leq p$, $c^\top v_j \geq 0$ for all j with $1 \leq j \leq q$, and $c^\top z < \alpha$.*

Proof. We use Farkas Lemma, Version II (Lemma 3.16). Observe that (a) is equivalent to the problem: Find $(u, t) \in \mathbb{R}^{p+q}$, so that

$$\begin{pmatrix} u \\ t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ Y & V \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ z \end{pmatrix},$$

which is exactly Farkas II(a). Now, the second alternative of Farkas II says that there is no solution as above if there is some $(-\alpha, c) \in \mathbb{R}^{d+1}$ so that

$$(-\alpha, c^\top) \begin{pmatrix} 1 \\ z \end{pmatrix} < 0 \quad \text{and} \quad (-\alpha, c^\top) \begin{pmatrix} \mathbb{I} & 0 \\ Y & V \end{pmatrix} \geq (\mathbb{O}, \mathbb{O}).$$

These are equivalent to

$$-\alpha + c^\top z < 0, \quad -\alpha \mathbb{I} + c^\top Y \geq \mathbb{O}, \quad c^\top V \geq \mathbb{O},$$

namely, $c^\top z < \alpha$, $c^\top Y \geq \alpha \mathbb{I}$ and $c^\top V \geq \mathbb{O}$, which are indeed the conditions of Farkas IV(b), in matrix form. $\qquad\square$

Observe that Farkas IV can be viewed as a separation criterion for polyhedral sets. This version subsumes Farkas I and Farkas II.

Chapter 5

Projective Spaces, Projective Polyhedra, Polar Duality w.r.t. a Nondegenerate Quadric

5.1 Projective Spaces

The fact that not just points but also vectors are needed to deal with unbounded polyhedra is a hint that perhaps the notions of polytope and polyhedra can be unified by "going projective". However, we have to be careful because projective geometry does not accomodate well the notion of convexity. This is because convexity has to do with convex combinations, but the essense of projective geometry is that everything is defined up to *non-zero* scalars, without any requirement that these scalars be positive.

It is possible to develop a theory of *oriented projective geometry* (due to J. Stolfi [38]) in wich convexity is nicely accomodated. However, in this approach, every point comes as a pair, (positive point, negative point), and although it is a very elegant theory, we find it a bit unwieldy. However, since all we really want is to "embed" $\mathbb{E}^d$ into its *projective completion*, $\mathbb{P}^d$, so that we can deal with "points at infinity" and "normal points" in a uniform manner in particular, with respect to projective transformations, we will content ourselves with a definition of the notion of a projective polyhedron using the notion of polyhedral cone. Thus, we will not attempt to define a general notion of convexity.

We begin with a "crash course" on (real) projective spaces. There are many texts on projective geometry. We suggest starting with Gallier [20] and then move on to far more comprehensive treatments such as Berger (Geometry II) [6] or Samuel [35].

Definition 5.1. The *(real) projective space*, $\mathbb{RP}^n$, is the set of all lines through the origin in $\mathbb{R}^{n+1}$, i.e., the set of one-dimensional subspaces of $\mathbb{R}^{n+1}$ (where $n \geq 0$). Since a one-dimensional subspace, $L \subseteq \mathbb{R}^{n+1}$, is spanned by any nonzero vector, $u \in L$, we can view $\mathbb{RP}^n$ as the set of equivalence classes of nonzero vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence

relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \in \mathbb{R}, \ \lambda \neq 0.$$

We have the projection, $p \colon (\mathbb{R}^{n+1} - \{0\}) \to \mathbb{RP}^n$, given by $p(u) = [u]_\sim$, the equivalence class of u modulo $\sim$. Write $[u]$ (or $\langle u \rangle$) for the line,

$$[u] = \{\lambda u \mid \lambda \in \mathbb{R}\},$$

defined by the nonzero vector, u. Note that $[u]_\sim = [u] - \{0\}$, for every $u \neq 0$, so the map $[u]_\sim \mapsto [u]$ is a bijection which allows us to identify $[u]_\sim$ and $[u]$. Thus, we will use both notations interchangeably as convenient.

The projective space, $\mathbb{RP}^n$, is sometimes denoted $\mathbb{P}(\mathbb{R}^{n+1})$. Since every line, L, in $\mathbb{R}^{n+1}$ intersects the sphere S^n in two antipodal points, we can view $\mathbb{RP}^n$ as the quotient of the sphere S^n by identification of antipodal points. We call this the *spherical model* of $\mathbb{RP}^n$.

A more subtle construction consists in considering the (upper) half-sphere instead of the sphere, where the upper half-sphere S^n_+ is set of points on the sphere S^n such that $x_{n+1} \geq 0$. This time, every line through the center intersects the (upper) half-sphere in a single point, except on the boundary of the half-sphere, where it intersects in two antipodal points a_+ and a_-. Thus, the projective space $\mathbb{RP}^n$ is the quotient space obtained from the (upper) half-sphere S^n_+ by identifying antipodal points a_+ and a_- on the boundary of the half-sphere. We call this model of $\mathbb{RP}^n$ the *half-spherical model*.

When $n = 2$, we get a circle. When $n = 3$, the upper half-sphere is homeomorphic to a closed disk (say, by orthogonal projection onto the xy-plane), and $\mathbb{RP}^2$ is in bijection with a closed disk in which antipodal points on its boundary (a unit circle) have been identified. This is hard to visualize! In this model of the real projective space, projective lines are great semicircles on the upper half-sphere, with antipodal points on the boundary identified. Boundary points correspond to points at infinity. By orthogonal projection, these great semicircles correspond to semiellipses, with antipodal points on the boundary identified. Traveling along such a projective "line," when we reach a boundary point, we "wrap around"! In general, the upper half-sphere S^n_+ is homeomorphic to the closed unit ball in $\mathbb{R}^n$, whose boundary is the $(n-1)$-sphere S^{n-1}. For example, the projective space $\mathbb{RP}^3$ is in bijection with the closed unit ball in $\mathbb{R}^3$, with antipodal points on its boundary (the sphere S^2) identified!

Another useful way of "visualizing" $\mathbb{RP}^n$ is to use the hyperplane, $H_{n+1} \subseteq \mathbb{R}^{n+1}$, of equation $x_{n+1} = 1$. Observe that for every line, $[u]$, through the origin in $\mathbb{R}^{n+1}$, if u does not belong to the hyperplane, $H_{n+1}(0) \cong \mathbb{R}^n$, of equation $x_{n+1} = 0$, then $[u]$ intersects H_{n+1} is a unique point, namely,

$$\left(\frac{u_1}{u_{n+1}}, \ldots, \frac{u_n}{u_{n+1}}, 1 \right),$$

where $u = (u_1, \ldots, u_{n+1})$. The lines, $[u]$, for which $u_{n+1} = 0$ are "points at infinity". Observe that the set of lines in $H_{n+1}(0) \cong \mathbb{R}^n$ is the set of points of the projective space, $\mathbb{RP}^{n-1}$, and

so, $\mathbb{RP}^n$ can be written as the disjoint union

$$\mathbb{RP}^n = \mathbb{R}^n \amalg \mathbb{RP}^{n-1}.$$

We can repeat the above analysis on $\mathbb{RP}^{n-1}$ and so we can think of $\mathbb{RP}^n$ as the disjoint union

$$\mathbb{RP}^n = \mathbb{R}^n \amalg \mathbb{R}^{n-1} \amalg \cdots \amalg \mathbb{R}^1 \amalg \mathbb{R}^0,$$

where $\mathbb{R}^0 = \{0\}$ consist of a single point. The above shows that there is an embedding, $\mathbb{R}^n \hookrightarrow \mathbb{RP}^n$, given by $(u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n, 1)$.

It will also be very useful to use homogeneous coordinates. Given any point, $p = [u]_\sim \in \mathbb{RP}^n$, the set

$$\{(\lambda u_1, \ldots, \lambda u_{n+1}) \mid \lambda \neq 0\}$$

is called the set of *homogeneous coordinates* of p. Since $u \neq 0$, observe that for all homogeneous coordinates, $(u_1, \ldots, u_{n+1})$, for p, some u_i must be non-zero. The traditional notation for the homogeneous coordinates of a point, $p = [u]_\sim$, is

$$(u_1 : \cdots : u_n : u_{n+1}).$$

There is a useful bijection between certain kinds of subsets of $\mathbb{R}^{d+1}$ and subsets of $\mathbb{RP}^d$. For any subset, S, of $\mathbb{R}^{d+1}$, let

$$-S = \{-u \mid u \in S\}.$$

Geometrically, $-S$ is the reflexion of S about 0. Note that for *any* nonempty subset, $S \subseteq \mathbb{R}^{d+1}$, with $S \neq \{0\}$, the sets S, $-S$ and $S \cup -S$ all induce the *same* set of points in projective space, $\mathbb{RP}^d$, since

$$
\begin{aligned}
p(S - \{0\}) &= \{[u]_\sim \mid u \in S - \{0\}\} \\
&= \{[-u]_\sim \mid u \in S - \{0\}\} \\
&= \{[u]_\sim \mid u \in -S - \{0\}\} = p((-S) - \{0\}) \\
&= \{[u]_\sim \mid u \in S - \{0\}\} \cup \{[u]_\sim \mid u \in (-S) - \{0\}\} = p((S \cup -S) - \{0\}),
\end{aligned}
$$

because $[u]_\sim = [-u]_\sim$. Using these facts we obtain a bijection between subsets of $\mathbb{RP}^d$ and certain subsets of $\mathbb{R}^{d+1}$.

We say that a set, $S \subseteq \mathbb{R}^{d+1}$, is *symmetric* iff $S = -S$. Obviously, $S \cup -S$ is symmetric for any set, S. Say that a subset, $C \subseteq \mathbb{R}^{d+1}$, is a *double cone* iff for every $u \in C - \{0\}$, the entire line, $[u]$, spanned by u is contained in C. We exclude the trivial double cone, $C = \{0\}$, since the trivial vector space does not yield a projective space. Thus, every double cone can be viewed as a set of lines through 0. Note that a double cone is symmetric. Given any nonempty subset, $S \subseteq \mathbb{RP}^d$, let $v(S) \subseteq \mathbb{R}^{d+1}$ be the set of vectors,

$$v(S) = \bigcup_{[u]_\sim \in S} [u]_\sim \cup \{0\}.$$

Note that $v(S)$ is a double cone.

Proposition 5.1. *The map, $v\colon S \mapsto v(S)$, from the set of nonempty subsets of $\mathbb{RP}^d$ to the set of nonempty, nontrivial double cones in $\mathbb{R}^{d+1}$ is a bijection.*

Proof. We already noted that $v(S)$ is nontrivial double cone. Consider the map,

$$\text{ps}\colon S \mapsto p(S) = \{[u]_\sim \in \mathbb{RP}^d \mid u \in S - \{0\}\}.$$

We leave it as an easy exercise to check that $\text{ps} \circ v = \text{id}$ and $v \circ \text{ps} = \text{id}$, which shows that v and ps are mutual inverses.   □

Given any subspace, $X \subseteq \mathbb{R}^{n+1}$, with $\dim X = k+1 \geq 1$ and $0 \leq k \leq n$, a *k-dimensional projective subspace* of $\mathbb{RP}^n$ is the image, $Y = p(X - \{0\})$, of $X - \{0\}$ under the projection p. We often write $Y = \mathbb{P}(X)$. When $k = n - 1$, we say that Y is a *projective hyperplane* or simply a *hyperplane*. When $k = 1$, we say that Y is a *projective line* or simply a *line*. It is easy to see that every projective hyperplane, H, is the kernel (zero set) of some linear equation of the form

$$a_1 x_1 + \cdots + a_{n+1} x_{n+1} = 0,$$

where one of the a_i is nonzero, in the sense that

$$H = \{(x_1\colon \cdots \colon x_{n+1}) \in \mathbb{RP}^n \mid a_1 x_1 + \cdots + a_{n+1} x_{n+1} = 0\}.$$

Conversely, the kernel of any such linear equation defines a projective hyperplane. Furthermore, given a projective hyperplane, $H \subseteq \mathbb{RP}^n$, the linear equation defining H is unique up to a nonzero scalar.

For any i, with $1 \leq i \leq n + 1$, the set

$$U_i = \{(x_1\colon \cdots \colon x_{n+1}) \in \mathbb{RP}^n \mid x_i \neq 0\}$$

is a subset of $\mathbb{RP}^n$ called an *affine patch* of $\mathbb{RP}^n$. We have a bijection, $\varphi_i\colon U_i \to \mathbb{R}^n$, between U_i and $\mathbb{R}^n$ given by

$$\varphi_i\colon (x_1\colon \cdots \colon x_{n+1}) \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

This map is well defined because if $(y_1, \ldots, y_{n+1}) \sim (x_1, \ldots, x_{n+1})$, that is, $(y_1, \ldots, y_{n+1}) = \lambda(x_1, \ldots, x_{n+1})$, with $\lambda \neq 0$, then

$$\frac{y_j}{y_i} = \frac{\lambda x_j}{\lambda x_i} = \frac{x_j}{x_i} \qquad (1 \leq j \leq n + 1),$$

since $\lambda \neq 0$ and $x_i, y_i \neq 0$. The inverse, $\psi_i\colon \mathbb{R}^n \to U_i \subseteq \mathbb{RP}^n$, of φ_i is given by

$$\psi_i\colon (x_1, \cdots, x_n) \mapsto (x_1\colon \cdots x_{i-1}\colon 1\colon x_i\colon \cdots \colon x_n).$$

Observe that the bijection, φ_i, between U_i and $\mathbb{R}^n$ can also be viewed as the bijection

$$(x_1 \colon \cdots \colon x_{n+1}) \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right),$$

between U_i and the hyperplane, $H_i \subseteq \mathbb{R}^{n+1}$, of equation $x_i = 1$. We will make heavy use of these bijections. For example, for any subset, $S \subseteq \mathbb{RP}^n$, the "view of S from the patch U_i", $S \restriction U_i$, is in bijection with $v(S) \cap H_i$, where $v(S)$ is the double cone associated with S (see Proposition 5.1).

The affine patches, $U_1, \ldots, U_{n+1}$, cover the projective space $\mathbb{RP}^n$, in the sense that every $(x_1 \colon \cdots \colon x_{n+1}) \in \mathbb{RP}^n$ belongs to one of the U_i's, as not all $x_i = 0$. The U_i's turn out to be open subsets of $\mathbb{RP}^n$ and they have nonempty overlaps. When we restrict ourselves to one of the U_i, we have an "affine view of $\mathbb{RP}^n$ from U_i". In particular, on the affine patch U_{n+1}, we have the "standard view" of $\mathbb{R}^n$ embedded into $\mathbb{RP}^n$ as H_{n+1}, the hyperplane of equation $x_{n+1} = 1$. The complement, $H_i(0)$, of U_i in $\mathbb{RP}^n$ is the (projective) hyperplane of equation $x_i = 0$ (a copy of $\mathbb{RP}^{n-1}$). With respect to the affine patch, U_i, the hyperplane, $H_i(0)$, plays the role of *hyperplane (of points) at infinity*.

From now on, for simplicity of notation, we will write $\mathbb{P}^n$ for $\mathbb{RP}^n$. We need to define projective maps. Such maps are induced by linear maps.

Definition 5.2. Any injective linear map, $h \colon \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$, induces a map, $\mathbb{P}(h) \colon \mathbb{P}^m \to \mathbb{P}^n$, defined by

$$\mathbb{P}(h)([u]_\sim) = [h(u)]_\sim$$

and called a *projective map*. When $m = n$ and h is bijective, the map $\mathbb{P}(h)$ is also bijective and it is called a *projectivity*.

We have to check that this definition makes sense, that is, it is compatible with the equivalence relation, $\sim$. For this, assume that $u \sim v$, that is

$$v = \lambda u,$$

with $\lambda \neq 0$ (of course, $u, v \neq 0$). As h is linear, we get

$$h(v) = h(\lambda u) = \lambda h(u),$$

that is, $h(u) \sim h(v)$, which shows that $[h(u)]_\sim$ does not depend on the representative chosen in the equivalence class of $[u]_\sim$. It is also easy to check that whenever two linear maps, h_1 and h_2, induce the same projective map, *i.e.*, if $\mathbb{P}(h_1) = \mathbb{P}(h_2)$, then there is a nonzero scalar, λ, so that $h_2 = \lambda h_1$.

Why did we require h to be injective? Because if h has a nontrivial kernel, then, any nonzero vector, $u \in \mathrm{Ker}\,(h)$, is mapped to 0, but as 0 does **not** correspond to any point of $\mathbb{P}^n$, the map $\mathbb{P}(h)$ is undefined on $\mathbb{P}(\mathrm{Ker}\,(h))$.

In some case, we allow projective maps induced by non-injective linear maps h. In this case, $\mathbb{P}(h)$ is a map whose domain is $\mathbb{P}^n - \mathbb{P}(\mathrm{Ker}\,(h))$. An example is the map, $\sigma_N \colon \mathbb{P}^3 \to \mathbb{P}^2$, given by

$$(x_1 \colon x_2 \colon x_3 \colon x_4) \mapsto (x_1 \colon x_2 \colon x_4 - x_3),$$

which is undefined at the point $(0 \colon 0 \colon 1 \colon 1)$. This map is the "homogenization" of the central projection (from the north pole, $N = (0,0,1)$) from $\mathbb{E}^3$ onto $\mathbb{E}^2$.

> Although a projective map, $f \colon \mathbb{P}^m \to \mathbb{P}^n$, is induced by some linear map, h, the map f is **not** linear! This is because linear combinations of points in $\mathbb{P}^m$ *do not make any sense!*

Another way of defining functions (possibly partial) between projective spaces involves using homogeneous polynomials. If $p_1(x_1, \ldots, x_{m+1})$, $\ldots$, $p_{n+1}(x_1, \ldots, x_{m+1})$ are $n+1$ homogeneous polynomials all of the same degree, d, and if these $n+1$ polynomials do not vanish simultaneously, then we claim that the function, f, given by

$$f(x_1 \colon \cdots \colon x_{m+1}) = (p_1(x_1, \ldots, x_{m+1}) \colon \cdots \colon p_{n+1}(x_1, \ldots, x_{m+1}))$$

is indeed a well-defined function from $\mathbb{P}^m$ to $\mathbb{P}^n$. Indeed, if $(y_1, \ldots, y_{m+1}) \sim (x_1, \ldots, x_{m+1})$, that is, $(y_1, \ldots, y_{m+1}) = \lambda(x_1, \ldots, x_{m+1})$, with $\lambda \neq 0$, as the p_i are homogeneous of degree d,

$$p_i(y_1, \ldots, y_{m+1}) = p_i(\lambda x_1, \ldots, \lambda x_{m+1}) = \lambda^d p_i(x_1, \ldots, x_{m+1}),$$

and so,

$$
\begin{aligned}
f(y_1 \colon \cdots \colon y_{m+1}) &= (p_1(y_1, \ldots, y_{m+1}) \colon \cdots \colon p_{n+1}(y_1, \ldots, y_{m+1})) \\
&= (\lambda^d p_1(x_1, \ldots, x_{m+1}) \colon \cdots \colon \lambda^d p_{n+1}(x_1, \ldots, x_{m+1})) \\
&= \lambda^d (p_1(x_1, \ldots, x_{m+1}) \colon \cdots \colon p_{n+1}(x_1, \ldots, x_{m+1})) \\
&= \lambda^d f(x_1 \colon \cdots \colon x_{m+1}),
\end{aligned}
$$

which shows that $f(y_1 \colon \cdots \colon y_{m+1}) \sim f(x_1 \colon \cdots \colon x_{m+1})$, as required.

For example, the map, $\tau_N \colon \mathbb{P}^2 \to \mathbb{P}^3$, given by

$$(x_1 \colon x_2, \colon x_3) \mapsto (2x_1 x_3 \colon 2x_2 x_3 \colon x_1^2 + x_2^2 - x_3^2 \colon x_1^2 + x_2^2 + x_3^2),$$

is well-defined. It turns out to be the "homogenization" of the inverse stereographic map from $\mathbb{E}^2$ to S^2 (see Section 8.5). Observe that

$$\tau_N(x_1 \colon x_2 \colon 0) = (0 \colon 0 \colon x_1^2 + x_2^2 \colon x_1^2 + x_2^2) = (0 \colon 0 \colon 1 \colon 1),$$

that is, τ_N maps all the points at infinity (in $H_3(0)$) to the "north pole", $(0 \colon 0 \colon 1 \colon 1)$. However, when $x_3 \neq 0$, we can prove that τ_N is injective (in fact, its inverse is σ_N, defined earlier).

Most interesting subsets of projective space arise as the collection of zeros of a (finite) set of homogeneous polynomials. Let us begin with a single homogeneous polynomial, $p(x_1, \ldots, x_{n+1})$, of degree d and set

$$V(p) = \{(x_1 \colon \cdots \colon x_{n+1}) \in \mathbb{P}^n \mid p(x_1, \ldots, x_{n+1}) = 0\}.$$

As usual, we need to check that this definition does not depend on the specific representative chosen in the equivalence class of $[(x_1, \ldots, x_{n+1})]_\sim$. If $(y_1, \ldots, y_{n+1}) \sim (x_1, \ldots, x_{n+1})$, that is, $(y_1, \ldots, y_{n+1}) = \lambda(x_1, \ldots, x_{n+1})$, with $\lambda \neq 0$, as p is homogeneous of degree d,

$$p(y_1, \ldots, y_{n+1}) = p(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d p(x_1, \ldots, x_{n+1}),$$

and as $\lambda \neq 0$,

$$p(y_1, \ldots, y_{n+1}) = 0 \quad \text{iff} \quad p(x_1, \ldots, x_{n+1}) = 0,$$

which shows that $V(p)$ is well defined. For a set of homogeneous polynomials (not necessarily of the same degree), $\mathcal{E} = \{p_1(x_1, \ldots, x_{n+1}), \ldots, p_s(x_1, \ldots, x_{n+1})\}$, we set

$$V(\mathcal{E}) = \bigcap_{i=1}^{s} V(p_i) = \{(x_1 \colon \cdots \colon x_{n+1}) \in \mathbb{P}^n \mid p_i(x_1, \ldots, x_{n+1}) = 0, \ i = 1 \ldots, s\}.$$

The set, $V(\mathcal{E})$, is usually called the *projective variety* defined by $\mathcal{E}$ (or *cut out by $\mathcal{E}$*). When $\mathcal{E}$ consists of a single polynomial, p, the set $V(p)$ is called a (projective) *hypersurface*. For example, if

$$p(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2,$$

then $V(p)$ is the *projective sphere* in $\mathbb{P}^3$, also denoted S^2. Indeed, if we "look" at $V(p)$ on the affine patch U_4, where $x_4 \neq 0$, we know that this amounts to setting $x_4 = 1$, and we do get the set of points $(x_1, x_2, x_3, 1) \in U_4$ satisfying $x_1^2 + x_2^2 + x_3^2 - 1 = 0$, our usual 2-sphere! However, if we look at $V(p)$ on the patch U_1, where $x_1 \neq 0$, we see the quadric of equation $1 + x_2^2 + x_3^2 = x_4^2$, which is not a sphere but a hyperboloid of two sheets! Nevertheless, if we pick $x_4 = 0$ as the plane at infinity, note that the projective sphere does not have points at infinity since the only *real* solution of $x_1^2 + x_2^2 + x_3^2 = 0$ is $(0, 0, 0)$, but $(0, 0, 0, 0)$ does not correspond to any point of $\mathbb{P}^3$.

Another example is given by

$$q = (x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3 x_4,$$

for which $V(q)$ corresponds to a paraboloid in the patch U_4. Indeed, if we set $x_4 = 1$, we get the set of points in U_4 satisfying $x_3 = x_1^2 + x_2^2$. For this reason, we denote $V(q)$ by $\mathcal{P}$ and called it a *(projective) paraboloid*.

Given any homogeneous polynomial, $F(x_1, \ldots, x_{d+1})$, we will also make use of the *hypersurface cone*, $C(F) \subseteq \mathbb{R}^{d+1}$, defined by

$$C(F) = \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid F(x_1, \ldots, x_{d+1}) = 0\}.$$

Observe that $V(F) = \mathbb{P}(C(F))$.

Remark: Every variety, $V(\mathcal{E})$, defined by a *set* of polynomials,
$\mathcal{E} = \{p_1(x_1, \ldots, x_{n+1}), \ldots, p_s(x_1, \ldots, x_{n+1})\}$, is also the hypersurface defined by the *single* polynomial equation,

$$p_1^2 + \cdots + p_s^2 = 0.$$

This fact, peculiar to the real field, $\mathbb{R}$, is a mixed blessing. On the one-hand, the study of varieties is reduced to the study of hypersurfaces. On the other-hand, this is a hint that we should expect that such a study will be hard.

Perhaps to the surprise of the novice, there is a bijective projective map (a projectivity) sending S^2 to $\mathcal{P}$. This map, θ, is given by

$$\theta(x_1 \colon x_2 \colon x_3 \colon x_4) = (x_1 \colon x_2 \colon x_3 + x_4 \colon x_4 - x_3),$$

whose inverse is given by

$$\theta^{-1}(x_1 \colon x_2 \colon x_3 \colon x_4) = \left(x_1 \colon x_2 \colon \frac{x_3 - x_4}{2} \colon \frac{x_3 + x_4}{2} \right).$$

Indeed, if $(x_1 \colon x_2 \colon x_3 \colon x_4)$ satisfies

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0,$$

and if $(z_1 \colon z_2 \colon z_3 \colon z_4) = \theta(x_1 \colon x_2 \colon x_3 \colon x_4)$, then from above,

$$(x_1 \colon x_2 \colon x_3 \colon x_4) = \left(z_1 \colon z_2 \colon \frac{z_3 - z_4}{2} \colon \frac{z_3 + z_4}{2} \right),$$

and by plugging the right-hand sides in the equation of the sphere, we get

$$
\begin{aligned}
z_1^2 + z_2^2 + \left(\frac{z_3 - z_4}{2}\right)^2 - \left(\frac{z_3 + z_4}{2}\right)^2 &= z_1^2 + z_2^2 + \frac{1}{4}(z_3^2 + z_4^2 - 2z_3 z_4 - (z_3^2 + z_4^2 + 2z_3 z_4)) \\
&= z_1^2 + z_2^2 - z_3 z_4 = 0,
\end{aligned}
$$

which is the equation of the paraboloid, $\mathcal{P}$.

5.2   Projective Polyhedra

Following the "projective doctrine" which consists in replacing points by lines through the origin, that is, to "conify" everything, we will define a projective polyhedron as any set of points in $\mathbb{P}^d$ induced by a polyhedral cone in $\mathbb{R}^{d+1}$. To do so, it is preferable to consider cones as sets of positive combinations of vectors (see Definition 4.3). Just to refresh our

memory, a set, $C \subseteq \mathbb{R}^d$, is a $\mathcal{V}$-*cone* or *polyhedral cone* if C is the positive hull of a finite set of vectors, that is,

$$C = \text{cone}(\{u_1, \ldots, u_p\}),$$

for some vectors, $u_1, \ldots, u_p \in \mathbb{R}^d$. An $\mathcal{H}$-*cone* is any subset of $\mathbb{R}^d$ given by a finite intersection of closed half-spaces cut out by hyperplanes through 0.

A good place to learn about cones (and much more) is Fulton [19]. See also Ewald [18].

By Theorem 4.18, $\mathcal{V}$-cones and $\mathcal{H}$-cones form the same collection of convex sets (for every $d \geq 0$). Naturally, we can think of these cones as sets of rays (half-lines) of the form

$$\langle u \rangle_+ = \{\lambda u \mid \lambda \in \mathbb{R}, \ \lambda \geq 0\},$$

where $u \in \mathbb{R}^d$ is any *nonzero* vector. We exclude the trivial cone, $\{0\}$, since 0 does not define any point in projective space. When we "go projective", each ray corresponds to the full line, $\langle u \rangle$, spanned by u which can be expressed as

$$\langle u \rangle = \langle u \rangle_+ \cup -\langle u \rangle_+,$$

where $-\langle u \rangle_+ = \langle u \rangle_- = \{\lambda u \mid \lambda \in \mathbb{R}, \ \lambda \leq 0\}$. Now, if $C \subseteq \mathbb{R}^d$ is a polyhedral cone, obviously $-C$ is also a polyhedral cone and the set $C \cup -C$ consists of the union of the two polyhedral cones C and $-C$. Note that $C \cup -C$ can be viewed as the set of all lines determined by the nonzero vectors in C (and $-C$). It is a double cone. Unless C is a closed half-space, $C \cup -C$ is not convex. It seems perfectly natural to define a projective polyhedron as any set of lines induced by a set of the form $C \cup -C$, where C is a polyhedral cone.

Definition 5.3. A *projective polyhedron* is any subset, $P \subseteq \mathbb{P}^d$, of the form

$$P = p((C \cup -C) - \{0\}) = p(C - \{0\}),$$

where C is any polyhedral cone ($\mathcal{V}$ or $\mathcal{H}$ cone) in $\mathbb{R}^{d+1}$ (with $C \neq \{0\}$). We write $P = \mathbb{P}(C \cup -C)$ or $P = \mathbb{P}(C)$.

It is important to observe that because $C \cup -C$ is a double cone there is a bijection between nontrivial double polyhedral cones and projective polyhedra. So, projective polyhedra are equivalent to double polyhedral cones. However, the projective interpretation of the lines induced by $C \cup -C$ as points in $\mathbb{P}^d$ makes the study of projective polyhedra geometrically more interesting.

Projective polyhedra inherit many of the properties of cones but we have to be careful because we are really dealing with double cones, $C \cup -C$, and not cones. As a consequence, there are a few unpleasant surprises, for example, the fact that the collection of projective polyhedra is **not** closed under intersection!

Before dealing with these issues, let us show that every "standard" polyhedron, $P \subseteq \mathbb{E}^d$, has a natural projective completion, $\widetilde{P} \subseteq \mathbb{P}^d$, such that on the affine patch U_{d+1} (where $x_{d+1} \neq$

0), $\widetilde{P} \upharpoonright U_{d+1} = P$. For this, we use our theorem on the Polyhedron–Cone Correspondence (Theorem 4.19, part (2)).

Let $A = X + U$, where X is a set of points in $\mathbb{E}^d$ and U is a cone in $\mathbb{R}^d$. For every point, $x \in X$, and every vector, $u \in U$, let

$$\widehat{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \qquad \widehat{u} = \begin{pmatrix} u \\ 0 \end{pmatrix},$$

and let $\widehat{X} = \{\widehat{x} \mid x \in X\}$, $\widehat{U} = \{\widehat{u} \mid u \in U\}$ and $\widehat{A} = \{\widehat{a} \mid a \in A\}$, with

$$\widehat{a} = \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

Then,

$$C(A) = \operatorname{cone}(\{\widehat{X} \cup \widehat{U}\})$$

is a cone in $\mathbb{R}^{d+1}$ such that

$$\widehat{A} = C(A) \cap H_{d+1},$$

where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$. If we set $\widetilde{A} = \mathbb{P}(C(A))$, then we get a subset of $\mathbb{P}^d$ and in the patch U_{d+1}, the set $\widetilde{A} \upharpoonright U_{d+1}$ is in bijection with the intersection $(C(A) \cup -C(A)) \cap H_{d+1} = \widehat{A}$, and thus, in bijection with A. We call $\widetilde{A}$ the *projective completion of A*. We have an injection, $A \longrightarrow \widetilde{A}$, given by

$$(a_1, \ldots, a_d) \mapsto (a_1 : \cdots : a_d : 1),$$

which is just the map, $\psi_{d+1} \colon \mathbb{R}^d \to U_{d+1}$. What the projective completion does is to add to A the "points at infinity" corresponding to the vectors in U, that is, the points of $\mathbb{P}^d$ corresponding to the lines in the cone, U. In particular, if $X = \operatorname{conv}(Y)$ and $U = \operatorname{cone}(V)$, for some finite sets $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$, then $P = \operatorname{conv}(Y) + \operatorname{cone}(V)$ is a $\mathcal{V}$-polyhedron and $\widetilde{P} = \mathbb{P}(C(P))$ is a projective polyhedron. The projective polyhedron, $\widetilde{P} = \mathbb{P}(C(P))$, is called the *projective completion* of P.

Observe that if C is a closed half-space in $\mathbb{R}^{d+1}$, then $P = \mathbb{P}(C \cup -C) = \mathbb{P}^d$. Now, if $C \subseteq \mathbb{R}^{d+1}$ is a polyhedral cone and C is contained in a closed half-space, it is still possible that C contains some nontrivial linear subspace and we would like to understand this situation.

The first thing to observe is that $U = C \cap (-C)$ is the largest linear subspace contained in C. If $C \cap (-C) = \{0\}$, we say that C is a *pointed* or *strongly convex* cone. In this case, one immediately realizes that 0 is an extreme point of C and so, there is a hyperplane, H, through 0 so that $C \cap H = \{0\}$, that is, except for its apex, C lies in one of the open half-spaces determined by H. As a consequence, by a linear change of coordinates, we may assume that this hyperplane is H_{d+1} and so, for every projective polyhedron, $P = \mathbb{P}(C)$, if C is pointed then there is an affine patch (say, U_{d+1}) where P has no points at infinity, that is, P is a polytope! On the other hand, from another patch, U_i, as $P \upharpoonright U_i$ is in bijection

with $(C \cup -C) \cap H_i$, the projective polyhedron P viewed on U_i may consist of *two* disjoint polyhedra.

The situation is very similar to the classical theory of projective conics or quadrics (for example, see Brannan, Esplen and Gray, [10]). The case where C is a pointed cone corresponds to the nondegenerate conics or quadrics. In the case of the conics, depending how we slice a cone, we see an ellipse, a parabola or a hyperbola. For projective polyhedra, when we slice a polyhedral double cone, $C \cup -C$, we may see a polytope (*elliptic type*) a single unbounded polyhedron (*parabolic type*) or two unbounded polyhedra (*hyperbolic type*).

Now, when $U = C \cap (-C) \neq \{0\}$, the polyhedral cone, C, contains the linear subspace, U, and if $C \neq \mathbb{R}^{d+1}$, then for every hyperplane, H, such that C is contained in one of the two closed half-spaces determined by H, the subspace $U \cap H$ is nontrivial. An example is the cone, $C \subseteq \mathbb{R}^3$, determined by the intersection of two planes through 0 (a wedge). In this case, U is equal to the line of intersection of these two planes. Also observe that $C \cap (-C) = C$ iff $C = -C$, that is, iff C is a linear subspace.

The situation where $C \cap (-C) \neq \{0\}$ is reminiscent of the case of cylinders in the theory of quadric surfaces (see [10] or Berger [6]). Now, every cylinder can be viewed as the ruled surface defined as the family of lines orthogonal to a plane and touching some nondegenerate conic. A similar decomposition holds for polyhedral cones as shown below in a proposition borrowed from Ewald [18] (Chapter V, Lemma 1.6). We should warn the reader that we have some doubts about the proof given there and so, we offer a different proof adapted from the proof of Lemma 16.2 in Barvinok [3]. Given any two subsets, $V, W \subseteq \mathbb{R}^d$, as usual, we write $V + W = \{v + w \mid v \in V, \ w \in W\}$ and $v + W = \{v + w \mid w \in W\}$, for any $v \in \mathbb{R}^d$.

Proposition 5.2. *For every polyhedral cone, $C \subseteq \mathbb{R}^d$, if $U = C \cap (-C)$, then there is some pointed cone, C_0, so that U and C_0 are orthogonal and*

$$C = U + C_0,$$

with $\dim(U) + \dim(C_0) = \dim(C)$.

Proof. We already know that $U = C \cap (-C)$ is the largest linear subspace of C. Let $U^\perp$ be the orthogonal complement of U in $\mathbb{R}^d$ and let $\pi \colon \mathbb{R}^d \to U^\perp$ be the orthogonal projection onto $U^\perp$. By Proposition 4.12, the projection, $C_0 = \pi(C)$, of C onto $U^\perp$ is a polyhedral cone. We claim that C_0 is pointed and that

$$C = U + C_0.$$

Since $\pi^{-1}(v) = v + U$ for every $v \in C_0$, we have $U + C_0 \subseteq C$. On the other hand, by definition of C_0, we also have $C \subseteq U + C_0$, so $C = U + C_0$. If C_0 was not pointed, then it would contain a linear subspace, V, of dimension at least 1 but then, $U + V$ would be a linear subspace of C of dimension strictly greater than U, which is impossible. Finally, $\dim(U) + \dim(C_0) = \dim(C)$ is obvious by orthogonality. $\qquad\square$

The linear subspace, $U = C \cap (-C)$, is called the *cospan* of C. Both U and C_0 are uniquely determined by C. To a great extent, Proposition reduces the study of non-pointed cones to the study of pointed cones. We propose to call the projective polyhedra of the form $P = \mathbb{P}(C)$, where C is a cone with a non-trivial cospan (a non-pointed cone) a *projective polyhedral cylinder*, by analogy with the quadric surfaces. We also propose to call the projective polyhedra of the form $P = \mathbb{P}(C)$, where C is a pointed cone, a *projective polytope* (or *nondegenerate projective polyhedron*).

The following propositions show that projective polyhedra behave well under projective maps and intersection with a hyperplane:

Proposition 5.3. *Given any projective map,* $h \colon \mathbb{P}^m \to \mathbb{P}^n$, *for any projective polyhedron,* $P \subseteq \mathbb{P}^m$, *the image,* $h(P)$, *of* P *is a projective polyhedron in* $\mathbb{P}^n$. *Even if* $h \colon \mathbb{P}^m \to \mathbb{P}^n$ *is a partial map but* h *is defined on* P, *then* $h(P)$ *is a projective polyhedron.*

Proof. The projective map, $h \colon \mathbb{P}^m \to \mathbb{P}^n$, is of the form $h = \mathbb{P}(\widehat{h})$, for some injective linear map, $\widehat{h} \colon \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$. Moreover, the projective polyhedron, P, is of the form $P = \mathbb{P}(C)$, for some polyhedral cone, $C \subseteq \mathbb{R}^{m+1}$, with $C = \mathrm{cone}(\{u_1, \ldots, u_p\})$, for some nonzero vector $u_i \in \mathbb{R}^{m+1}$. By definition,

$$\mathbb{P}(h)(P) = \mathbb{P}(h)(\mathbb{P}(C)) = \mathbb{P}(\widehat{h}(C)).$$

As $\widehat{h}$ is linear,

$$\widehat{h}(C) = \widehat{h}(\mathrm{cone}(\{u_1, \ldots, u_p\})) = \mathrm{cone}(\{\widehat{h}(u_1), \ldots, \widehat{h}(u_p)\}).$$

If we let $\widehat{C} = \mathrm{cone}(\{\widehat{h}(u_1), \ldots, \widehat{h}(u_p)\})$, then $\widehat{h}(C) = \widehat{C}$ is a polyhedral cone and so,

$$\mathbb{P}(h)(P) = \mathbb{P}(\widehat{h}(C)) = \mathbb{P}(\widehat{C})$$

is a projective cone. This argument does not depend on the injectivity of $\widehat{h}$, as long as $C \cap \mathrm{Ker}\,(\widehat{h}) = \{0\}$. $\square$

Proposition 5.3 together with earlier arguments shows that every projective polytope, $P \subseteq \mathbb{P}^d$, is equivalent under some suitable projectivity to another projective polytope, P', which is a polytope when viewed in the affine patch, U_{d+1}. This property is similar to the fact that every (non-degenerate) projective conic is projectively equivalent to an ellipse.

Since the notion of a face is defined for arbitrary polyhedra it is also defined for cones. Consequently, we can define the notion of a face for projective polyhedra. Given a projective polyhedron, $P \subseteq \mathbb{P}^d$, where $P = \mathbb{P}(C)$ for some polyhedral cone (uniquely determined by P), $C \subseteq \mathbb{R}^{d+1}$, a *face of* P is any subset of P of the form $\mathbb{P}(F) = p(F - \{0\})$, for any nontrivial face, $F \subseteq C$, of C ($F \neq \{0\}$). Consequently, we say that $\mathbb{P}(F)$ is a *vertex* iff $\dim(F) = 1$, an *edge* iff $\dim(F) = 2$ and a *facet* iff $\dim(F) = \dim(C) - 1$. The projective polyhedron, P, and the empty set are the *improper* faces of P. If C is strongly convex, then it is easy

to prove that C is generated by its edges (its one-dimensional faces, these are rays) in the sense that any set of nonzero vectors spanning these edges generates C (using positive linear combinations). As a consequence, if C is strongly convex, we may say that P is "spanned" by its vertices, since P is equal to $\mathbb{P}$(all positive combinations of vectors representing its edges).

Remark: Even though we did not define the notion of convex combination of points in $\mathbb{P}^d$, the notion of projective polyhedron gives us a way to mimic certain properties of convex sets in the framework of projective geometry. That's because every projective polyhedron corresponds to a unique polyhedral cone.

If our projective polyhedron is the completion, $\widetilde{P} = \mathbb{P}(C(P)) \subseteq \mathbb{P}^d$, of some polyhedron, $P \subseteq \mathbb{R}^d$, then each face of the cone, $C(P)$, is of the form $C(F)$, where F is a face of P and so, each face of $\widetilde{P}$ is of the form $\mathbb{P}(C(F))$, for some face, F, of P. In particular, in the affine patch, U_{d+1}, the face, $\mathbb{P}(C(F))$, is in bijection with the face, F, of P. We will usually identify $\mathbb{P}(C(F))$ and F.

We now consider the intersection of projective polyhedra but first, let us make some general remarks about the intersection of subsets of $\mathbb{P}^d$. Given any two nonempty subsets, $\mathbb{P}(S)$ and $\mathbb{P}(S')$, of $\mathbb{P}^d$ what is $\mathbb{P}(S) \cap \mathbb{P}(S')$? It is tempting to say that

$$\mathbb{P}(S) \cap \mathbb{P}(S') = \mathbb{P}(S \cap S'),$$

but unfortunately this is generally false! The problem is that $\mathbb{P}(S) \cap \mathbb{P}(S')$ is the set of *all lines* determined by vectors both in S and S' but there may be some line spanned by some vector $u \in (-S) \cap S'$ or $u \in S \cap (-S')$ such that u does not belong to $S \cap S'$ or $-(S \cap S')$.

Observe that

$$\begin{aligned}
-(-S) &= S \\
-(S \cap S') &= (-S) \cap (-S').
\end{aligned}$$

Then, the correct intersection is given by

$$\begin{aligned}
(S \cup -S) \cap (S' \cup -S') &= (S \cap S') \cup ((-S) \cap (-S')) \cup (S \cap (-S')) \cup ((-S) \cap S') \\
&= (S \cap S') \cup -(S \cap S') \cup (S \cap (-S')) \cup -(S \cap (-S')),
\end{aligned}$$

which is the union of two double cones (except for 0, which belongs to both). Therefore,

$$\mathbb{P}(S) \cap \mathbb{P}(S') = \mathbb{P}(S \cap S') \cup \mathbb{P}(S \cap (-S')) = \mathbb{P}(S \cap S') \cup \mathbb{P}((-S) \cap S'),$$

since $\mathbb{P}(S \cap (-S')) = \mathbb{P}((-S) \cap S')$.

Furthermore, if S' is symmetric (*i.e.*, $S' = -S'$), then

$$\begin{aligned}
(S \cup -S) \cap (S' \cup -S') &= (S \cup -S) \cap S' \\
&= (S \cap S') \cup ((-S) \cap S') \\
&= (S \cap S') \cup -(S \cap (-S')) \\
&= (S \cap S') \cup -(S \cap S').
\end{aligned}$$

Thus, if either S or S' is symmetric, it is true that

$$\mathbb{P}(S) \cap \mathbb{P}(S') = \mathbb{P}(S \cap S').$$

Now, if C is a pointed polyhedral cone then $C \cap (-C) = \{0\}$. Consequently, for any other polyhedral cone, C', we have $(C \cap C') \cap ((-C) \cap C') = \{0\}$. Using these facts we obtain the following result:

Proposition 5.4. *Let* $P = \mathbb{P}(C)$ *and* $P' = \mathbb{P}(C')$ *be any two projective polyhedra in* $\mathbb{P}^d$. *If* $\mathbb{P}(C) \cap \mathbb{P}(C') \neq \emptyset$, *then the following properties hold:*

(1)
$$\mathbb{P}(C) \cap \mathbb{P}(C') = \mathbb{P}(C \cap C') \cup \mathbb{P}(C \cap (-C')),$$

the union of two projective polyhedra. If C *or* C' *is a pointed cone i.e.,* P *or* P' *is a projective polytope, then* $\mathbb{P}(C \cap C')$ *and* $\mathbb{P}(C \cap (-C'))$ *are disjoint.*

(2) If $P' = H$, *for some hyperplane,* $H \subseteq \mathbb{P}^d$, *then* $P \cap H$ *is a projective polyhedron.*

Proof. We already proved (1) so only (2) remains to be proved. Of course, we may assume that $P \neq \mathbb{P}^d$. This time, using the equivalence theorem of $\mathcal{V}$-cones and $\mathcal{H}$-cones (Theorem 4.18), we know that P is of the form $P = \mathbb{P}(C)$, with $C = \bigcap_{i=1}^{p} C_i$, where the C_i are closed half-spaces in $\mathbb{R}^{d+1}$. Moreover, $H = \mathbb{P}(\widehat{H})$, for some hyperplane, $\widehat{H} \subseteq \mathbb{R}^{d+1}$, through 0. Now, as $\widehat{H}$ is symmetric,
$$P \cap H = \mathbb{P}(C) \cap \mathbb{P}(\widehat{H}) = \mathbb{P}(C \cap \widehat{H}).$$

Consequently,

$$
\begin{aligned}
P \cap H &= \mathbb{P}(C \cap \widehat{H}) \\
&= \mathbb{P}\left(\left(\bigcap_{i=1}^{p} C_i \right) \cap \widehat{H} \right).
\end{aligned}
$$

However, $\widehat{H} = \widehat{H}_+ \cap \widehat{H}_-$, where $\widehat{H}_+$ and $\widehat{H}_-$ are the two closed half-spaces determined by $\widehat{H}$ and so,

$$\widehat{C} = \left(\bigcap_{i=1}^{p} C_i \right) \cap \widehat{H} = \left(\bigcap_{i=1}^{p} C_i \right) \cap \widehat{H}_+ \cap \widehat{H}_-$$

is a polyhedral cone. Therefore, $P \cap H = \mathbb{P}(\widehat{C})$ is a projective polyhedron. $\square$

We leave it as an instructive exercise to find explicit examples where $P \cap P'$ consists of two disjoint projective polyhedra in $\mathbb{P}^1$ (or $\mathbb{P}^2$).

Proposition 5.4 can be sharpened a little.

Proposition 5.5. *Let $P = \mathbb{P}(C)$ and $P' = \mathbb{P}(C')$ be any two projective polyhedra in $\mathbb{P}^d$. If $\mathbb{P}(C) \cap \mathbb{P}(C') \neq \emptyset$, then*

$$\mathbb{P}(C) \cap \mathbb{P}(C') = \mathbb{P}(C \cap C') \cup \mathbb{P}(C \cap (-C')),$$

the union of two projective polyhedra. If $C = -C$, i.e., C is a linear subspace (or if C' is a linear subspace), then

$$\mathbb{P}(C) \cap \mathbb{P}(C') = \mathbb{P}(C \cap C').$$

Furthermore, if either C or C' is pointed, the above projective polyhedra are disjoint, else if C and C' both have nontrivial cospan and $\mathbb{P}(C \cap C')$ and $\mathbb{P}(C \cap (-C'))$ intersect then

$$\mathbb{P}(C \cap C') \cap \mathbb{P}(C \cap (-C')) = \mathbb{P}(C \cap (C' \cap (-C'))) \cup \mathbb{P}(C' \cap (C \cap (-C))).$$

Finally, if the two projective polyhedra on the right-hand side intersect, then

$$\mathbb{P}(C \cap (C' \cap (-C'))) \cap \mathbb{P}(C' \cap (C \cap (-C))) = \mathbb{P}((C \cap (-C)) \cap (C' \cap (-C'))).$$

Proof. Left as a simple exercise in boolean algebra. □

In preparation for Section 8.6, we also need the notion of tangent space at a point of a variety.

5.3   Tangent Spaces of Hypersurfaces and Projective Hypersurfaces

Since we only need to consider the case of hypersurfaces we restrict attention to this case (but the general case is a straightforward generalization). Let us begin with a hypersurface of equation $p(x_1, \ldots, x_d) = 0$ in $\mathbb{R}^d$, that is, the set

$$S = V(p) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid p(x_1, \ldots, x_d) = 0\},$$

where $p(x_1, \ldots, x_d)$ is a polynomial of total degree, m.

Pick any point $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Recall that there is a version of the Taylor expansion formula for polynomials such that, for any polynomial, $p(x_1, \ldots, x_d)$, of total degree m, for every $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$, we have

$$
\begin{aligned}
p(a + h) &= p(a) + \sum_{1 \leq |\alpha| \leq m} \frac{D^\alpha p(a)}{\alpha!} h^\alpha \\
&= p(a) + \sum_{i=1}^d p_{x_i}(a) h_i + \sum_{2 \leq |\alpha| \leq m} \frac{D^\alpha p(a)}{\alpha!} h^\alpha,
\end{aligned}
$$

where we use the *multi-index notation*, with $\alpha = (i_1, \ldots, i_d) \in \mathbb{N}^d$, $|\alpha| = i_1 + \cdots + i_d$, $\alpha! = i_1! \cdots i_d!$, $h^\alpha = h_1^{i_1} \cdots h_d^{i_d}$,

$$D^\alpha p(a) = \frac{\partial^{i_1 + \cdots + i_d} p}{\partial x_1^{i_1} \cdots \partial x_d^{i_d}}(a),$$

and

$$p_{x_i}(a) = \frac{\partial p}{\partial x_i}(a).$$

Consider any line, ℓ, through a, given parametrically by

$$\ell = \{a + th \mid t \in \mathbb{R}\},$$

with $h \neq 0$ and say $a \in S$ is a point on the hypersurface, $S = V(p)$, which means that $p(a) = 0$. The intuitive idea behind the notion of the tangent space to S at a is that it is the set of lines that intersect S at a in a point of *multiplicity at least two*, which means that the equation giving the intersection, $S \cap \ell$, namely

$$p(a + th) = p(a_1 + th_1, \ldots, a_d + th_d) = 0,$$

is of the form

$$t^2 q(a, h)(t) = 0,$$

where $q(a, h)(t)$ is some polynomial in t. Using Taylor's formula, as $p(a) = 0$, we have

$$p(a + th) = t \sum_{i=1}^{d} p_{x_i}(a)h_i + t^2 q(a, h)(t),$$

for some polynomial, $q(a, h)(t)$. From this, we see that a is an intersection point of multiplicity at least 2 iff

$$\sum_{i=1}^{d} p_{x_i}(a)h_i = 0. \tag{†}$$

Consequently, if $\nabla p(a) = (p_{x_1}(a), \ldots, p_{x_d}(a)) \neq 0$ (that is, if the gradient of p at a is nonzero), we see that ℓ intersects S at a in a point of multiplicity at least 2 iff h belongs to the hyperplane of equation (†).

Definition 5.4. Let $S = V(p)$ be a hypersurface in $\mathbb{R}^d$. For any point, $a \in S$, if $\nabla p(a) \neq 0$, then we say that a is a *non-singular* point of S. When a is nonsingular, the (*affine*) *tangent space*, $T_a(S)$ (or simply, $T_a S$), to S at a is the hyperplane through a of equation

$$\sum_{i=1}^{d} p_{x_i}(a)(x_i - a_i) = 0.$$

Observe that the hyperplane of the direction of $T_a S$ is the hyperplane through 0 and parallel to $T_a S$ given by

$$\sum_{i=1}^{d} p_{x_i}(a)x_i = 0.$$

When $\nabla p(a) = 0$, we either say that $T_a S$ is undefined or we set $T_a S = \mathbb{R}^d$.

We now extend the notion of tangent space to projective varieties. As we will see, this amounts to homogenizing and the result turns out to be simpler than the affine case!

So, let $S = V(F) \subseteq \mathbb{P}^d$ be a projective hypersurface, which means that

$$S = V(F) = \{(x_1 : \cdots : x_{d+1}) \in \mathbb{P}^d \mid F(x_1, \ldots, x_{d+1}) = 0\},$$

where $F(x_1, \ldots, x_{d+1})$ is a homogeneous polynomial of total degree, m. Again, we say that a point, $a \in S$, is *non-singular* iff $\nabla F(a) = (F_{x_1}(a), \ldots, F_{x_{d+1}}(a)) \neq 0$. For every $i = 1, \ldots, d+1$, let

$$z_j^{\lceil i} = \frac{x_j}{x_i},$$

where $j = 1, \ldots, d+1$ and $j \neq i$, and let $f^{\lceil i}$ be the result of "dehomogenizing" F at i, that is,

$$f^{\lceil i}(z_1^{\lceil i}, \ldots, z_{i-1}^{\lceil i}, z_{i+1}^{\lceil i}, \ldots, z_{d+1}^{\lceil i}) = F(z_1^{\lceil i}, \ldots, z_{i-1}^{\lceil i}, 1, z_{i+1}^{\lceil i}, \ldots, z_{d+1}^{\lceil i}).$$

We define the *(projective) tangent space*, $T_a S$, to a at S as the hyperplane, H, such that for each affine patch, U_i where $a_i \neq 0$, if we let

$$a_j^{\lceil i} = \frac{a_j}{a_i},$$

where $j = 1, \ldots, d+1$ and $j \neq i$, then the restriction, $H \restriction U_i$, of H to U_i is the affine hyperplane tangent to $S \restriction U_i$ given by

$$\sum_{\substack{j=1 \\ j \neq i}}^{d+1} f_{z_j^{\lceil i}}^{\lceil i}(a^{\lceil i})(z_j^{\lceil i} - a_j^{\lceil i}) = 0.$$

Thus, on the affine patch, U_i, the tangent space, $T_a S$, is given by the homogeneous equation

$$\sum_{\substack{j=1 \\ j \neq i}}^{d+1} f_{z_j^{\lceil i}}^{\lceil i}(a^{\lceil i})(x_j - a_j^{\lceil i} x_i) = 0.$$

This looks awful but we can make it pretty if we remember that F is a homogeneous polynomial of degree m and that we have the *Euler relation*:

$$\sum_{j=1}^{d+1} F_{x_j}(a)a_j = mF(a),$$

for every $a = (a_1, \ldots, a_{d+1}) \in \mathbb{R}^{d+1}$. Using this, we can come up with a clean equation for our projective tangent hyperplane. It is enough to carry out the computations for $i = d + 1$. Our tangent hyperplane has the equation

$$\sum_{j=1}^{d} F_{x_j}(a_1^{\lceil d+1}, \ldots, a_d^{\lceil d+1}, 1)(x_j - a_j^{\lceil d+1} x_{d+1}) = 0,$$

that is,

$$\sum_{j=1}^{d} F_{x_j}(a_1^{\lceil d+1}, \ldots, a_d^{\lceil d+1}, 1)x_j + \sum_{j=1}^{d} F_{x_j}(a_1^{\lceil d+1}, \ldots, a_d^{\lceil d+1}, 1)(-a_j^{\lceil d+1} x_{d+1}) = 0.$$

As $F(x_1, \ldots, x_{d+1})$ is homogeneous of degree m, and as $a_{d+1} \neq 0$ on U_{d+1}, we have

$$a_{d+1}^m F(a_1^{\lceil d+1}, \ldots, a_d^{\lceil d+1}, 1) = F(a_1, \ldots, a_d, a_{d+1}),$$

so from the above equation we get

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})x_j + \sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})(-a_j^{\lceil d+1} x_{d+1}) = 0. \qquad (*)$$

Since $a \in S$, we have $F(a) = 0$, so the Euler relation yields

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})a_j + F_{x_{d+1}}(a_1, \ldots, a_{d+1})a_{d+1} = 0,$$

which, by dividing by a_{d+1} and multiplying by x_{d+1}, yields

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})(-a_j^{\lceil d+1} x_{d+1}) = F_{x_{d+1}}(a_1, \ldots, a_{d+1})x_{d+1},$$

and by plugging this in $(*)$, we get

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})x_j + F_{x_{d+1}}(a_1, \ldots, a_{d+1})x_{d+1} = 0.$$

Consequently, the tangent hyperplane to S at a is given by the equation

$$\sum_{j=1}^{d+1} F_{x_j}(a)x_j = 0.$$

Definition 5.5. Let $S = V(F)$ be a hypersurface in $\mathbb{P}^d$, where $F(x_1, \ldots, x_{d+1})$ is a homogeneous polynomial. For any point, $a \in S$, if $\nabla F(a) \neq 0$, then we say that a is a *non-singular* point of S. When a is nonsingular, the (*projective*) *tangent space*, $T_a(S)$ (or simply, $T_a S$), to S at a is the hyperplane through a of equation

$$\sum_{i=1}^{d+1} F_{x_i}(a)x_i = 0.$$

For example, if we consider the sphere, $S^2 \subseteq \mathbb{P}^3$, of equation

$$x^2 + y^2 + z^2 - w^2 = 0,$$

the tangent plane to S^2 at $a = (a_1, a_2, a_3, a_4)$ is given by

$$a_1 x + a_2 y + a_3 z - a_4 w = 0.$$

Remark: If $a \in S = V(F)$, as $F(a) = \sum_{i=1}^{d+1} F_{x_i}(a)a_i = 0$ (by Euler), the equation of the tangent plane, $T_a S$, to S at a can also be written as

$$\sum_{i=1}^{d+1} F_{x_i}(a)(x_i - a_i) = 0.$$

Now, if $a = (a_1 : \cdots : a_d : 1)$ is a point in the affine patch U_{d+1}, then the equation of the intersection of $T_a S$ with U_{d+1} is obtained by setting $a_{d+1} = x_{d+1} = 1$, that is

$$\sum_{i=1}^{d} F_{x_i}(a_1, \ldots, a_d, 1)(x_i - a_i) = 0,$$

which is just the equation of the affine hyperplane to $S \cap U_{d+1}$ at $a \in U_{d+1}$.

It will be convenient to adopt the following notational convention: Given any point, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, written as a row vector, we let $\mathbf{x}$ denote the corresponding column vector such that $\mathbf{x}^\top = x$.

Projectivities behave well with respect to hypersurfaces and their tangent spaces. Let $S = V(F) \subseteq \mathbb{P}^d$ be a projective hypersurface, where F is a homogeneous polynomial of degree m and let $h \colon \mathbb{P}^d \to \mathbb{P}^d$ be a projectivity (a bijective projective map). Assume that h is induced by the invertible $(d+1) \times (d+1)$ matrix, $A = (a_{ij})$, and write $A^{-1} = (a_{ij}^{-1})$. For any hyperplane, $H \subseteq \mathbb{R}^{d+1}$, if φ is any linear from defining φ, *i.e.*, $H = \text{Ker}(\varphi)$, then

$$
\begin{aligned}
h(H) &= \{h(x) \in \mathbb{R}^{d+1} \mid \varphi(x) = 0\} \\
&= \{y \in \mathbb{R}^{d+1} \mid (\exists x \in \mathbb{R}^{d+1})(y = h(x), \varphi(x) = 0)\} \\
&= \{y \in \mathbb{R}^{d+1} \mid (\varphi \circ h^{-1})(y) = 0\}.
\end{aligned}
$$

Consequently, if H is given by

$$\alpha_1 x_1 + \cdots + \alpha_{d+1} x_{d+1} = 0$$

and if we write $\alpha = (\alpha_1, \ldots, \alpha_{d+1})$, then $h(H)$ is the hyperplane given by the equation

$$\alpha A^{-1} \mathbf{y} = 0.$$

Similarly,

$$
\begin{aligned}
h(S) &= \{h(x) \in \mathbb{R}^{d+1} \mid F(x) = 0\} \\
&= \{y \in \mathbb{R}^{d+1} \mid (\exists x \in \mathbb{R}^{d+1})(y = h(x),\ F(x) = 0)\} \\
&= \{y \in \mathbb{R}^{d+1} \mid F((A^{-1}\mathbf{y})^{\top}) = 0\}
\end{aligned}
$$

is the hypersurface defined by the polynomial

$$G(x_1, \ldots, x_{d+1}) = F\left(\sum_{j=1}^{d+1} a_{1j}^{-1} x_j, \ldots, \sum_{j=1}^{d+1} a_{d+1\,j}^{-1} x_j \right).$$

Furthermore, using the chain rule, we get

$$(G_{x_1}, \ldots, G_{x_{d+1}}) = (F_{x_1}, \ldots, F_{x_{d+1}})A^{-1},$$

which shows that a point, $a \in S$, is non-singular iff its image, $h(a) \in h(S)$, is non-singular on $h(S)$. This also shows that

$$h(T_a S) = T_{h(a)} h(S),$$

that is, the projectivity, h, preserves tangent spaces. In summary, we have

Proposition 5.6. *Let $S = V(F) \subseteq \mathbb{P}^d$ be a projective hypersurface, where F is a homogeneous polynomial of degree m and let $h \colon \mathbb{P}^d \to \mathbb{P}^d$ be a projectivity (a bijective projective map). Then, $h(S)$ is a hypersurface in $\mathbb{P}^d$ and a point, $a \in S$, is nonsingular for S iff $h(a)$ is nonsingular for $h(S)$. Furthermore,*

$$h(T_a S) = T_{h(a)} h(S),$$

that is, the projectivity, h, preserves tangent spaces.

Remark: If $h \colon \mathbb{P}^m \to \mathbb{P}^n$ is a projective map, say induced by an injective linear map given by the $(n+1) \times (m+1)$ matrix, $A = (a_{ij})$, given any hypersurface, $S = V(F) \subseteq \mathbb{P}^n$, we can define the *pull-back*, $h^*(S) \subseteq \mathbb{P}^m$, of S, by

$$h^*(S) = \{x \in \mathbb{P}^m \mid F(h(x)) = 0\}.$$

This is indeed a hypersurface because $F(x_1, \ldots, x_{n+1})$ is a homogenous polynomial and $h^*(S)$ is the zero locus of the homogeneous polynomial

$$G(x_1, \ldots, x_{m+1}) = F\left(\sum_{j=1}^{m+1} a_{1j} x_j, \ldots, \sum_{j=1}^{m+1} a_{n+1\,j} x_j \right).$$

If $m = n$ and h is a projectivity, then we have

$$h(S) = (h^{-1})^*(S).$$

5.4 Quadrics (Affine, Projective) and Polar Duality

The case where $S = V(\Phi) \subseteq \mathbb{P}^d$ is a hypersurface given by a homogeneous polynomial, $\Phi(x_1, \ldots, x_{d+1})$, of degree 2 will come up a lot and deserves a little more attention. In this case, if we write $x = (x_1, \ldots, x_{d+1})$, then $\Phi(x) = \Phi(x_1, \ldots, x_{d+1})$ is completely determined by a $(d+1) \times (d+1)$ symmetric matrix, say $F = (f_{ij})$, and we have

$$\Phi(x) = \mathbf{x}^\top F \mathbf{x} = \sum_{i,j=1}^{d+1} f_{ij} x_i x_j.$$

Since F is symmetric, we can write

$$\Phi(x) = \sum_{i,j=1}^{d+1} f_{ij} x_i x_j = \sum_{i=1}^{d+1} f_{ii} x_i^2 + 2 \sum_{\substack{i,j=1 \\ i<j}}^{d+1} f_{ij} x_i x_j.$$

The *polar form*, $\varphi(x, y)$, of $\Phi(x)$, is given by

$$\varphi(x, y) = \mathbf{x}^\top F \mathbf{y} = \sum_{i,j=1}^{d+1} f_{ij} x_i y_j,$$

where $x = (x_1, \ldots, x_{d+1})$ and $y = (y_1, \ldots, y_{d+1})$. Of course,

$$2\varphi(x, y) = \Phi(x + y) - \Phi(x) - \Phi(y).$$

We also check immediately that

$$2\varphi(x, y) = 2\mathbf{x}^\top F \mathbf{y} = \sum_{j=1}^{d+1} \frac{\partial \Phi(x)}{\partial x_j} y_j,$$

and so,

$$\left(\frac{\partial \Phi(x)}{\partial x_1}, \ldots, \frac{\partial \Phi(x)}{\partial x_{d+1}} \right) = 2\mathbf{x}^\top F.$$

The hypersurface, $S = V(\Phi) \subseteq \mathbb{P}^d$, is called a *(projective) (hyper-)quadric surface*. We say that a quadric surface, $S = V(\Phi)$, is *nondegenerate* iff the matrix, F, defining Φ is invertible.

For example, the sphere, $S^d \subseteq \mathbb{P}^{d+1}$, is the nondegenerate quadric given by

$$\mathbf{x}^\top \begin{pmatrix} I_{d+1} & \mathbf{0} \\ \mathbb{O} & -1 \end{pmatrix} \mathbf{x} = 0$$

and the paraboloid, $\mathcal{P} \subseteq \mathbb{P}^{d+1}$, is the nongenerate quadric given by

$$\mathbf{x}^\top \begin{pmatrix} I_d & \mathbf{0} & \mathbf{0} \\ \mathbb{O} & 0 & -\frac{1}{2} \\ \mathbb{O} & -\frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = 0.$$

If $h\colon \mathbb{P}^d \to \mathbb{P}^d$ is a projectivity induced by some invertible matrix, $A = (a_{ij})$, and if $S = V(\Phi)$ is a quadric defined by the matrix F, we immediately check that $h(S)$ is the quadric defined by the matrix $(A^{-1})^\top F A^{-1}$. Furthermore, as A is invertible, we see that S is nondegenerate iff $h(S)$ is nondegenerate.

Observe that polar duality w.r.t. the sphere, S^{d-1}, can be expressed by

$$X^* = \left\{ x \in \mathbb{R}^d \mid (\forall y \in X) \left((\mathbf{x}^\top, 1) \begin{pmatrix} I_d & \mathbf{0} \\ \mathbb{O} & -1 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \leq 0 \right) \right\},$$

where X is any subset of $\mathbb{R}^d$. The above suggests generalizing polar duality with respect to any nondegenerate quadric.

Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric given by the homogeneous polynomial Φ with corresponding polar form φ and matrix $F = (f_{ij})$. Then, we know that φ induces a natural duality between $\mathbb{R}^{d+1}$ and $(\mathbb{R}^{d+1})^*$, namely, for every $u \in \mathbb{R}^{d+1}$, if $\varphi_u \in (\mathbb{R}^{d+1})^*$ is the linear form given by

$$\varphi_u(v) = \varphi(u, v)$$

for every $v \in \mathbb{R}^{d+1}$, then the map $u \mapsto \varphi_u$, from $\mathbb{R}^{d+1}$ to $(\mathbb{R}^{d+1})^*$, is a linear isomorphism.

Definition 5.6. Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. For any $u \in \mathbb{R}^{d+1}$, with $u \neq 0$, the set

$$u^\dagger = \{v \in \mathbb{R}^{d+1} \mid \varphi(u, v) = 0\} = \{v \in \mathbb{R}^{d+1} \mid \varphi_u(v) = 0\} = \mathrm{Ker}\ \varphi_u$$

is a hyperplane called the *polar of u (w.r.t. Q)*.

In terms of the matrix representation of Q, the polar of u is given by the equation

$$\mathbf{u}^\top F \mathbf{x} = 0,$$

or

$$\sum_{j=1}^{d+1} \frac{\partial \Phi(u)}{\partial x_j} x_j = 0.$$

Going over to $\mathbb{P}^d$, we say that $\mathbb{P}(u^\dagger)$ is the *polar (hyperplane)* of the point $a = [u] \in \mathbb{P}^d$ and we write $a^\dagger$ for $\mathbb{P}(u^\dagger)$.

Note that the equation of the polar hyperplane, $a^\dagger$, of a point, $a \in \mathbb{P}^d$, is identical to the equation of the tangent plane to Q at a, except that a is not necessarily on Q. However, if $a \in Q$, then the polar of a is indeed the tangent hyperplane, $T_a Q$, to Q at a.

Proposition 5.7. *Let $Q = V(\Phi(x_1, \ldots, x_{d+1})) \subseteq \mathbb{P}^d$ be a nondegenerate quadric with corresponding polar form, φ, and matrix, F. Then, every point, $a \in Q$, is nonsingular.*

Proof. Since

$$\left(\frac{\partial\Phi(a)}{\partial x_1}, \ldots, \frac{\partial\Phi(a)}{\partial x_{d+1}}\right) = 2\mathbf{a}^\top F,$$

if $a \in Q$ is singular, then $\mathbf{a}^\top F = 0$ with $a \neq 0$, contradicting the fact that F is invertible. $\square$

The reader should prove the following simple proposition:

Proposition 5.8. *Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. Then, the following properties hold: For any two points, $a, b \in \mathbb{P}^d$,*

(1) $a \in b^\dagger$ iff $b \in a^\dagger$;

(2) $a \in a^\dagger$ iff $a \in Q$;

(3) Q does not contain any hyperplane.

Remark: As in the case of the sphere, if Q is a nondegenerate quadric and $a \in \mathbb{P}^d$ is any point such that the polar hyperplane, $a^\dagger$, intersects Q, then there is a nice geometric interpretation for $a^\dagger$. Observe that for every $b \in Q \cap a^\dagger$, the polar hyperplane, $b^\dagger$, is the tangent hyperplane, T_bQ, to Q at b and that $a \in T_bQ$. Also, if $a \in T_bQ$ for any $b \in Q$, as $b^\dagger = T_bQ$, then $b \in a^\dagger$. Therefore, $Q \cap a^\dagger$ is the set of contact points of all the tangent hyperplanes to Q passing through a.

Every hyperplane, $H \subseteq \mathbb{P}^d$, is the polar of a single point, $a \in \mathbb{P}^d$. Indeed, if H is defined by a nonzero linear form, $f \in (\mathbb{R}^{d+1})^*$, as Φ is nondegenerate, there is a unique $u \in \mathbb{R}^{d+1}$, with $u \neq 0$, so that $f = \varphi_u$, and as φ_u vanishes on H, we see that H is the polar of the point $a = [u]$. If H is also the polar of another point, $b = [v]$, then φ_v vanishes on H, which means that

$$\varphi_v = \lambda\varphi_u = \varphi_{\lambda u},$$

with $\lambda \neq 0$ and this implies $v = \lambda u$, that is, $a = [u] = [v] = b$ and the pole of H is indeed unique.

Definition 5.7. Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. The *polar dual (w.r.t. Q)*, X^*, of a subset, $X \subseteq \mathbb{R}^{d+1}$, is given by

$$X^* = \{v \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\varphi(u, v) \leq 0)\}.$$

For every subset, $X \subseteq \mathbb{P}^d$, we let

$$X^* = \mathbb{P}((v(X))^*),$$

where $v(X)$ is the unique double cone associated with X as in Proposition 5.1.

Observe that X^* is always a cone, even if $X \subseteq \mathbb{R}^{d+1}$ is not. By analogy with the Euclidean case, for any nonzero vector, $u \in \mathbb{R}^{d+1}$, let

$$(u^{\dagger})_- = \{v \in \mathbb{R}^{d+1} \mid \varphi(u, v) \leq 0\}.$$

Now, we have the following version of Proposition 4.3:

Proposition 5.9. *Let* $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ *be a nondegenerate quadric with corresponding polar form,* φ, *and matrix,* $F = (f_{ij})$. *For any nontrivial polyhedral cone,* $C = \mathrm{cone}(u_1, \ldots, u_p)$, *where* $u_i \in \mathbb{R}^{d+1}$, $u_i \neq 0$, *we have*

$$C^* = \bigcap_{i=1}^{p} (u_i^{\dagger})_-.$$

If U is the $(d+1) \times p$ matrix whose i^{th} column is u_i, then we can also write

$$C^* = P(U^{\top} F, \mathbf{0}),$$

where

$$P(U^{\top} F, \mathbf{0}) = \{v \in \mathbb{R}^{d+1} \mid U^{\top} F \mathbf{v} \leq \mathbf{0}\}.$$

Consequently, the polar dual of a polyhedral cone w.r.t. a nondegenerate quadric is a polyhedral cone.

Proof. The proof is essentially the same as the proof of Proposition 4.3. As

$$C = \mathrm{cone}(u_1, \ldots, u_p) = \{\lambda_1 u_1 + \cdots + \lambda_p u_p \mid \lambda_i \geq 0, \ 1 \leq i \leq p\},$$

we have

$$
\begin{aligned}
C^* &= \{v \in \mathbb{R}^{d+1} \mid (\forall u \in C)(\varphi(u, v) \leq 0)\} \\
&= \{v \in \mathbb{R}^{d+1} \mid \varphi(\lambda_1 u_1 + \cdots + \lambda_p u_p, v) \leq 0, \ \lambda_i \geq 0, \ 1 \leq i \leq p\} \\
&= \{v \in \mathbb{R}^{d+1} \mid \lambda_1 \varphi(u_1, v) + \cdots + \lambda_p \varphi(u_p, v) \leq 0, \ \lambda_i \geq 0, \ 1 \leq i \leq p\} \\
&= \bigcap_{i=1}^{p} \{v \in \mathbb{R}^{d+1} \mid \varphi(u_i, v) \leq 0\} \\
&= \bigcap_{i=1}^{p} (u_i^{\dagger})_-.
\end{aligned}
$$

By the equivalence theorem for $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra, we conclude that C^* is a polyhedral cone. $\qquad\square$

Proposition 5.9 allows us to make the following definition:

Definition 5.8. Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. Given any projective polyhedron, $P = \mathbb{P}(C)$, where C is a polyhedral cone, the *polar dual (w.r.t. Q)*, P^*, of P is the projective polyhedron

$$P^* = \mathbb{P}(C^*).$$

We also show that projectivities behave well with respect to polar duality.

Proposition 5.10. *Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ, and matrix, $F = (f_{ij})$. For every projectivity, $h \colon \mathbb{P}^d \to \mathbb{P}^d$, if h is induced by the linear map, $\widehat{h}$, given by the invertible matrix, $A = (a_{ij})$, for every subset, $X \subseteq \mathbb{R}^{d+1}$, we have*

$$\widehat{h}(X^*) = (\widehat{h}(X))^*,$$

where on the left-hand side, X^* is the polar dual of X w.r.t. Q and on the right-hand side, $(\widehat{h}(X))^*$ is the polar dual of $\widehat{h}(X)$ w.r.t. the nondegenerate quadric, $h(Q)$, given by the matrix $(A^{-1})^\top F A^{-1}$. Consequently, if $X \neq \{0\}$, then

$$h((\mathbb{P}(X))^*) = (h(\mathbb{P}(X)))^*$$

and for every projective polyhedron, P, we have

$$h(P^*) = (h(P))^*.$$

Proof. As

$$X^* = \{v \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\mathbf{u}^\top F \mathbf{v} \leq 0)\},$$

we have

$$
\begin{aligned}
\widehat{h}(X^*) &= \{\widehat{h}(v) \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\mathbf{u}^\top F \mathbf{v} \leq 0)\} \\
&= \{y \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\mathbf{u}^\top F A^{-1} \mathbf{y} \leq 0)\} \\
&= \{y \in \mathbb{R}^{d+1} \mid (\forall x \in \widehat{h}(X))(\mathbf{x}^\top (A^{-1})^\top F A^{-1} \mathbf{y} \leq 0)\} \\
&= (\widehat{h}(X))^*,
\end{aligned}
$$

where $(\widehat{h}(X))^*$ is the polar dual of $\widehat{h}(X)$ w.r.t. the quadric whose matrix is $(A^{-1})^\top F A^{-1}$, that is, the polar dual w.r.t. $h(Q)$.

The second part of the proposition follows immediately by setting $X = C$, where C is the polyhedral cone defining the projective polyhedron, $P = \mathbb{P}(C)$. $\qquad\square$

We will also need the notion of an affine quadric and polar duality with respect to an affine quadric. Fortunately, the properties we need in the affine case are easily derived from the projective case using the "trick" that the affine space, $\mathbb{E}^d$, can be viewed as the hyperplane, $H_{d+1} \subseteq \mathbb{R}^{d+1}$, of equation, $x_{d+1} = 1$ and that its associated vector space, $\mathbb{R}^d$, can be viewed as the hyperplane, $H_{d+1}(0) \subseteq \mathbb{R}^{d+1}$, of equation $x_{d+1} = 0$. A point, $a \in \mathbb{A}^d$, corresponds to

the vector, $\widehat{a} = \binom{a}{1} \in \mathbb{R}^{d+1}$, and a vector, $u \in \mathbb{R}^d$, corresponds to the vector, $\widehat{u} = \binom{u}{0} \in \mathbb{R}^{d+1}$. This way, the projective space, $\mathbb{P}^d = \mathbb{P}(\mathbb{R}^{d+1})$, is the natural *projective completion* of $\mathbb{E}^d$, which is isomorphic to the affine patch U_{d+1} where $x_{d+1} \neq 0$. The hyperplane, $x_{d+1} = 0$, is the "hyperplane at infinity" in $\mathbb{P}^d$.

If we write $x = (x_1, \ldots, x_d)$, a polynomial, $\Phi(x) = \Phi(x_1, \ldots, x_d)$, of degree 2 can be written as

$$\Phi(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j + 2 \sum_{i=1}^{d} b_i x_i + c,$$

where $A = (a_{ij})$ is a symmetric matrix. If we write $b^\top = (b_1, \ldots, b_d)$, then we have

$$\Phi(x) = (\mathbf{x}^\top, 1) \begin{pmatrix} A & b \\ b^\top & c \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \widehat{\mathbf{x}}^\top \begin{pmatrix} A & b \\ b^\top & c \end{pmatrix} \widehat{\mathbf{x}}.$$

Therefore, as in the projective case, Φ is completely determined by a $(d+1) \times (d+1)$ symmetric matrix, say $F = (f_{ij})$, and we have

$$\Phi(x) = (\mathbf{x}^\top, 1) F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \widehat{\mathbf{x}}^\top F \widehat{\mathbf{x}}.$$

We say that $Q \subseteq \mathbb{R}^d$ is a *nondegenerate affine quadric* iff

$$Q = V(\Phi) = \left\{ x \in \mathbb{R}^d \mid (\mathbf{x}^\top, 1) F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \right\},$$

where F is symmetric and invertible. Given any point $a \in \mathbb{R}^d$, the *polar hyperplane*, $a^\dagger$, of a w.r.t. Q is defined by

$$a^\dagger = \left\{ x \in \mathbb{R}^d \mid (\mathbf{a}^\top, 1) F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \right\}.$$

From a previous discussion, the equation of the polar hyperplane, $a^\dagger$, is

$$\sum_{i=1}^{d} \frac{\partial \Phi(a)}{\partial x_i} (x_i - a_i) = 0.$$

Given any subset, $X \subseteq \mathbb{R}^d$, the *polar dual*, X^*, of X is defined by

$$X^* = \left\{ y \in \mathbb{R}^d \mid (\forall x \in X) \left((\mathbf{x}^\top, 1) F \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \leq 0 \right) \right\}.$$

As noted before, polar duality with respect to the affine sphere, $S^d \subseteq \mathbb{R}^{d+1}$, corresponds to the case where

$$F = \begin{pmatrix} I_d & \mathbf{0} \\ \mathbb{O} & -1 \end{pmatrix}$$

and polar duality with respect to the affine paraboloid $\mathcal{P} \subseteq \mathbb{R}^{d+1}$, corresponds to the case where

$$F = \begin{pmatrix} I_{d-1} & \mathbf{0} & \mathbf{0} \\ \mathbb{O} & 0 & -\frac{1}{2} \\ \mathbb{O} & -\frac{1}{2} & 0 \end{pmatrix}.$$

We will need the following version of Proposition 4.14:

Proposition 5.11. *Let Q be a nondegenerate affine quadric given by the $(d+1) \times (d+1)$ symmetric matrix, F, let $\{y_1, \ldots, y_p\}$ be any set of points in $\mathbb{E}^d$ and let $\{v_1, \ldots, v_q\}$ be any set of nonzero vectors in $\mathbb{R}^d$. If $\widehat{Y}$ is the $(d+1) \times p$ matrix whose i^{th} column is $\widehat{y}_i$ and V is the $(d+1) \times q$ matrix whose j^{th} column is $\widehat{v}_j$, then*

$$(\operatorname{conv}(\{y_1, \ldots, y_p\}) \cup \operatorname{cone}(\{v_1, \ldots, v_q\}))^* = P(\widehat{Y}^\top F, \mathbf{0}; \widehat{V}^\top F, \mathbf{0}),$$

with

$$P(\widehat{Y}^\top F, \mathbf{0}; \widehat{V}^\top F, \mathbf{0}) = \left\{ x \in \mathbb{R}^d \mid \widehat{Y}^\top F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \leq \mathbf{0}, \ \widehat{V}^\top F \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \leq \mathbf{0} \right\}.$$

Proof. The proof is immediately adpated from that of Proposition 4.14. □

Using Proposition 5.11, we can prove the following Proposition showing that projective completion and polar duality commute:

Proposition 5.12. *Let Q be a nondegenerate affine quadric given by the $(d+1) \times (d+1)$ symmetric, invertible matrix, F. For every polyhedron, $P \subseteq \mathbb{R}^d$, we have*

$$\widetilde{P^*} = (\widetilde{P})^*,$$

where on the right-hand side, we use polar duality w.r.t. the nondegenerate projective quadric, $\widetilde{Q}$, defined by F.

Proof. By definition, we have $\widetilde{P} = \mathbb{P}(C(P))$, $(\widetilde{P})^* = \mathbb{P}((C(P))^*)$ and $\widetilde{P^*} = \mathbb{P}(C(P^*))$. Therefore, it suffices to prove that

$$(C(P))^* = C(P^*).$$

Now, $P = \operatorname{conv}(Y) + \operatorname{cone}(V)$, for some finite set of points, Y, and some finite set of vectors, V, and we know that

$$C(P) = \operatorname{cone}(\widehat{Y} \cup \widehat{V}).$$

From Proposition 5.9,

$$(C(P))^* = \{v \in \mathbb{R}^{d+1} \mid \widehat{Y}^\top F \mathbf{v} \leq \mathbf{0}, \ \widehat{V}^\top F \mathbf{v} \leq \mathbf{0}\}$$

and by Proposition 5.11,

$$P^* = \left\{ x \in \mathbb{R}^d \mid \widehat{Y}^\top F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \leq \mathbf{0}, \ \widehat{V}^\top F \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \leq \mathbf{0} \right\}.$$

But, by definition of $C(P^*)$ (see Section 4.4, especially Proposition 4.19), the hyperplanes cutting out $C(P^*)$ are obtained by homogenizing the equations of the hyperplanes cutting out P^* and so,

$$C(P^*) = \left\{ \begin{pmatrix} \mathbf{x} \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid \widehat{Y}^\top F \begin{pmatrix} \mathbf{x} \\ x_{d+1} \end{pmatrix} \leq \mathbf{0}, \ \widehat{V}^\top F \begin{pmatrix} \mathbf{x} \\ x_{d+1} \end{pmatrix} \leq \mathbf{0} \right\} = (C(P))^*,$$

as claimed. □

Remark: If $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ is a projective or an affine quadric, it is obvious that

$$V(\Phi(x_1, \ldots, x_{d+1})) = V(\lambda \Phi(x_1, \ldots, x_{d+1}))$$

for every $\lambda \neq 0$. This raises the following question: If

$$Q = V(\Phi_1(x_1, \ldots, x_{d+1})) = V(\Phi_2(x_1, \ldots, x_{d+1})),$$

what is the relationship between Φ_1 and Φ_2?

The answer depends crucially on the field over which projective space or affine space is defined (*i.e.*, whether $Q \subseteq \mathbb{RP}^d$ or $Q \subseteq \mathbb{CP}^d$ in the projective case or whether $Q \subseteq \mathbb{R}^{d+1}$ or $Q \subseteq \mathbb{C}^{d+1}$ in the affine case). For example, over $\mathbb{R}$, the polynomials $\Phi_1(x_1, x_2, x_3) = x_1^2 + x_2^2$ and $\Phi_2(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2$ both define the point $(0\colon 0\colon 1) \in \mathbb{P}^2$, since the only real solution of Φ_1 and Φ_2 are of the form $(0, 0, z)$. However, if Q has some nonsingular point, the following can be proved (see Samuel [35], Theorem 46 (Chapter 3)):

Theorem 5.13. *Let $Q = V(\Phi(x_1, \ldots, x_{d+1})$ be a projective or an affine quadric, over $\mathbb{RP}^d$ or $\mathbb{R}^{d+1}$. If Q has a nonsingular point, then for every polynonial, Φ', such that $Q = V(\Phi'(x_1, \ldots, x_{d+1})$, there is some $\lambda \neq 0$ ($\lambda \in \mathbb{R}$) so that $\Phi' = \lambda\Phi$.*

In particular, Theorem 5.13 shows that the equation of a nondegenerate quadric is unique up to a scalar.

Actually, more is true. It turns out that if we allow complex solutions, that is, if $Q \subseteq \mathbb{CP}^d$ in the projective case or $Q \subseteq \mathbb{C}^{d+1}$ in the affine case, then $Q = V(\Phi_1) = V(\Phi_2)$ always implies $\Phi_2 = \lambda\Phi_1$ for some $\lambda \in \mathbb{C}$, with $\lambda \neq 0$. In the real case, the above holds (for some $\lambda \in \mathbb{R}$, with $\lambda \neq 0$) unless Q is an affine subspace (resp. a projective subspace) of dimension at most $d - 1$ (resp. of dimension at most $d - 2$). Even in this case, there is a bijective affine map, f, (resp. a bijective projective map, h), such that $\Phi_2 = \Phi_1 \circ f^{-1}$ (resp. $\Phi_2 = \Phi_1 \circ h^{-1}$). A proof of these facts (and more) can be found in Tisseron [42] (Chapter 3).

We now have everything we need for a rigorous presentation of the material of Section 8.6. For a comprehensive treatment of the affine and projective quadrics and related material, the reader should consult Berger (Geometry II) [6] or Samuel [35].

Chapter 6

Basics of Combinatorial Topology

6.1   Simplicial and Polyhedral Complexes

In order to study and manipulate complex shapes it is convenient to discretize these shapes and to view them as the union of simple building blocks glued together in a "clean fashion". The building blocks should be simple geometric objects, for example, points, lines segments, triangles, tehrahedra and more generally simplices, or even convex polytopes. We will begin by using simplices as building blocks. The material presented in this chapter consists of the most basic notions of combinatorial topology, going back roughly to the 1900-1930 period and it is covered in nearly every algebraic topology book (certainly the "classics"). A classic text (slightly old fashion especially for the notation and terminology) is Alexandrov [1], Volume 1 and another more "modern" source is Munkres [30]. An excellent treatment from the point of view of computational geometry can be found is Boissonnat and Yvinec [8], especially Chapters 7 and 10. Another fascinating book covering a lot of the basics but devoted mostly to three-dimensional topology and geometry is Thurston [41].

Recall that a simplex is just the convex hull of a finite number of affinely independent points. We also need to define faces, the boundary, and the interior of a simplex.

Definition 6.1. Let $\mathcal{E}$ be any normed affine space, say $\mathcal{E} = \mathbb{E}^m$ with its usual Euclidean norm. Given any $n+1$ affinely independent points $a_0, \ldots, a_n$ in $\mathcal{E}$, the *n-simplex (or simplex)* σ *defined by* $a_0, \ldots, a_n$ is the convex hull of the points $a_0, \ldots, a_n$, that is, the set of all convex combinations $\lambda_0 a_0 + \cdots + \lambda_n a_n$, where $\lambda_0 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$ for all i, $0 \leq i \leq n$. We call n the *dimension* of the n-simplex σ, and the points $a_0, \ldots, a_n$ are the *vertices* of σ. Given any subset $\{a_{i_0}, \ldots, a_{i_k}\}$ of $\{a_0, \ldots, a_n\}$ (where $0 \leq k \leq n$), the k-simplex generated by $a_{i_0}, \ldots, a_{i_k}$ is called a *k-face* or simply a *face* of σ. A face s of σ is a *proper face* if $s \neq \sigma$ (we agree that the empty set is a face of any simplex). For any vertex a_i, the face generated by $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ (i.e., omitting a_i) is called the *face opposite* a_i. Every face that is an $(n-1)$-simplex is called a *boundary face* or *facet*. The union of the boundary faces is the *boundary of* σ, denoted by $\partial\sigma$, and the complement of $\partial\sigma$ in σ is the *interior* Int $\sigma = \sigma - \partial\sigma$ of σ. The interior Int σ of σ is sometimes called an *open simplex*.

It should be noted that for a 0-simplex consisting of a single point $\{a_0\}$, $\partial\{a_0\} = \emptyset$, and Int $\{a_0\} = \{a_0\}$. Of course, a 0-simplex is a single point, a 1-simplex is the line segment (a_0, a_1), a 2-simplex is a triangle (a_0, a_1, a_2) (with its interior), and a 3-simplex is a tetrahedron (a_0, a_1, a_2, a_3) (with its interior). The inclusion relation between any two faces σ and τ of some simplex, s, is written $\sigma \preceq \tau$.

We now state a number of properties of simplices, whose proofs are left as an exercise. Clearly, a point x belongs to the boundary $\partial\sigma$ of σ iff at least one of its barycentric coordinates $(\lambda_0, \ldots, \lambda_n)$ is zero, and a point x belongs to the interior Int σ of σ iff all of its barycentric coordinates $(\lambda_0, \ldots, \lambda_n)$ are positive, i.e., $\lambda_i > 0$ for all i, $0 \leq i \leq n$. Then, for every $x \in \sigma$, there is a unique face s such that $x \in$ Int s, the face generated by those points a_i for which $\lambda_i > 0$, where $(\lambda_0, \ldots, \lambda_n)$ are the barycentric coordinates of x.

A simplex σ is convex, arcwise connected, compact, and closed. The interior Int σ of a simplex is convex, arcwise connected, open, and σ is the closure of Int σ.

We now put simplices together to form more complex shapes, following Munkres [30]. The intuition behind the next definition is that the building blocks should be "glued cleanly".

Definition 6.2. A *simplicial complex in* $\mathbb{E}^m$ (for short, a *complex* in $\mathbb{E}^m$) is a set K consisting of a (finite or infinite) set of simplices in $\mathbb{E}^m$ satisfying the following conditions:

(1) Every face of a simplex in K also belongs to K.

(2) For any two simplices σ_1 and σ_2 in K, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Every k-simplex, $\sigma \in K$, is called a *k-face* (or *face*) of K. A 0-face $\{v\}$ is called a *vertex* and a 1-face is called an *edge*. The *dimension* of the simplicial complex K is the maximum of the dimensions of all simplices in K. If $\dim K = d$, then every face of dimension d is called a *cell* and every face of dimension $d - 1$ is called a *facet*.

Condition (2) guarantees that the various simplices forming a complex intersect nicely. It is easily shown that the following condition is equivalent to condition (2):

(2′) For any two distinct simplices σ_1, σ_2, Int $\sigma_1 \cap$ Int $\sigma_2 = \emptyset$.

Remarks:

1. A simplicial complex, K, is a combinatorial object, namely, a *set* of simplices satisfying certain conditions but not a subset of $\mathbb{E}^m$. However, every complex, K, yields a subset of $\mathbb{E}^m$ called the geometric realization of K and denoted $|K|$. This object will be defined shortly and should not be confused with the complex. Figure 6.1 illustrates this aspect of the definition of a complex. For clarity, the two triangles (2-simplices) are drawn as disjoint objects even though they share the common edge, (v_2, v_3) (a 1-simplex) and similarly for the edges that meet at some common vertex.
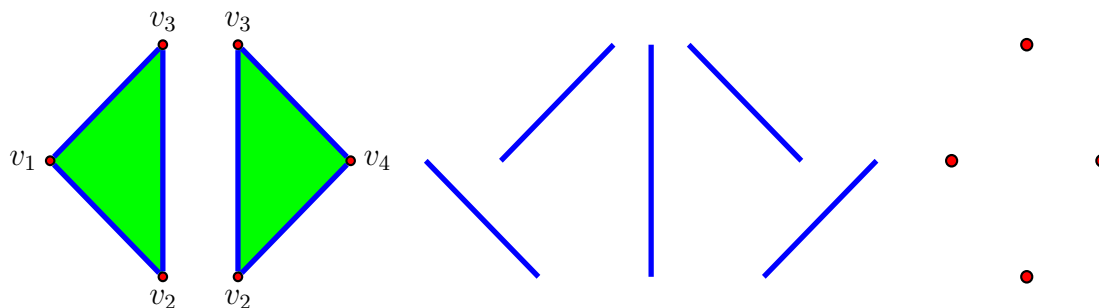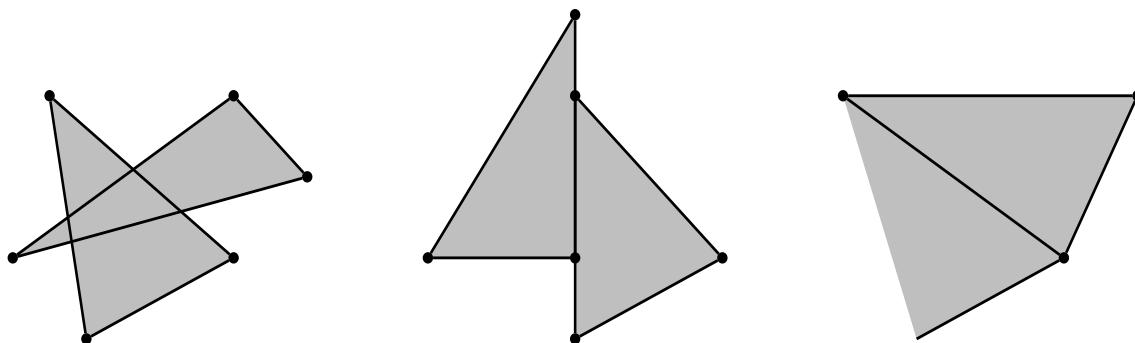
Figure 6.1: A set of simplices forming a complex

Figure 6.2: Collections of simplices not forming a complex

2. Some authors define a *facet* of a complex, K, of dimension d to be a d-simplex in K, as opposed to a $(d-1)$-simplex, as we did. This practice is not consistent with the notion of facet of a polyhedron and this is why we prefer the terminology *cell* for the d-simplices in K.

3. It is important to note that in order for a complex, K, of dimension d to be realized in $\mathbb{E}^m$, the dimension of the "ambient space", m, must be big enough. For example, there are 2-complexes that can't be realized in $\mathbb{E}^3$ or even in $\mathbb{E}^4$. There has to be enough room in order for condition (2) to be satisfied. It is not hard to prove that $m = 2d + 1$ is always sufficient. Sometimes, $2d$ works, for example in the case of surfaces (where $d = 2$).

Some collections of simplices violating some of the conditions of Definition 6.2 are shown in Figure 6.2. On the left, the intersection of the two 2-simplices is neither an edge nor a vertex of either triangle. In the middle case, two simplices meet along an edge which is not an edge of either triangle. On the right, there is a missing edge and a missing vertex.

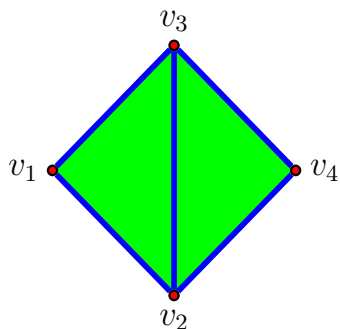Some "legal" simplicial complexes are shown in Figure 6.4.

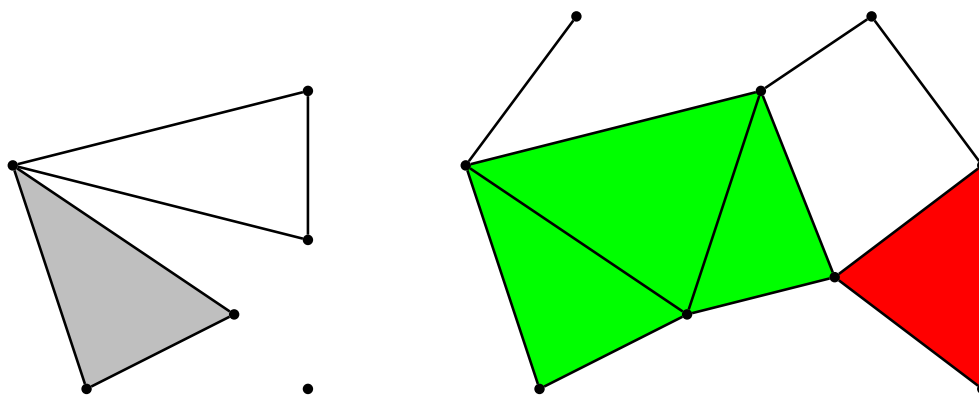Figure 6.3: The geometric realization of the complex of Figure 6.1

Figure 6.4: Examples of simplicial complexes

The union $|K|$ of all the simplices in K is a subset of $\mathbb{E}^m$. We can define a topology on $|K|$ by defining a subset F of $|K|$ to be closed iff $F \cap \sigma$ is closed in σ for every face $\sigma \in K$. It is immediately verified that the axioms of a topological space are indeed satisfied. The resulting topological space $|K|$ is called the *geometric realization of* K. The geometric realization of the complex from Figure 6.1 is shown in Figure 6.3.

Obviously, $|\sigma| = \sigma$ for every simplex, σ. Also, note that distinct complexes may have the same geometric realization. In fact, all the complexes obtained by subdividing the simplices of a given complex yield the same geometric realization.

A *polytope* is the geometric realization of some simplicial complex. A polytope of dimension 1 is usually called a *polygon*, and a polytope of dimension 2 is usually called a *polyhedron*. When K consists of infinitely many simplices we usually require that K be *locally finite*, which means that every vertex belongs to finitely many faces. If K is locally finite, then its geometric realization, $|K|$, is locally compact.

In the sequel, we will consider only finite simplicial complexes, that is, complexes K
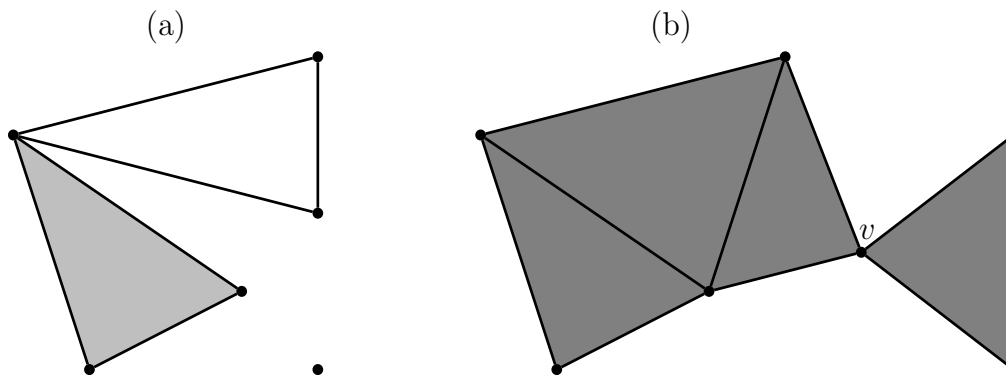
Figure 6.5: (a) A complex that is not pure. (b) A pure complex

consisting of a finite number of simplices. In this case, the topology of $|K|$ defined above is identical to the topology induced from $\mathbb{E}^m$. Also, for any simplex σ in K, $\mathrm{Int}\,\sigma$ coincides with the interior $\mathring{\sigma}$ of σ in the topological sense, and $\partial\sigma$ coincides with the boundary of σ in the topological sense.

Definition 6.3. Given any complex, K_2, a subset $K_1 \subseteq K_2$ of K_2 is a *subcomplex* of K_2 iff it is also a complex. For any complex, K, of dimension d, for any i with $0 \le i \le d$, the subset

$$K^{(i)} = \{\sigma \in K \mid \dim \sigma \le i\}$$

is called the *i-skeleton* of K. Clearly, $K^{(i)}$ is a subcomplex of K. We also let

$$K^i = \{\sigma \in K \mid \dim \sigma = i\}.$$

Observe that K^0 is the set of vertices of K and K^i is not a complex. A simplicial complex, K_1 is a *subdivision* of a complex K_2 iff $|K_1| = |K_2|$ and if every face of K_1 is a subset of some face of K_2. A complex K of dimension d is *pure* (or *homogeneous*) iff every face of K is a face of some d-simplex of K (i.e., some cell of K). A complex is *connected* iff $|K|$ is connected.

It is easy to see that a complex is connected iff its 1-skeleton is connected. The intuition behind the notion of a pure complex, K, of dimension d is that a pure complex is the result of gluing pieces all having the same dimension, namely, d-simplices. For example, in Figure 6.5, the complex on the left is not pure but the complex on the right is pure of dimension 2.

Most of the shapes that we will be interested in are well approximated by pure complexes, in particular, surfaces or solids. However, pure complexes may still have undesirable "singularities" such as the vertex, v, in Figure 6.5(b). The notion of link of a vertex provides a technical way to deal with singularities.
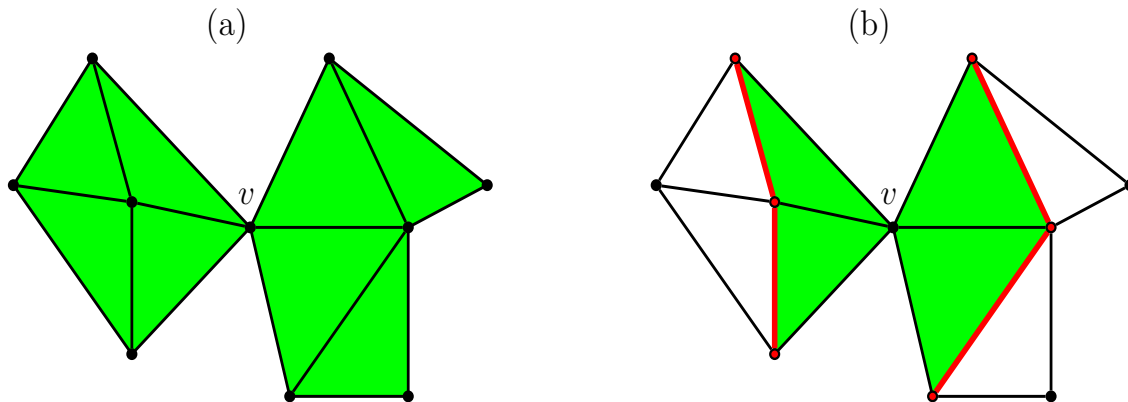
Figure 6.6: (a) A complex. (b) Star and Link of v

Definition 6.4. Let K be any complex and let σ be any face of K. The *star*, $\mathrm{St}(\sigma)$ (or if we need to be very precise, $\mathrm{St}(\sigma, K)$), of σ is the subcomplex of K consisting of all faces, τ, containing σ and of all faces of τ, *i.e.*,

$$\mathrm{St}(\sigma) = \{s \in K \mid (\exists \tau \in K)(\sigma \preceq \tau \quad \text{and} \quad s \preceq \tau)\}.$$

The *link*, $\mathrm{Lk}(\sigma)$ (or $\mathrm{Lk}(\sigma, K)$) of σ is the subcomplex of K consisting of all faces in $\mathrm{St}(\sigma)$ that do not intersect σ, i.e.,

$$\mathrm{Lk}(\sigma) = \{\tau \in K \mid \tau \in \mathrm{St}(\sigma) \quad \text{and} \quad \sigma \cap \tau = \emptyset\}.$$

To simplify notation, if $\sigma = \{v\}$ is a vertex we write $\mathrm{St}(v)$ for $\mathrm{St}(\{v\})$ and $\mathrm{Lk}(v)$ for $\mathrm{Lk}(\{v\})$. Figure 6.6 shows:

(a) A complex (on the left).

(b) The star of the vertex v, indicated in gray and the link of v, shown as thicker lines.

If K is pure and of dimension d, then $\mathrm{St}(\sigma)$ is also pure of dimension d and if $\dim \sigma = k$, then $\mathrm{Lk}(\sigma)$ is pure of dimension $d - k - 1$.

For technical reasons, following Munkres [30], besides defining the complex, $\mathrm{St}(\sigma)$, it is useful to introduce the *open star* of σ, denoted $\mathrm{st}(\sigma)$, defined as the subspace of $|K|$ consisting of the union of the interiors, $\mathrm{Int}(\tau) = \tau - \partial \tau$, of all the faces, τ, containing, σ. According to this definition, the open star of σ is not a complex but instead a subset of $|K|$.

Note that

$$\overline{\mathrm{st}(\sigma)} = |\mathrm{St}(\sigma)|,$$

that is, the closure of $\text{st}(\sigma)$ is the geometric realization of the complex $\text{St}(\sigma)$. Then, $\text{lk}(\sigma) = |\text{Lk}(\sigma)|$ is the union of the simplices in $\text{St}(\sigma)$ that are disjoint from σ. If σ is a vertex, v, we have

$$\text{lk}(v) = \overline{\text{st}(v)} - \text{st}(v).$$

However, beware that if σ is not a vertex, then $\text{lk}(\sigma)$ is properly contained in $\overline{\text{st}(\sigma)} - \text{st}(\sigma)$!

One of the nice properties of the open star, $\text{st}(\sigma)$, of σ is that it is open. To see this, observe that for any point, $a \in |K|$, there is a unique smallest simplex, $\sigma = (v_0, \ldots, v_k)$, such that $a \in \text{Int}(\sigma)$, that is, such that

$$a = \lambda_0 v_0 + \cdots + \lambda_k v_k$$

with $\lambda_i > 0$ for all i, with $0 \le i \le k$ (and of course, $\lambda_0 + \cdots + \lambda_k = 1$). (When $k = 0$, we have $v_0 = a$ and $\lambda_0 = 1$.) For every arbitrary vertex, v, of K, we define $t_v(a)$ by

$$t_v(a) = \begin{cases} \lambda_i & \text{if } v = v_i, \text{ with } 0 \le i \le k, \\ 0 & \text{if } v \notin \{v_0, \ldots, v_k\}. \end{cases}$$

Using the above notation, observe that

$$\text{st}(v) = \{a \in |K| \mid t_v(a) > 0\}$$

and thus, $|K| - \text{st}(v)$ is the union of all the faces of K that do not contain v as a vertex, obviously a closed set. Thus, $\text{st}(v)$ is open in $|K|$. It is also quite clear that $\text{st}(v)$ is path connected. Moreover, for any k-face, σ, of K, if $\sigma = (v_0, \ldots, v_k)$, then

$$\text{st}(\sigma) = \{a \in |K| \mid t_{v_i}(a) > 0, \quad 0 \le i \le k\},$$

that is,

$$\text{st}(\sigma) = \text{st}(v_0) \cap \cdots \cap \text{st}(v_k).$$

Consequently, $\text{st}(\sigma)$ is open and path connected.

Unfortunately, the "nice" equation

$$\text{St}(\sigma) = \text{St}(v_0) \cap \cdots \cap \text{St}(v_k)$$

is false! (and anagolously for $\text{Lk}(\sigma)$.) For a counter-example, consider the boundary of a tetrahedron with one face removed.

Recall that in $\mathbb{E}^d$, the *(open) unit ball*, B^d, is defined by

$$B^d = \{x \in \mathbb{E}^d \mid \|x\| < 1\},$$

the *closed unit ball*, $\overline{B}^d$, is defined by

$$\overline{B}^d = \{x \in \mathbb{E}^d \mid \|x\| \le 1\},$$

and the *$(d-1)$-sphere*, S^{d-1}, by

$$S^{d-1} = \{x \in \mathbb{E}^d \mid \|x\| = 1\}.$$

Obviously, S^{d-1} is the boundary of $\overline{B}^d$ (and B^d).

Definition 6.5. Let K be a pure complex of dimension d and let σ be any k-face of K, with $0 \leq k \leq d - 1$. We say that σ is *nonsingular* iff the geometric realization, $\mathrm{lk}(\sigma)$, of the link of σ is homeomorphic to either S^{d-k-1} or to $\overline{B}^{d-k-1}$; this is written as $\mathrm{lk}(\sigma) \approx S^{d-k-1}$ or $\mathrm{lk}(\sigma) \approx \overline{B}^{d-k-1}$, where $\approx$ means homeomorphic.

In Figure 6.6, note that the link of v is not homeomorphic to S^1 or B^1, so v is singular.

It will also be useful to express $\mathrm{St}(v)$ in terms of $\mathrm{Lk}(v)$, where v is a vertex, and for this, we define yet another notion of cone.

Definition 6.6. Given any complex, K, in $\mathbb{E}^n$, if $\dim K = d < n$, for any point, $v \in \mathbb{E}^n$, such that v does not belong to the affine hull of $|K|$, the *cone on K with vertex v*, denoted, $v * K$, is the complex consisting of all simplices of the form $(v, a_0, \ldots, a_k)$ and their faces, where $(a_0, \ldots, a_k)$ is any k-face of K. If $K = \emptyset$, we set $v * K = v$.

It is not hard to check that $v * K$ is indeed a complex of dimension $d + 1$ containing K as a subcomplex.

Remark: Unfortunately, the word "cone" is overloaded. It might have been better to use the locution *pyramid* instead of cone as some authors do (for example, Ziegler). However, since we have been following Munkres [30], a standard reference in algebraic topology, we decided to stick with the terminology used in that book, namely, "cone".

The following proposition is also easy to prove:

Proposition 6.1. *For any complex, K, of dimension d and any vertex, $v \in K$, we have*

$$\mathrm{St}(v) = v * \mathrm{Lk}(v).$$

More generally, for any face, σ, of K, we have

$$\overline{\mathrm{st}(\sigma)} = |\mathrm{St}(\sigma)| \approx \sigma \times |v * \mathrm{Lk}(\sigma)|,$$

for every $v \in \sigma$ and

$$\overline{\mathrm{st}(\sigma)} - \mathrm{st}(\sigma) = \partial \sigma \times |v * \mathrm{Lk}(\sigma)|,$$

for every $v \in \partial \sigma$.

Figure 6.7 shows a 3-dimensional complex. The link of the edge (v_6, v_7) is the pentagon $P = (v_1, v_2, v_3, v_4, v_5) \approx S^1$. The link of the vertex v_7 is the cone $v_6 * P \approx B^2$. The link of (v_1, v_2) is $(v_6, v_7) \approx B^1$ and the link of v_1 is the union of the triangles (v_2, v_6, v_7) and (v_5, v_6, v_7), which is homeomorphic to B^2.

Given a pure complex, it is necessary to distinguish between two kinds of faces.

Definition 6.7. Let K be any pure complex of dimension d. A k-face, σ, of K is a *boundary* or *external* face iff it belongs to a single cell (i.e., a d-simplex) of K and otherwise it is called an *internal* face ($0 \leq k \leq d - 1$). The *boundary* of K, denoted $\mathrm{bd}(K)$, is the subcomplex of K consisting of all boundary facets of K together with their faces.
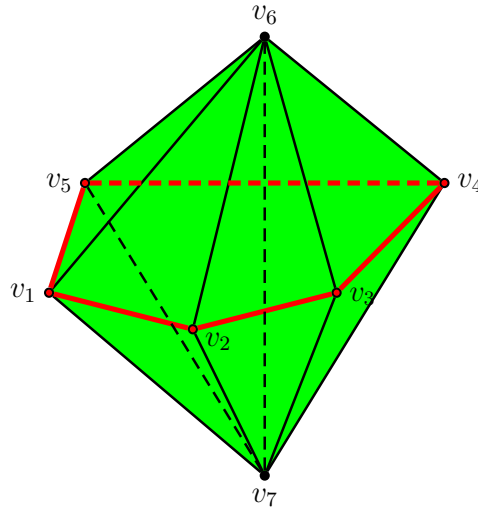
Figure 6.7: More examples of links and stars

It is clear by definition that $\mathrm{bd}(K)$ is a pure complex of dimension $d - 1$. Even if K is connected, $\mathrm{bd}(K)$ is not connected, in general. For example, if K is a 2-complex in the plane, the boundary of K usually consists of several simple closed polygons (i.e, 1 dimensional complexes homeomorphic to the circle, S^1).

Proposition 6.2. *Let K be any pure complex of dimension d. For any k-face, σ, of K the boundary complex, $\mathrm{bd}(\mathrm{Lk}(\sigma))$, is nonempty iff σ is a boundary face of K ($0 \leq k \leq d - 2$). Furthermore, $\mathrm{Lk}_{\mathrm{bd}(K)}(\sigma) = \mathrm{bd}(\mathrm{Lk}(\sigma))$ for every face, σ, of $\mathrm{bd}(K)$, where $\mathrm{Lk}_{\mathrm{bd}(K)}(\sigma)$ denotes the link of σ in $\mathrm{bd}(K)$.*

Proof. Let F be any facet of K containing σ. We may assume that $F = (v_0, \ldots, v_{d-1})$ and $\sigma = (v_0, \ldots, v_k)$, in which case, $F' = (v_{k+1}, \ldots, v_{d-1})$ is a $(d - k - 2)$-face of K and by definition of $\mathrm{Lk}(\sigma)$, we have $F' \in \mathrm{Lk}(\sigma)$. Now, every cell (i.e., d-simplex), s, containing F is of the form $s = \mathrm{conv}(F \cup \{v\})$ for some vertex, v, and $s' = \mathrm{conv}(F' \cup \{v\})$ is a $(d-k-1)$-face in $\mathrm{Lk}(\sigma)$ containing F'. Consequently, F' is an external face of $\mathrm{Lk}(\sigma)$ iff F is an external facet of K, establishing the proposition. The second statement follows immediately from the proof of the first. $\qquad\square$

Proposition 6.2 shows that if every face of K is nonsingular, then the link of every internal face is a sphere whereas the link of every external face is a ball. The following proposition shows that for any pure complex, K, nonsingularity of all the vertices is enough to imply that every open star is homeomorphic to B^d:

Proposition 6.3. *Let K be any pure complex of dimension d. If every vertex of K is nonsingular, then $\mathrm{st}(\sigma) \approx B^d$ for every k-face, σ, of K ($1 \leq k \leq d - 1$).*

Proof. Let σ be any k-face of K and assume that σ is generated by the vertices $v_0, \ldots, v_k$, with $1 \leq k \leq d-1$. By hypothesis, $\mathrm{lk}(v_i)$ is homeomorphic to either S^{d-1} or $\overline{B}^{d-1}$. Then, it is easy to show that in either case, we have

$$|v_i * \mathrm{Lk}(v_i)| \approx \overline{B}^d,$$

and by Proposition 6.1, we get

$$|\mathrm{St}(v_i)| \approx \overline{B}^d.$$

Consequently, $\mathrm{st}(v_i) \approx B^d$. Furthermore,

$$\mathrm{st}(\sigma) = \mathrm{st}(v_0) \cap \cdots \cap \mathrm{st}(v_k) \approx B^d$$

and so, $\mathrm{st}(\sigma) \approx B^d$, as claimed. $\qquad\square$

Here are more useful propositions about pure complexes without singularities.

Proposition 6.4. *Let K be any pure complex of dimension d. If every vertex of K is nonsingular, then for every point, $a \in |K|$, there is an open subset, $U \subseteq |K|$, containing a such that $U \approx B^d$ or $U \approx B^d \cap \mathbb{H}^d$, where $\mathbb{H}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_d \geq 0\}$.*

Proof. We already know from Proposition 6.3 that $\mathrm{st}(\sigma) \approx B^d$, for every $\sigma \in K$. So, if $a \in \sigma$ and σ is not a boundary face, we can take $U = \mathrm{st}(\sigma) \approx B^d$. If σ is a boundary face, then $|\sigma| \subseteq |\mathrm{bd}(\mathrm{St}(\sigma))|$ and it can be shown that we can take $U = B^d \cap \mathbb{H}^d$. $\qquad\square$

Proposition 6.5. *Let K be any pure complex of dimension d. If every facet of K is nonsingular, then every facet of K, is contained in at most two cells (d-simplices).*

Proof. If $|K| \subseteq \mathbb{E}^d$, then this is an immediate consequence of the definition of a complex. Otherwise, consider $\mathrm{lk}(\sigma)$. By hypothesis, either $\mathrm{lk}(\sigma) \approx B^0$ or $\mathrm{lk}(\sigma) \approx S^0$. As $B^0 = \{0\}$, $S^0 = \{-1, 1\}$ and $\dim \mathrm{Lk}(\sigma) = 0$, we deduce that $\mathrm{Lk}(\sigma)$ has either one or two points, which proves that σ belongs to at most two d-simplices. $\qquad\square$

Proposition 6.6. *Let K be any pure and connected complex of dimension d. If every face of K is nonsingular, then for every pair of cells (d-simplices), σ and σ', there is a sequence of cells, $\sigma_0, \ldots, \sigma_p$, with $\sigma_0 = \sigma$ and $\sigma_p = \sigma'$, and such that σ_i and σ_{i+1} have a common facet, for $i = 0, \ldots, p-1$.*

Proof. We proceed by induction on d, using the fact that the links are connected for $d \geq 2$. $\qquad\square$

Proposition 6.7. *Let K be any pure complex of dimension d. If every facet of K is nonsingular, then the boundary, $\mathrm{bd}(K)$, of K is a pure complex of dimension $d-1$ with an empty boundary. Furthermore, if every face of K is nonsingular, then every face of $\mathrm{bd}(K)$ is also nonsingular.*

Proof. Left as an exercise. □

The building blocks of simplicial complexes, namely, simplicies, are in some sense mathematically ideal. However, in practice, it may be desirable to use a more flexible set of building blocks. We can indeed do this and use convex polytopes as our building blocks.

Definition 6.8. A *polyhedral complex in* $\mathbb{E}^m$ (for short, a *complex in* $\mathbb{E}^m$) is a set, K, consisting of a (finite or infinite) set of convex polytopes in $\mathbb{E}^m$ satisfying the following conditions:

(1) Every face of a polytope in K also belongs to K.

(2) For any two polytopes σ_1 and σ_2 in K, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Every polytope, $\sigma \in K$, of dimension k, is called a k-*face* (or *face*) of K. A 0-face $\{v\}$ is called a *vertex* and a 1-face is called an *edge*. The *dimension* of the polyhedral complex K is the maximum of the dimensions of all polytopes in K. If $\dim K = d$, then every face of dimension d is called a *cell* and every face of dimension $d-1$ is called a *facet*.

Remark: Since the building blocks of a polyhedral complex are convex *polytopes* it might be more appropriate to use the term "polytopal complex" rather than "polyhedral complex" and some authors do that. On the other hand, most of the traditional litterature uses the terminology *polyhedral complex* so we will stick to it. There is a notion of complex where the building blocks are cones but these are called *fans*.

Every convex polytope, P, yields two natural polyhedral complexes:

(i) The polyhedral complex, $\mathcal{K}(P)$, consisting of P together with all of its faces. This complex has a single cell, namely, P itself.

(ii) The *boundary complex*, $\mathcal{K}(\partial P)$, consisting of all faces of P other than P itself. The cells of $\mathcal{K}(\partial P)$ are the facets of P.

The notions of k-skeleton and pureness are defined just as in the simplicial case. The notions of star and link are defined for polyhedral complexes just as they are defined for simplicial complexes except that the word "face" now means face of a polytope. Now, by Theorem 4.7, every polytope, σ, is the convex hull of its vertices. Let $\text{vert}(\sigma)$ denote the set of vertices of σ. Then, we have the following crucial observation: Given any polyhedral complex, K, for every point, $x \in |K|$, there is a *unique* polytope, $\sigma_x \in K$, such that $x \in \text{Int}(\sigma_x) = \sigma_x - \partial \sigma_x$. We define a function, $t \colon V \to \mathbb{R}_+$, that tests whether x belongs to the interior of any face (polytope) of K having v as a vertex as follows: For every vertex, v, of K,

$$t_v(x) = \begin{cases} 1 & \text{if } v \in \text{vert}(\sigma_x) \\ 0 & \text{if } v \notin \text{vert}(\sigma_x), \end{cases}$$

where σ_x is the unique face of K such that $x \in \mathrm{Int}(\sigma_x)$.

Now, just as in the simplicial case, the open star, $\mathrm{st}(v)$, of a vertex, $v \in K$, is given by

$$\mathrm{st}(v) = \{x \in |K| \mid t_v(x) = 1\}$$

and it is an open subset of $|K|$ (the set $|K| - \mathrm{st}(v)$ is the union of the polytopes of K that do not contain v as a vertex, a closed subset of $|K|$). Also, for any face, σ, of K, the open star, $\mathrm{st}(\sigma)$, of σ is given by

$$\mathrm{st}(\sigma) = \{x \in |K| \mid t_v(x) = 1, \text{ for all } v \in \mathrm{vert}(\sigma)\} = \bigcap_{v \in \mathrm{vert}(\sigma)} \mathrm{st}(v).$$

Therefore, $\mathrm{st}(\sigma)$ is also open in $|K|$.

The next proposition is another result that seems quite obvious, yet a rigorous proof is more involved that we might think. In fact, the only place that I am aware of where a proof is mentioned is the survey article by Carl Lee, Subdivisions and Triangulations of Polytopes (Chapter 17), in Goodman and O'Rourke [22]. Actually, the "proof" that Lee is referring to is a proof sketch whose details are "left to the reader." It turns out that a proof can be given using an inductive construction described in Grünbaum [24] (Chapter 5).

The proposition below states that a convex polytope can always be cut up into simplices, that is, it can be subdivided into a simplicial complex. In other words, every convex polytope can be triangulated. This implies that simplicial complexes are as general as polyhedral complexes.

One should be warned that even though, in the plane, every bounded region (not necessarily convex) whose boundary consists of a finite number of closed polygons (polygons homeomorphic to the circle, S^1) can be triangulated, this is no longer true in three dimensions!

Proposition 6.8. *Every convex d-polytope, P, can be subdivided into a simplicial complex without adding any new vertices, i.e., every convex polytope can be triangulated.*

Proof sketch. . It would be tempting to proceed by induction on the dimension, d, of P but we do not know any correct proof of this kind. Instead, we proceed by induction on the number, p, of vertices of P. Since $\dim(P) = d$, we must have $p \geq d + 1$. The case $p = d + 1$ corresponds to a simplex, so the base case holds.

For $p > d + 1$, we can pick some vertex, $v \in P$, such that the convex hull, Q, of the remaining $p - 1$ vertices still has dimension d. Then, by the induction hypothesis, Q, has a simplicial subdivision. Now, we say that a facet, F, of Q is *visible from* v iff v and the interior of Q are strictly separated by the supporting hyperplane of F. Then, we add the d-simplices, $\mathrm{conv}(F \cup \{v\}) = v * F$, for every facet, F, of Q visible from v to those in the triangulation of Q. We claim that the resulting collection of simplices (with their faces) constitutes a simplicial complex subdividing P.

This is the part of the proof that requires some details. We say that v *is beneath a facet F of Q* iff v belongs to the open half–space determined by the supporting hyperplane of F which contains the interior of Q. We make use of a theorem of Grünbaum [24] (Theorem 1, Chapter 5, Section 5.2) which states the following:

Theorem (Grünbaum). *If P and Q are two polytopes as above with $P = \text{conv}(Q \cup \{v\})$, then the following properties hold:*

(i) *A face F of Q is a face of P iff there is a facet F' of Q such that $F \subseteq F'$ and v is beneath F'.*

(ii) *If F is a face of Q, then $F^* = \text{conv}(F \cup \{v\})$ is a face of P iff either*

    (a) *$v \in \text{aff}(F)$; or*

    (b) *among the facets of Q containing F there is at least one such that v is beneath it, and at least one which is visible from v.*

Moreover, each face of P if of one and only one of those types.

The above theorem implies that the new simplices that need to be added to form a triangulation of P are the convex hulls $\text{conv}(F \cup \{v\})$ associated with facets F of Q visible from v. The reader should check that everything really works out! $\qquad\square$

With all this preparation, it is now quite natural to define combinatorial manifolds.

6.2   Combinatorial and Topological Manifolds

The notion of pure complex without singular faces turns out to be a very good "discrete" approximation of the notion of (topological) manifold because of its highly computational nature. This motivates the following definition:

Definition 6.9. A *combinatorial d-manifold* is any space, X, homeomorphic to the geometric realization, $|K| \subseteq \mathbb{E}^n$, of some pure (simplicial or polyhedral) complex, K, of dimension d whose faces are all nonsingular. If the link of every k-face of K is homeomorphic to the sphere S^{d-k-1}, we say that X is a combinatorial manifold *without boundary*, else it is a combinatorial manifold *with boundary*.

Other authors use the term *triangulation* for what we call a combinatorial manifold.

It is easy to see that the connected components of a combinatorial 1-manifold are either simple closed polygons or simple chains ("simple" means that the interiors of distinct edges are disjoint). A combinatorial 2-manifold which is connected is also called a *combinatorial surface* (with or without boundary). Proposition 6.7 immediately yields the following result:

Proposition 6.9. *If X is a combinatorial d-manifold with boundary, then $\text{bd}(X)$ is a combinatorial $(d-1)$-manifold without boundary.*

Now, because we are assuming that X sits in some Euclidean space, $\mathbb{E}^n$, the space X is Hausdorff and second-countable. (Recall that a topological space is second-countable iff there is a countable family, $\{U_i\}_{i \geq 0}$, of open sets of X such that every open subset of X is the union of open sets from this family.) Since it is desirable to have a good match between manifolds and combinatorial manifolds, we are led to the definition below.

Recall that
$$\mathbb{H}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_d \geq 0\}.$$

Definition 6.10. For any $d \geq 1$, a *(topological) d-manifold with boundary* is a second-countable, topological Hausdorff space M, together with an open cover, $(U_i)_{i \in I}$, of open sets in M and a family, $(\varphi_i)_{i \in I}$, of homeomorphisms, $\varphi_i \colon U_i \to \Omega_i$, where each Ω_i is some open subset of $\mathbb{H}^d$ in the subset topology. Each pair (U, φ) is called a *coordinate system*, or *chart*, of M, each homeomorphism $\varphi_i \colon U_i \to \Omega_i$ is called a *coordinate map*, and its inverse $\varphi_i^{-1} \colon \Omega_i \to U_i$ is called a *parameterization* of U_i. The family $(U_i, \varphi_i)_{i \in I}$ is often called an *atlas* for M. A *(topological) bordered surface* is a connected 2-manifold with boundary. If for every homeomorphism, $\varphi_i \colon U_i \to \Omega_i$, the open set $\Omega_i \subseteq \mathbb{H}^d$ is actually an open set in $\mathbb{R}^d$ (which means that $x_d > 0$ for every $(x_1, \ldots, x_d) \in \Omega_i$), then we say that M is a *d-manifold*.

Note that a d-manifold is also a d-manifold with boundary.

If $\varphi_i \colon U_i \to \Omega_i$ is some homeomorphism onto some open set Ω_i of $\mathbb{H}^d$ in the subset topology, some $p \in U_i$ may be mapped into $\mathbb{R}^{d-1} \times \mathbb{R}_+$, or into the "boundary" $\mathbb{R}^{d-1} \times \{0\}$ of $\mathbb{H}^d$. Letting $\partial\mathbb{H}^d = \mathbb{R}^{d-1} \times \{0\}$, it can be shown using homology that if some coordinate map, φ, defined on p maps p into $\partial\mathbb{H}^d$, then every coordinate map, ψ, defined on p maps p into $\partial\mathbb{H}^d$.

Thus, M is the disjoint union of two sets ∂M and $\mathrm{Int}\, M$, where ∂M is the subset consisting of all points $p \in M$ that are mapped by some (in fact, all) coordinate map, φ, defined on p into $\partial\mathbb{H}^d$, and where $\mathrm{Int}\, M = M - \partial M$. The set ∂M is called the *boundary* of M, and the set $\mathrm{Int}\, M$ is called the *interior* of M, even though this terminology clashes with some prior topological definitions. A good example of a bordered surface is the Möbius strip. The boundary of the Möbius strip is a circle.

The boundary ∂M of M may be empty, but $\mathrm{Int}\, M$ is nonempty. Also, it can be shown using homology that the integer d is unique. It is clear that $\mathrm{Int}\, M$ is open and a d-manifold, and that ∂M is closed. If $p \in \partial M$, and φ is some coordinate map defined on p, since $\Omega = \varphi(U)$ is an open subset of $\partial\mathbb{H}^d$, there is some open half ball B_{o+}^d centered at $\varphi(p)$ and contained in Ω which intersects $\partial\mathbb{H}^d$ along an open ball B_o^{d-1}, and if we consider $W = \varphi^{-1}(B_{o+}^d)$, we have an open subset of M containing p which is mapped homeomorphically onto B_{o+}^d in such that way that every point in $W \cap \partial M$ is mapped onto the open ball B_o^{d-1}. Thus, it is easy to see that ∂M is a $(d-1)$-manifold.

Proposition 6.10. *Every combinatorial d-manifold is a d-manifold with boundary.*

Proof. This is an immediate consequence of Proposition 6.4. $\square$

Is the converse of Proposition 6.10 true?

It turns out that answer is yes for $d = 1, 2, 3$ but **no** for $d \geq 4$. This is not hard to prove for $d = 1$. For $d = 2$ and $d = 3$, this is quite hard to prove; among other things, it is necessary to prove that triangulations exist and this is very technical. For $d \geq 4$, not every manifold can be triangulated (in fact, this is undecidable!).

What if we assume that M is a triangulated manifold, which means that $M \approx |K|$, for some pure d-dimensional complex, K?

Surprisingly, for $d \geq 5$, there are triangulated manifolds whose links are not spherical (i.e., not homeomorphic to $\overline{B}^{d-k-1}$ or S^{d-k-1}), see Thurston [41].

Fortunately, we will only have to deal with $d = 2, 3$! Another issue that must be addressed is orientability.

Assume that we fix a total ordering of the vertices of a complex, K. Let $\sigma = (v_0, \ldots, v_k)$ be any simplex. Recall that every permutation (of $\{0, \ldots, k\}$) is a product of *transpositions*, where a transposition swaps two distinct elements, say i and j, and leaves every other element fixed. Furthermore, for any permutation, π, the parity of the number of transpositions needed to obtain π only depends on π and it called the *signature* of π. We say that *two permutations are equivalent* iff they have the same signature. Consequently, there are two equivalence classes of permutations: Those of even signature and those of odd signature. Then, an *orientation* of σ is the choice of one of the two equivalence classes of permutations of its vertices. If σ has been given an orientation, then we denote by $-\sigma$ the result of assigning the other orientation to it (we call it the *opposite orientation*).

For example, $(0, 1, 2)$ has the two orientation classes:

$$\{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \quad \text{and} \quad \{(2, 1, 0), (1, 0, 2), (0, 2, 1)\}.$$

Definition 6.11. Let $X \approx |K|$ be a combinatorial d-manifold. We say that X is *orientable* if it is possible to assign an orientation to all of its cells (d-simplices) so that whenever two cells σ_1 and σ_2 have a common facet, σ, the two orientations induced by σ_1 and σ_2 on σ are opposite. A combinatorial d-manifold together with a specific orientation of its cells is called an *oriented manifold*. If X is not orientable we say that it is *non-orientable*.

Remark: It is possible to define the notion of orientation of a manifold but this is quite technical and we prefer to avoid digressing into this matter. This shows another advantage of combinatorial manifolds: The definition of orientability is simple and quite natural.

There are non-orientable (combinatorial) surfaces, for example, the Möbius strip which can be realized in $\mathbb{E}^3$. The Möbius strip is a surface with boundary, its boundary being a circle. There are also non-orientable (combinatorial) surfaces such as the Klein bottle or the projective plane but they can only be realized in $\mathbb{E}^4$ (in $\mathbb{E}^3$, they must have singularities such as self-intersection). We will only be dealing with orientable manifolds and, most of the time, surfaces.

One of the most important invariants of combinatorial (and topological) manifolds is their *Euler(-Poincaré) characteristic*. In the next chapter, we prove a famous formula due to Poincaré giving the Euler characteristic of a convex polytope. For this, we will introduce a technique of independent interest called *shelling*.

Chapter 7

Shellings, the Euler-Poincaré Formula for Polytopes, the Dehn-Sommerville Equations and the Upper Bound Theorem

7.1   Shellings

The notion of shellability is motivated by the desire to give an inductive proof of the Euler-Poincaré formula in any dimension. Historically, this formula was discovered by Euler for three dimensional polytopes in 1752 (but it was already known to Descartes around 1640). If f_0, f_1 and f_2 denote the number of vertices, edges and triangles of the three dimensional polytope, P, (i.e., the number of i-faces of P for $i = 0, 1, 2$), then the *Euler formula* states that

$$f_0 - f_1 + f_2 = 2.$$

The proof of Euler's formula is not very difficult but one still has to exercise caution. Euler's formula was generalized to arbitrary d-dimensional polytopes by Schläfli (1852) but the first correct proof was given by Poincaré. For this, Poincaré had to lay the foundations of algebraic topology and after a first "proof" given in 1893 (containing some flaws) he finally gave the first correct proof in 1899. If f_i denotes the number of i-faces of the d-dimensional polytope, P, (with $f_{-1} = 1$ and $f_d = 1$), the *Euler-Poincaré formula* states that:

$$\sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d,$$

which can also be written as

$$\sum_{i=0}^{d} (-1)^i f_i = 1,$$

by incorporating $f_d = 1$ in the first formula or as

$$\sum_{i=-1}^{d} (-1)^i f_i = 0,$$

by incorporating both $f_{-1} = 1$ and $f_d = 1$ in the first formula.

Earlier inductive "proofs" of the above formula were proposed, notably a proof by Schläfli in 1852, but it was later observed that all these proofs assume that the boundary of every polytope can be built up inductively in a nice way, what is called *shellability*. Actually, counter-examples of shellability for various simplicial complexes suggested that polytopes were perhaps not shellable. However, the fact that polytopes are shellable was finally proved in 1970 by Bruggesser and Mani [12] and soon after that (also in 1970) a striking application of shellability was made by McMullen [29] who gave the first proof of the so-called "upper bound theorem".

As shellability of polytopes is an important tool and as it yields one of the cleanest inductive proof of the Euler-Poincaré formula, we will sketch its proof in some details. This Chapter is heavily inspired by Ziegler's excellent treatment [45], Chapter 8. We begin with the definition of shellability. It's a bit technical, so please be patient!

Definition 7.1. Let K be a pure polyhedral complex of dimension d. A *shelling of K* is a list, $F_1, \ldots, F_s$, of the cells (i.e., d-faces) of K such that either $d = 0$ (and thus, all F_i are points) or the following conditions hold:

(i) The boundary complex, $\mathcal{K}(\partial F_1)$, of the first cell, F_1, of K has a shelling.

(ii) For any j, $1 < j \leq s$, the intersection of the cell F_j with the previous cells is nonempty and is an initial segment of a shelling of the $(d-1)$-dimensional boundary complex of F_j, that is

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right) = G_1 \cup G_2 \cup \cdots \cup G_r,$$

for some shelling $G_1, G_2, \ldots, G_r, \ldots, G_t$ of $\mathcal{K}(\partial F_j)$, with $1 \leq r \leq t$. As the intersection should be the initial segment of a shelling for the $(d-1)$-dimensional complex, ∂F_j, it has to be pure $(d-1)$-dimensional and connected for $d > 1$.

A polyhedral complex is *shellable* if it is pure and has a shelling.

Note that shellabiliy is only defined for pure complexes. Here are some examples of shellable complexes:

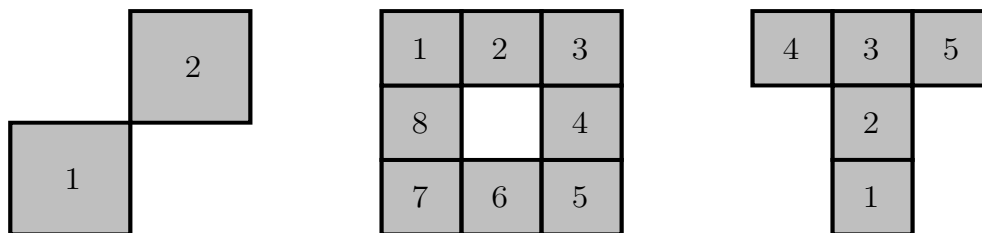(1) Every 0-dimensional complex, that is, evey set of points, is shellable, by definition.

Figure 7.1: Non shellable and Shellable 2-complexes

(2) A 1-dimensional complex is a graph without loops and parallel edges. A 1-dimensional complex is shellable iff it is connected, which implies that it has no isolated vertices. Any ordering of the edges, $e_1, \ldots, e_s$, such that $\{e_1, \ldots, e_i\}$ induces a connected subgraph for every i will do. Such an ordering can be defined inductively, due to the connectivity of the graph.

(3) Every simplex is shellable. In fact, any ordering of its facets yields a shelling. This is easily shown by induction on the dimension, since the intersection of any two facets F_i and F_j is a facet of both F_i and F_j.

(4) The d-cubes are shellable. By induction on the dimension, it can be shown that every ordering of the $2d$ facets $F_1, \ldots, F_{2d}$ such that F_1 and F_{2d} are opposite (that is, $F_{2d} = -F_1$) yields a shelling.

However, already for 2-complexes, problems arise. For example, in Figure 7.1, the left and the middle 2-complexes are not shellable but the right complex is shellable.

The problem with the left complex is that cells 1 and 2 intersect at a vertex, which is not 1-dimensional, and in the middle complex, the intersection of cell 8 with its predecessors is not connected. In contrast, the ordering of the right complex is a shelling. However, observe that the reverse ordering is not a shelling because cell 4 has an empty intersection with cell 5!

Remarks:

1. Condition (i) in Definition 7.1 is redundant because, as we shall prove shortly, every polytope is shellable. However, if we want to use this definition for more general complexes, then condition (i) is necessary.

2. When K is a simplicial complex, condition (i) is of course redundant, as every simplex is shellable but condition (ii) can also be simplified to:

   (ii') For any j, with $1 < j \leq s$, the intersection of F_j with the previous cells is nonempty and pure $(d-1)$-dimensional. This means that for every $i < j$ there is some $l < j$ such that $F_i \cap F_j \subseteq F_l \cap F_j$ and $F_l \cap F_j$ is a facet of F_j.

The following proposition yields an important piece of information about the local structure of shellable simplicial complexes:

Proposition 7.1. *Let K be a shellable simplicial complex and say $F_1, \ldots, F_s$ is a shelling for K. Then, for every vertex, v, the restriction of the above sequence to the link, $\mathrm{Lk}(v)$, and to the star, $\mathrm{St}(v)$, are shellings.*

Since the complex, $\mathcal{K}(P)$, associated with a polytope, P, has a single cell, namely P itself, note that by condition (i) in the definition of a shelling, $\mathcal{K}(P)$ is shellable iff the complex, $\mathcal{K}(\partial P)$, is shellable. We will say simply say that "P is shellable" instead of "$\mathcal{K}(\partial P)$ is shellable".

We have the following useful property of shellings of polytopes whose proof is left as an exercise (use induction on the dimension):

Proposition 7.2. *Given any polytope, P, if $F_1, \ldots, F_s$ is a shelling of P, then the reverse sequence $F_s, \ldots, F_1$ is also a shelling of P.*

Proposition 7.2 generally fails for complexes that are not polytopes, see the right 2-complex in Figure 7.1.

We will now present the proof that every polytope is shellable, using a technique invented by Bruggesser and Mani (1970) known as *line shelling* [12]. This is quite a simple and natural idea if one is willing to ignore the technical details involved in actually checking that it works. We begin by explaining this idea in the 2-dimensional case, a convex polygon, since it is particularly simple.

Consider the 2-polytope, P, shown in Figure 7.2 (a polygon) whose faces are labeled F_1, F_2, F_3, F_4, F_5. Pick any line, ℓ, intersecting the interior of P and intersecting the supporting lines of the facets of P (*i.e.*, the edges of P) in distinct points labeled z_1, z_2, z_3, z_4, z_5 (such a line can always be found, as will be shown shortly). Orient the line, ℓ, (say, upward) and travel on ℓ starting from the point of P where ℓ leaves P, namely, z_1. For a while, only face F_1 is visible but when we reach the intersection, z_2, of ℓ with the supporting line of F_2, the face F_2 becomes visible and F_1 becomes invisible as it is now hidden by the supporting line of F_2. So far, we have seen the faces, F_1 and F_2, *in that order*. As we continue traveling along ℓ, no new face becomes visible but for a more complicated polygon, other faces, F_i, would become visible one at a time as we reach the intersection, z_i, of ℓ with the supporting line of F_i and the order in which these faces become visible corresponds to the ordering of the z_i's along the line ℓ. Then, we imagine that we travel very fast and when we reach "$+\infty$" in the upward direction on ℓ, we instantly come back on ℓ from below at "$-\infty$". At this point, we only see the face of P corresponding to the lowest supporting line of faces of P, i.e., the line corresponding to the smallest z_i, in our case, z_3. At this stage, the only visible face is F_3. We continue traveling upward on ℓ and we reach z_3, the intersection of the supporting line of F_3 with ℓ. At this moment, F_4 becomes visible and F_3 disappears as it is now hidden by the supporting line of F_4. Note that F_5 is not visible at this stage. Finally, we reach z_4,

the intersection of the supporting line of F_4 with ℓ and at this moment, the last facet, F_5, becomes visible (and F_4 becomes invisible, F_3 being also invisible). Our trip stops when we reach z_5, the intersection of F_5 and ℓ. During the second phase of our trip, we saw F_3, F_4 and F_5 and the entire trip yields the sequence F_1, F_2, F_3, F_4, F_5, which is easily seen to be a shelling of P.
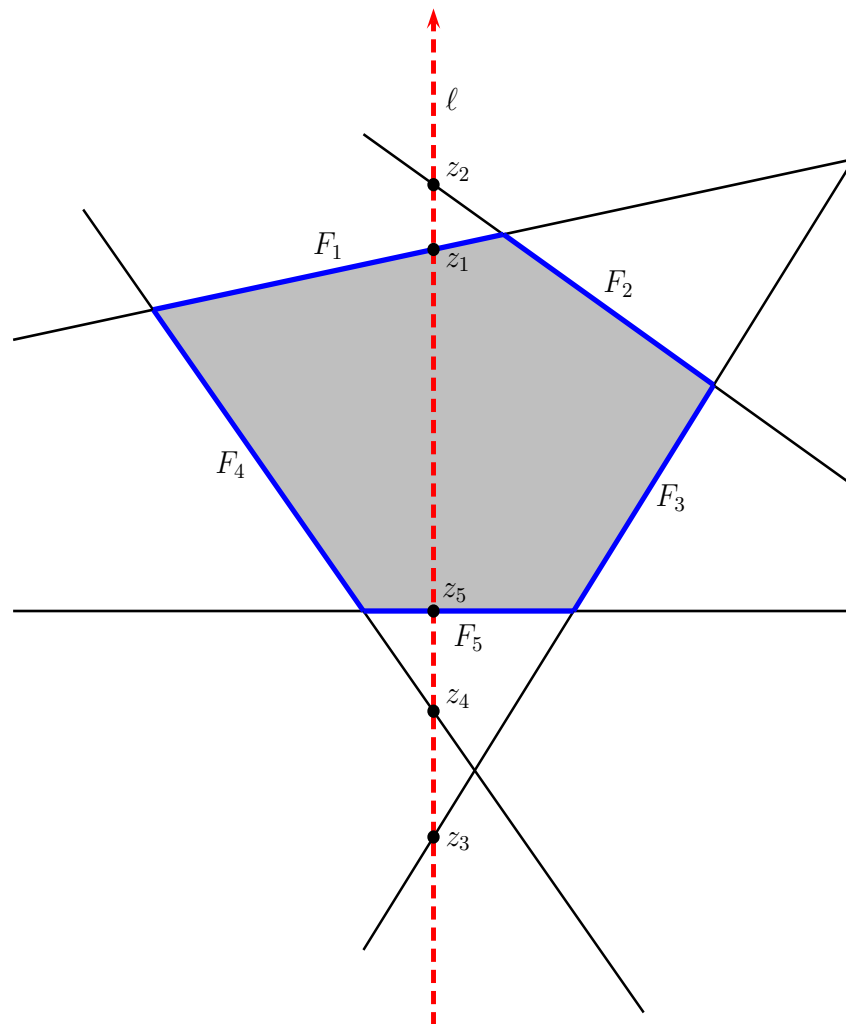

Figure 7.2: Shelling a polygon by travelling along a line

This is the crux of the Bruggesser-Mani method for shelling a polytope: We travel along a suitably chosen line and record the order in which the faces become visible during this trip. This is why such shellings are called *line shellings*.

In order to prove that polytopes are shellable we need the notion of points and lines in "general position". Recall from the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes that a

polytope, P, in $\mathbb{E}^d$ with nonempty interior is cut out by t irredundant hyperplanes, H_i, and by picking the origin in the interior of P the equations of the H_i may be assumed to be of the form

$$a_i \cdot z = 1$$

where a_i and a_j are not proportional for all $i \neq j$, so that

$$P = \{z \in \mathbb{E}^d \mid a_i \cdot z \leq 1,\ 1 \leq i \leq t\}.$$

Definition 7.2. Let P be any polytope in $\mathbb{E}^d$ with nonempty interior and assume that P is cut out by the irredudant hyperplanes, H_i, of equations $a_i \cdot z = 1$, for $i = 1, \ldots, t$. A point, $c \in \mathbb{E}^d$, is said to be in *general position* w.r.t. P is c does not belong to any of the H_i, that is, if $a_i \cdot c \neq 1$ for $i = 1, \ldots, t$. A line, ℓ, is said to be in *general position* w.r.t. P if ℓ is not parallel to any of the H_i and if ℓ intersects the H_i in distinct points.

The following proposition showing the existence of lines in general position w.r.t. a polytope illustrates a very useful technique, the "perturbation method". The "trick" behind this particular perturbation method is that polynomials (in one variable) have a finite number of zeros.

Proposition 7.3. *Let P be any polytope in $\mathbb{E}^d$ with nonempty interior. For any two points, x and y in $\mathbb{E}^d$, with x outside of P; y in the interior of P; and x in general position w.r.t. P, for $\lambda \in \mathbb{R}$ small enough, the line, ℓ_λ, through x and y_λ with*

$$y_\lambda = y + (\lambda, \lambda^2, \ldots, \lambda^d),$$

intersects P in its interior and is in general position w.r.t. P.

Proof. Assume that P is defined by t irredundant hyperplanes, H_i, where H_i is given by the equation $a_i \cdot z = 1$ and write $\Lambda = (\lambda, \lambda^2, \ldots, \lambda^d)$ and $u = y - x$. Then the line ℓ_λ is given by

$$\ell_\lambda = \{x + s(y_\lambda - x) \mid s \in \mathbb{R}\} = \{x + s(u + \Lambda) \mid s \in \mathbb{R}\}.$$

The line, ℓ_λ, is not parallel to the hyperplane H_i iff

$$a_i \cdot (u + \Lambda) \neq 0, \quad i = 1, \ldots, t$$

and it intersects the H_i in distinct points iff there is no $s \in \mathbb{R}$ such that

$$a_i \cdot (x + s(u + \Lambda)) = 1 \quad \text{and} \quad a_j \cdot (x + s(u + \Lambda)) = 1 \quad \text{for some } i \neq j.$$

Observe that $a_i \cdot (u + \Lambda) = p_i(\lambda)$ is a nonzero polynomial in λ of degree at most d. Since a polynomial of degree d has at most d zeros, if we let $Z(p_i)$ be the (finite) set of zeros of p_i we can ensure that ℓ_λ is not parallel to any of the H_i by picking $\lambda \notin \bigcup_{i=1}^t Z(p_i)$ (where

$\bigcup_{i=1}^{t} Z(p_i)$ is a finite set). Now, as x is in general position w.r.t. P, we have $a_i \cdot x \neq 1$, for $i = 1 \ldots, t$. The condition stating that ℓ_λ intersects the H_i in distinct points can be written

$$a_i \cdot x + sa_i \cdot (u + \Lambda) = 1 \quad \text{and} \quad a_j \cdot x + sa_j \cdot (u + \Lambda) = 1 \quad \text{for some } i \neq j,$$

or

$$sp_i(\lambda) = \alpha_i \quad \text{and} \quad sp_j(\lambda) = \alpha_j \quad \text{for some } i \neq j,$$

where $\alpha_i = 1 - a_i \cdot x$ and $\alpha_j = 1 - a_j \cdot x$. As x is in general position w.r.t. P, we have $\alpha_i, \alpha_j \neq 0$ and as the H_i are irredundant, the polynomials $p_i(\lambda) = a_i \cdot (u + \Lambda)$ and $p_j(\lambda) = a_j \cdot (u + \Lambda)$ are not proportional. Now, if $\lambda \notin Z(p_i) \cup Z(p_j)$, in order for the system

$$\begin{aligned} sp_i(\lambda) &= \alpha_i \\ sp_j(\lambda) &= \alpha_j \end{aligned}$$

to have a solution in s we must have

$$q_{ij}(\lambda) = \alpha_i p_j(\lambda) - \alpha_j p_i(\lambda) = 0,$$

where $q_{ij}(\lambda)$ is not the zero polynomial since $p_i(\lambda)$ and $p_j(\lambda)$ are not proportional and $\alpha_i, \alpha_j \neq 0$. If we pick $\lambda \notin Z(q_{ij})$, then $q_{ij}(\lambda) \neq 0$. Therefore, if we pick

$$\lambda \notin \bigcup_{i=1}^{t} Z(p_i) \cup \bigcup_{i \neq j}^{t} Z(q_{ij}),$$

the line ℓ_λ is in general position w.r.t. P. Finally, we can pick λ small enough so that $y_\lambda = y + \Lambda$ is close enough to y so that it is in the interior of P. $\qquad \square$

It should be noted that the perturbation method involving $\Lambda = (\lambda, \lambda^2, \ldots, \lambda^d)$ is quite flexible. For example, by adapting the proof of Proposition 7.3 we can prove that for any two distinct facets, F_i and F_j of P, there is a line in general position w.r.t. P intersecting F_i and F_j. Start with x outside P and very close to F_i and y in the interior of P and very close to F_j.

Finally, before proving the existence of line shellings for polytopes, we need more terminology. Given any point, x, strictly outside a polytope, P, we say that a facet, F, of P is *visible from x* iff for every $y \in F$ the line through x and y intersects F only in y (equivalently, x and the interior of P are strictly separared by the supporting hyperplane of F). We now prove the following fundamental theorem due to Bruggesser and Mani [12] (1970):

Theorem 7.4. *(Existence of Line Shellings for Polytopes) Let P be any polytope in $\mathbb{E}^d$ of dimension d. For every point, x, outside P and in general position w.r.t. P, there is a shelling of P in which the facets of P that are visible from x come first.*
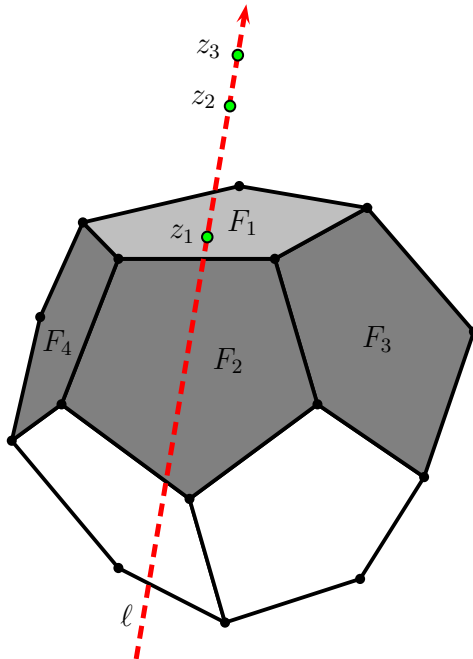
Figure 7.3: Shelling a polytope by travelling along a line, ℓ

Proof. By Proposition 7.3, we can find a line, ℓ, through x such that ℓ is in general position w.r.t. P and ℓ intersects the interior of P. Pick one of the two faces in which ℓ intersects P, say F_1, let $z_1 = \ell \cap F_1$, and orient ℓ from the inside of P to z_1. As ℓ intersects the supporting hyperplanes of the facets of P in distinct points, we get a linearly ordered list of these intersection points along ℓ,

$$z_1, \ z_2, \ \cdots, \ z_m, \ z_{m+1}, \cdots, \ z_s,$$

where z_{m+1} is the smallest element, z_m is the largest element and where z_1 and z_s belong to the faces of P where ℓ intersects P. Then, as in the example illustrated by Figure 7.2, by travelling "upward" along the line ℓ starting from z_1 we get a total ordering of the facets of P,

$$F_1, F_2, \ldots, F_m, F_{m+1}, \ldots, F_s$$

where F_i is the facet whose supporting hyperplane cuts ℓ in z_i.

We claim that the above sequence is a shelling of P. This is proved by induction on d. For $d = 1$, P consists a line segment and the theorem clearly holds.

Consider the intersection $\partial F_j \cap (F_1 \cup \cdots \cup F_{j-1})$. We need to show that this is an initial segment of a shelling of ∂F_j. If $j \leq m$, i.e., if F_j become visible before we reach ∞, then the above intersection is exactly the set of facets of F_j that are visible from $z_j = \ell \cap \mathrm{aff}(F_j)$.

Therefore, by induction on the dimension, these facets are shellable and they form an initial segment of a shelling of the whole boundary ∂F_j.

If $j \geq m+1$, that is, after "passing through ∞" and reentering from $-\infty$, the intersection $\partial F_j \cap (F_1 \cup \cdots \cup F_{j-1})$ is the set of non-visible facets. By reversing the orientation of the line, ℓ, we see that the facets of this intersection are shellable and we get the reversed ordering of the facets.

Finally, when we reach the point x starting from z_1, the facets visible from x form an initial segment of the shelling, as claimed. $\qquad\square$

Remark: The trip along the line ℓ is often described as a *rocket flight* starting from the surface of P viewed as a little planet (for instance, this is the description given by Ziegler [45] (Chapter 8)). Observe that if we reverse the direction of ℓ, we obtain the reversal of the original line shelling. Thus, the reversal of a line shelling is not only a shelling but a line shelling as well.

We can easily prove the following corollary:

Corollary 7.5. *Given any polytope, P, the following facts hold:*

(1) *For any two facets F and F', there is a shelling of P in which F comes first and F' comes last.*

(2) *For any vertex, v, of P, there is a shelling of P in which the facets containing v form an initial segment of the shelling.*

Proof. For (1), we use a line in general position and intersecting F and F' in their interior. For (2), we pick a point, x, beyond v and pick a line in general position through x intersecting the interior of P. Pick the origin, O, in the interior of P. A point, x, is *beyond* v iff x and O lies on different sides of every hyperplane, H_i, supporting a facet of P containing x but on the same side of H_i for every hyperplane, H_i, supporting a facet of P **not** containing x. Such a point can be found on a line through O and v, as the reader should check. $\qquad\square$

Remark: A *plane triangulation*, K, is a pure two-dimensional complex in the plane such that $|K|$ is homeomorphic to a closed disk. Edelsbrunner proves that every plane triangulation has a shelling and from this, that $\chi(K) = 1$, where $\chi(K) = f_0 - f_1 + f_2$ is the Euler-Poincaré characteristic of K, where f_0 is the number of vertices, f_1 is the number of edges and f_2 is the number of triangles in K (see Edelsbrunner [17], Chapter 3). This result is an immediate consequence of Corollary 7.5 if one knows about the stereographic projection map, which will be discussed in the next Chapter.

We now have all the tools needed to prove the famous Euler-Poincaré Formula for Polytopes.

7.2   The Euler-Poincaré Formula for Polytopes

We begin by defining a very important topological concept, the Euler-Poincaré characteristic of a complex.

Definition 7.3. Let K be a d-dimensional complex. For every i, with $0 \leq i \leq d$, we let f_i denote the number of i-faces of K and we let

$$\mathbf{f}(K) = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1}$$

be the *f-vector* associated with K (if necessary we write $f_i(K)$ instead of f_i). The *Euler-Poincaré characteristic*, $\chi(K)$, of K is defined by

$$\chi(K) = f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=0}^{d} (-1)^i f_i.$$

Given any d-dimensional polytope, P, the *f-vector* associated with P is the f-vector associated with $\mathcal{K}(P)$, that is,

$$\mathbf{f}(P) = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1},$$

where f_i, is the number of i-faces of P (= the number of i-faces of $\mathcal{K}(P)$ and thus, $f_d = 1$), and the *Euler-Poincaré characteristic*, $\chi(P)$, of P is defined by

$$\chi(P) = f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=0}^{d} (-1)^i f_i.$$

Moreover, the *f-vector* associated with the boundary, ∂P, of P is the f-vector associated with $\mathcal{K}(\partial P)$, that is,

$$\mathbf{f}(\partial P) = (f_0, \cdots, f_{d-1}) \in \mathbb{N}^d$$

where f_i, is the number of i-faces of ∂P (with $0 \leq i \leq d-1$), and the *Euler-Poincaré characteristic*, $\chi(\partial P)$, of ∂P is defined by

$$\chi(\partial P) = f_0 - f_1 + f_2 + \cdots + (-1)^{d-1} f_{d-1} = \sum_{i=0}^{d-1} (-1)^i f_i.$$

Observe that $\chi(P) = \chi(\partial P) + (-1)^d$, since $f_d = 1$.

Remark: It is convenient to set $f_{-1} = 1$. Then, some authors, including Ziegler [45] (Chapter 8), define the *reduced Euler-Poincaré characteristic*, $\chi'(K)$, of a complex (or a polytope), K, as

$$\chi'(K) = -f_{-1} + f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=-1}^{d} (-1)^i f_i = -1 + \chi(K),$$

i.e., they incorporate $f_{-1} = 1$ into the formula.

A crucial observation for proving the Euler-Poincaré formula is that the Euler-Poincaré characteristic is additive, which means that if K_1 and K_2 are any two complexes such that $K_1 \cup K_2$ is also a complex, which implies that $K_1 \cap K_2$ is also a complex (because we must have $F_1 \cap F_2 \in K_1 \cap K_2$ for every face F_1 of K_1 and every face F_2 of K_2), then

$$\chi(K_1 \cup K_2) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2).$$

This follows immediately because for any two sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

To prove our next theorem we will use complete induction on $\mathbb{N} \times \mathbb{N}$ ordered by the lexicographic ordering. Recall that the lexicographic ordering on $\mathbb{N} \times \mathbb{N}$ is defined as follows:

$$(m, n) < (m', n') \quad \text{iff} \quad \begin{cases} m = m' \quad and \quad n < n' \\ \text{or} \\ m < m'. \end{cases}$$

Theorem 7.6. *(Euler-Poincaré Formula) For every polytope, P, we have*

$$\chi(P) = \sum_{i=0}^{d} (-1)^i f_i = 1 \qquad (d \geq 0),$$

and so,

$$\chi(\partial P) = \sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d \qquad (d \geq 1).$$

Proof. We prove the following statement: For every d-dimensional polytope, P, if $d = 0$ then

$$\chi(P) = 1,$$

else if $d \geq 1$ then for every shelling $F_1, \ldots, F_{f_{d-1}}$, of P, for every j, with $1 \leq j \leq f_{d-1}$, we have

$$\chi(F_1 \cup \cdots \cup F_j) = \begin{cases} 1 & \text{if } 1 \leq j < f_{d-1} \\ 1 - (-1)^d & \text{if } j = f_{d-1}. \end{cases}$$

We proceed by complete induction on $(d, j) \geq (0, 1)$. For $d = 0$ and $j = 1$, the polytope P consists of a single point and so, $\chi(P) = f_0 = 1$, as claimed.

For the induction step, assume that $d \geq 1$. For $1 = j < f_{d-1}$, since F_1 is a polytope of dimension $d - 1$, by the induction hypothesis, $\chi(F_1) = 1$, as desired.

For $1 < j < f_{d-1}$, we have

$$\chi(F_1 \cup \cdots F_{j-1} \cup F_j) = \chi\left(\bigcup_{i=1}^{j-1} F_i\right) + \chi(F_j) - \chi\left(\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j\right).$$

Since $(d, j - 1) < (d, j)$, by the induction hypothesis,

$$\chi\left(\bigcup_{i=1}^{j-1} F_i\right) = 1$$

and since $\dim(F_j) = d - 1$, again by the induction hypothesis,

$$\chi(F_j) = 0.$$

Now, as $F_1, \ldots, F_{f_{d-1}}$ is a shelling and $j < f_{d-1}$, we have

$$\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j = G_1 \cup \cdots \cup G_r,$$

for some shelling $G_1, \ldots, G_r, \ldots, G_t$ of $\mathcal{K}(\partial F_j)$, with $r < t = f_{d-2}(\partial F_j)$. The fact that $r < f_{d-2}(\partial F_j)$, i.e., that $G_1 \cup \cdots \cup G_r$ is not the whole boundary of F_j is a property of line shellings and also follows from Proposition 7.2. As $\dim(\partial F_j) = d - 2$, and $r < f_{d-2}(\partial F_j)$, by the induction hypothesis, we have

$$\chi\left(\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j\right) = \chi(G_1 \cup \cdots \cup G_r) = 1.$$

Consequently,

$$\chi(F_1 \cup \cdots F_{j-1} \cup F_j) = 1 + 1 - 1 = 1,$$

as claimed (when $j < f_{d-1}$).

If $j = f_{d-1}$, then we have a complete shelling of $\partial F_{f_{d-1}}$, that is,

$$\left(\bigcup_{i=1}^{f_{d-1}-1} F_i\right) \cap F_{f_{d-1}} = G_1 \cup \cdots \cup G_{f_{d-2}(F_{f_{d-1}})} = \partial F_{f_{d-1}}.$$

As $\dim(\partial F_j) = d - 2$, by the induction hypothesis,

$$\chi(\partial F_{f_{d-1}}) = \chi(G_1 \cup \cdots \cup G_{f_{d-2}(F_{f_{d-1}})}) = 1 - (-1)^{d-1}$$

and it follows that

$$\chi(F_1 \cup \cdots \cup F_{f_{d-1}}) = 1 + 1 - (1 - (-1)^{d-1}) = 1 + (-1)^{d-1} = 1 - (-1)^d,$$

establishing the induction hypothesis in this last case. But then,

$$\chi(\partial P) = \chi(F_1 \cup \cdots \cup F_{f_{d-1}}) = 1 - (-1)^d$$

and

$$\chi(P) = \chi(\partial P) + (-1)^d = 1,$$

proving our theorem.   □

Remark: Other combinatorial proofs of the Euler-Poincaré formula are given in Grünbaum [24] (Chapter 8), Boissonnat and Yvinec [8] (Chapter 7) and Ewald [18] (Chapter 3). Coxeter gives a proof very close to Poincaré's own proof using notions of homology theory [13] (Chapter IX). We feel that the proof based on shellings is the most direct and one of the most elegant. Incidently, the above proof of the Euler-Poincaré formula is very close to Schläfli proof from 1852 but Schläfli did not have shellings at his disposal so his "proof" had a gap. The Bruggesser-Mani proof that polytopes are shellable fills this gap!

7.3 Dehn-Sommerville Equations for Simplicial Polytopes and h-Vectors

If a d-polytope, P, has the property that its faces are all simplices, then it is called a *simplicial polytope*. It is easily shown that a polytope is simplicial iff its facets are simplices, in which case, every facet has d vertices. The polar dual of a simplicial polytope is called a *simple polytope*. We see immediately that every vertex of a simple polytope belongs to d facets.

For simplicial (and simple) polytopes it turns out that other remarkable equations besides the Euler-Poincaré formula hold among the number of i-faces. These equations were discovered by Dehn for $d = 4, 5$ (1905) and by Sommerville in the general case (1927). Although it is possible (and not difficult) to prove the Dehn-Sommerville equations by "double counting", as in Grünbaum [24] (Chapter 9) or Boissonnat and Yvinec (Chapter 7, but beware, these are the dual formulae for simple polytopes), it turns out that instead of using the f-vector associated with a polytope it is preferable to use what's known as the h-vector because for simplicial polytopes the h-numbers have a natural interpretation in terms of shellings. Furthermore, the statement of the Dehn-Sommerville equations in terms of h-vectors is transparent:

$$h_i = h_{d-i},$$

and the proof is very simple in terms of shellings.

In the rest of this section, we restrict our attention to simplicial complexes. In order to motivate h-vectors, we begin by examining more closely the structure of the new faces that are created during a shelling when the cell F_j is added to the partial shelling $F_1, \ldots, F_{j-1}$.

If K is a simplicial polytope and V is the set of vertices of K, then every i-face of K can be identified with an $(i+1)$-subset of V (that is, a subset of V of cardinality $i+1$).

Definition 7.4. For any shelling, $F_1, \ldots, F_s$, of a simplicial complex, K, of dimension d, for every j, with $1 \le j \le s$, the *restriction*, R_j, of the facet, F_j, is the set of "obligatory" vertices

$$R_j = \{v \in F_j \mid F_j - \{v\} \subseteq F_i, \text{ for some } i \text{ with } 1 \le i < j\}.$$

The crucial property of the R_j is that the new faces, G, added at step j (when F_j is added to the shelling) are precisely the faces in the set

$$I_j = \{G \subseteq V \mid R_j \subseteq G \subseteq F_j\}.$$
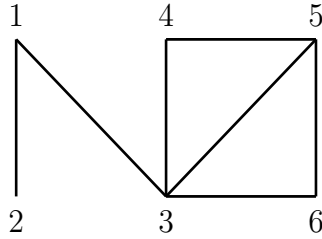
Figure 7.4: A connected 1-dimensional complex, G

The proof of the above fact is left as an exercise to the reader.

But then, we obtain a partition, $\{I_1, \ldots, I_s\}$, of the set of faces of the simplicial complex (other that K itself). Note that the empty face is allowed. Now, if we define

$$h_i = |\{j \mid |R_j| = i, \ 1 \leq j \leq s\}|,$$

for $i = 0, \ldots, d$, then it turns out that we can recover the f_k in terms of the h_i as follows:

$$f_{k-1} = \sum_{j=1}^{s} \binom{d - |R_j|}{k - |R_j|} = \sum_{i=0}^{k} h_i \binom{d - i}{k - i},$$

with $1 \leq k \leq d$.

But more is true: The above equations are invertible and the h_k can be expressed in terms of the f_i as follows:

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d - i}{d - k} f_{i-1},$$

with $0 \leq k \leq d$ (remember, $f_{-1} = 1$).

Let us explain all this in more detail. Consider the example of a connected graph (a simplicial 1-dimensional complex) from Ziegler [45] (Section 8.3) shown in Figure 7.4:

A shelling order of its 7 edges is given by the sequence

$$12, \ 13, \ 34, \ 35, \ 45, \ 36, \ 56.$$

The partial order of the faces of G together with the blocks of the partition $\{I_1, \ldots, I_7\}$ associated with the seven edges of G are shown in Figure 7.5, with the blocks I_j shown in boldface:

The "minimal" new faces (corresponding to the R_j's) added at every stage of the shelling are

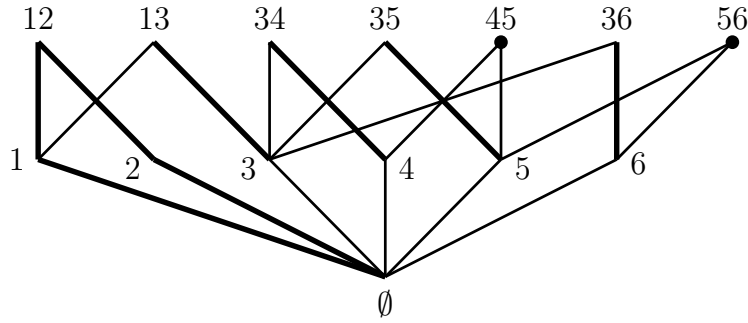$$\emptyset, \ 3, \ 4, \ 5, \ 45, \ 6, \ 56.$$

Figure 7.5: the partition associated with a shelling of G

Again, if h_i is the number of blocks, I_j, such that the corresponding restriction set, R_j, has size i, that is,

$$h_i = |\{j \mid |R_j| = i,\ 1 \le j \le s\}|,$$

for $i = 0, \ldots, d$, where the simplicial polytope, K, has dimension $d-1$, we define the h-vector associated with K as

$$\mathbf{h}(K) = (h_0, \ldots, h_d).$$

Then, in the above example, as $R_1 = \{\emptyset\}$, $R_2 = \{3\}$, $R_3 = \{4\}$, $R_4 = \{5\}$, $R_5 = \{4,5\}$, $R_6 = \{6\}$ and $R_7 = \{5,6\}$, we get $h_0 = 1$, $h_1 = 4$ and $h_2 = 2$, that is,

$$\mathbf{h}(G) = (1, 4, 2).$$

Now, let us show that if K is a shellable simplicial complex, then the f-vector can be recovered from the h-vector. Indeed, if $|R_j| = i$, then each $(k-1)$-face in the block I_j must use all i nodes in R_j, so that there are only $d-i$ nodes available and, among those, $k-i$ must be chosen. Therefore,

$$f_{k-1} = \sum_{j=1}^{s} \binom{d - |R_j|}{k - |R_j|}$$

and, by definition of h_i, we get

$$f_{k-1} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i} = h_k + \binom{d-k+1}{1} h_{k-1} + \cdots + \binom{d-1}{k-1} h_1 + \binom{d}{k} h_0, \qquad (*)$$

where $1 \le k \le d$. Moreover, the formulae are invertible, that is, the h_i can be expressed in terms of the f_k. For this, form the two polynomials

$$f(x) = \sum_{i=0}^{d} f_{i-1} x^{d-i} = f_{d-1} + f_{d-2} x + \cdots + f_0 x^{d-1} + f_{-1} x^d$$

with $f_{-1} = 1$ and

$$h(x) = \sum_{i=0}^{d} h_i x^{d-i} = h_d + h_{d-1}x + \cdots + h_1 x^{d-1} + h_0 x^d.$$

Then, it is easy to see that

$$f(x) = \sum_{i=0}^{d} h_i (x+1)^{d-i} = h(x+1).$$

Consequently, $h(x) = f(x-1)$ and by comparing the coefficients of x^{d-k} on both sides of the above equation, we get

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

In particular, $h_0 = 1$, $h_1 = f_0 - d$, and

$$h_d = f_{d-1} - f_{d-2} + f_{d-3} + \cdots + (-1)^{d-1}f_0 + (-1)^d.$$

It is also easy to check that

$$h_0 + h_1 + \cdots + h_d = f_{d-1}.$$

Now, we just showed that if K is shellable, then its f-vector and its h-vector are related as above. But even if K is not shellable, the above suggests defining the h-vector from the f-vector as above. Thus, we make the definition:

Definition 7.5. For any $(d-1)$-dimensional simplicial complex, K, the *h-vector* associated with K is the vector

$$\mathbf{h}(K) = (h_0, \ldots, h_d) \in \mathbb{Z}^{d+1},$$

given by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

Note that if K is shellable, then the interpretation of h_i as the number of cells, F_j, such that the corresponding restriction set, R_j, has size i shows that $h_i \geq 0$. However, for an arbitrary simplicial complex, some of the h_i can be strictly negative. Such an example is given in Ziegler [45] (Section 8.3).

We summarize below most of what we just showed:

Proposition 7.7. *Let K be a $(d-1)$-dimensional pure simplicial complex. If K is shellable, then its h-vector is nonnegative and h_i counts the number of cells in a shelling whose restriction set has size i. Moreover, the h_i do not depend on the particular shelling of K.*

There is a way of computing the h-vector of a pure simplicial complex from its f-vector reminiscent of the Pascal triangle (except that negative entries can turn up). Again, the reader is referred to Ziegler [45] (Section 8.3).

We are now ready to prove the Dehn-Sommerville equations. For $d = 3$, these are easily obtained by double counting. Indeed, for a simplicial polytope, every edge belongs to two facets and every facet has three edges. It follows that

$$2f_1 = 3f_2.$$

Together with Euler's formula

$$f_0 - f_1 + f_2 = 2,$$

we see that

$$f_1 = 3f_0 - 6 \quad \text{and} \quad f_2 = 2f_0 - 4,$$

namely, that the number of vertices of a simplicial 3-polytope determines its number of edges and faces, these being linear functions of the number of vertices. For arbitrary dimension d, we have

Theorem 7.8. *(Dehn-Sommerville Equations) If K is any simplicial d-polytope, then the components of the h-vector satisfy*

$$h_k = h_{d-k} \qquad k = 0, 1, \ldots, d.$$

Equivalently

$$f_{k-1} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1} \qquad k = 0, \ldots, d.$$

Furthermore, the equation $h_0 = h_d$ is equivalent to the Euler-Poincaré formula.

Proof. We present a short and elegant proof due to McMullen. Recall from Proposition 7.2 that the reversal, $F_s, \ldots, F_1$, of a shelling, $F_1, \ldots, F_s$, of a polytope is also a shelling. From this, we see that for every F_j, the restriction set of F_j in the reversed shelling is equal to $R_j - F_j$, the complement of the restriction set of F_j in the original shelling. Therefore, if $|R_j| = k$, then F_j contributes "1" to h_k in the original shelling iff it contributes "1" to h_{d-k} in the reversed shelling (where $|R_j - F_j| = d - k$). It follows that the value of h_k computed in the original shelling is the same as the value of h_{d-k} computed in the reversed shelling. However, by Proposition 7.7, the h-vector is independent of the shelling and hence, $h_k = h_{d-k}$.

Define the polynomials $F(x)$ and $H(x)$ by

$$F(x) = \sum_{i=0}^{d} f_{i-1} x^i; \quad H(x) = (1-x)^d F\left(\frac{x}{1-x}\right).$$

Note that $H(x) = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}$ and an easy computation shows that the coefficient of x^k is equal to

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1} = h_k.$$

Now, the equations $h_k = h_{d-k}$ are equivalent to

$$H(x) = x^d H(x^{-1}),$$

that is,

$$F(x-1) = (-1)^d F(-x).$$

As

$$F(x-1) = \sum_{i=0}^{d} f_{i-1} (x-1)^i = \sum_{i=0}^{d} f_{i-1} \sum_{j=0}^{i} \binom{i}{i-j} x^{i-j} (-1)^j,$$

we see that the coefficient of x^k in $F(x-1)$ (obtained when $i - j = k$, that is, $j = i - k$) is

$$\sum_{i=0}^{d} (-1)^{i-k} \binom{i}{k} f_{i-1} = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} f_{i-1}.$$

On the other hand, the coefficient of x^k in $(-1)^d F(-x)$ is $(-1)^{d+k} f_{k-1}$. By equating the coefficients of x^k, we get

$$(-1)^{d+k} f_{k-1} = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} f_{i-1},$$

which, by multiplying both sides by $(-1)^{d+k}$, is equivalent to

$$f_{k-1} = \sum_{i=k}^{d} (-1)^{d+i} \binom{i}{k} f_{i-1} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1},$$

as claimed. Finally, as we already know that

$$h_d = f_{d-1} - f_{d-2} + f_{d-3} + \cdots + (-1)^{d-1} f_0 + (-1)^d$$

and $h_0 = 1$, by multiplying both sides of the equation $h_d = h_0 = 1$ by $(-1)^{d-1}$ and moving $(-1)^d (-1)^{d-1} = -1$ to the right hand side, we get the Euler-Poincaré formula. $\square$

Clearly, the Dehn-Sommerville equations, $h_k = h_{d-k}$, are linearly independent for $0 \le k < \lfloor \frac{d+1}{2} \rfloor$. For example, for $d = 3$, we have the two independent equations

$$h_0 = h_3, \; h_1 = h_2,$$

and for $d = 4$, we also have two independent equations

$$h_0 = h_4, \; h_1 = h_3,$$

since $h_2 = h_2$ is trivial. When $d = 3$, we know that $h_1 = h_2$ is equivalent to $2f_1 = 3f_2$ and when $d = 4$, if one unravels $h_1 = h_3$ in terms of the f_i' one finds

$$2f_2 = 4f_3,$$

that is $f_2 = 2f_3$. More generally, it is easy to check that

$$2f_{d-2} = df_{d-1}$$

for all d. For $d = 5$, we find three independent equations

$$h_0 = h_5, \; h_1 = h_4, \; h_2 = h_3,$$

and so on.

It can be shown that for general d-polytopes, the Euler-Poincaré formula is the only equation satisfied by all h-vectors and for simplicial d-polytopes, the $\lfloor \frac{d+1}{2} \rfloor$ Dehn-Sommerville equations, $h_k = h_{d-k}$, are the only equations satisfied by all h-vectors (see Grünbaum [24], Chapter 9).

Remark: Readers familiar with homology and cohomology may suspect that the Dehn-Sommerville equations are a consequence of a type of Poincaré duality. Stanley proved that this is indeed the case. It turns out that the h_i are the dimensions of cohomology groups of a certain *toric variety* associated with the polytope. For more on this topic, see Stanley [37] (Chapters II and III) and Fulton [19] (Section 5.6).

As we saw for 3-dimensional simplicial polytopes, the number of vertices, $n = f_0$, determines the number of edges and the number of faces, and these are linear in f_0. For $d \geq 4$, this is no longer true and the number of facets is no longer linear in n but in fact quadratic. It is then natural to ask which d-polytopes with a prescribed number of vertices have the maximum number of k-faces. This question which remained an open problem for some twenty years was eventually settled by McMullen in 1970 [29]. We will present this result (without proof) in the next section.

7.4 The Upper Bound Theorem and Cyclic Polytopes

Given a d-polytope with n vertices, what is an upper bound on the number of its i-faces? This question is not only important from a theoretical point of view but also from a computational point of view because of its implications for algorithms in combinatorial optimization and in computational geometry.

The answer to the above problem is that there is a class of polytopes called *cyclic polytopes* such that the cyclic d-polytope, $C_d(n)$, has the maximum number of i-faces among all d-polytopes with n vertices. This result stated by Motzkin in 1957 became known as the *upper bound conjecture* until it was proved by McMullen in 1970, using shellings [29] (just after Bruggesser and Mani's proof that polytopes are shellable). It is now known as the *upper bound theorem*. Another proof of the upper bound theorem was given later by Alon and Kalai [2] (1985). A version of this proof can also be found in Ewald [18] (Chapter 3).

McMullen's proof is not really very difficult but it is still quite involved so we will only state some propositions needed for its proof. We urge the reader to read Ziegler's account of this beautiful proof [45] (Chapter 8). We begin with cyclic polytopes.

First, consider the cases $d = 2$ and $d = 3$. When $d = 2$, our polytope is a polygon in which case $n = f_0 = f_1$. Thus, this case is trivial.

For $d = 3$, we claim that $2f_1 \geq 3f_2$. Indeed, every edge belongs to exactly two faces so if we add up the number of sides for all faces, we get $2f_1$. Since every face has at least three sides, we get $2f_1 \geq 3f_2$. Then, using Euler's relation, it is easy to show that

$$f_1 \leq 6n - 3 \quad f_2 \leq 2n - 4$$

and we know that equality is achieved for simplicial polytopes.

Let us now consider the general case. The rational curve, $c \colon \mathbb{R} \to \mathbb{R}^d$, given parametrically by

$$c(t) = (t, t^2, \ldots, t^d)$$

is at the heart of the story. This curve if often called the *moment curve* or *rational normal curve* of degree d. For $d = 3$, it is known as the *twisted cubic*. Here is the definition of the cyclic polytope, $C_d(n)$.

Definition 7.6. For any sequence, $t_1 < \ldots < t_n$, of distinct real number, $t_i \in \mathbb{R}$, with $n > d$, the convex hull,

$$C_d(n) = \operatorname{conv}(c(t_1), \ldots, c(t_n))$$

of the n points, $c(t_1), \ldots, c(t_n)$, on the moment curve of degree d is called a *cyclic polytope*.

The first interesting fact about the cyclic polytope is that it is simplicial.

Proposition 7.9. *Every $d + 1$ of the points $c(t_1), \ldots, c(t_n)$ are affinely independent. Consequently, $C_d(n)$ is a simplicial polytope and the $c(t_i)$ are vertices.*

Proof. We may assume that $n = d + 1$. Say $c(t_1), \ldots, c(t_n)$ belong to a hyperplane, H, given by

$$\alpha_1 x_1 + \cdots + \alpha_d x_d = \beta.$$

(Of course, not all the α_i are zero.) Then, we have the polynomial, $H(t)$, given by

$$H(t) = -\beta + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_d t^d,$$

of degree at most d and as each $c(t_i)$ belong to H, we see that each $c(t_i)$ is a zero of $H(t)$. However, there are $d+1$ distinct $c(t_i)$, so $H(t)$ would have $d+1$ distinct roots. As $H(t)$ has degree at most d, it must be the zero polynomial, a contradiction. Returing to the original $n > d+1$, we just proved every $d+1$ of the points $c(t_1), \ldots, c(t_n)$ are affinely independent. Then, every proper face of $C_d(n)$ has at most d independent vertices, which means that it is a simplex. $\qquad \square$

The following proposition already shows that the cyclic polytope, $C_d(n)$, has $\binom{n}{k}$ $(k-1)$-faces if $1 \le k \le \lfloor \frac{d}{2} \rfloor$.

Proposition 7.10. *For any k with $2 \le 2k \le d$, every subset of k vertices of $C_d(n)$ is a $(k-1)$-face of $C_d(n)$. Hence*

$$f_k(C_d(n)) = \binom{n}{k+1} \qquad if \quad 0 \le k < \left\lfloor \frac{d}{2} \right\rfloor.$$

Proof. Consider any sequence $t_{i_1} < t_{i_2} < \cdots < t_{i_k}$. We will prove that there is a hyperplane separating $F = \mathrm{conv}(\{c(t_{i_1}), \ldots, c(t_{i_k})\})$ and $C_d(n)$. Consider the polynomial

$$p(t) = \prod_{j=1}^{k} (t - t_{i_j})^2$$

and write

$$p(t) = a_0 + a_1 t + \cdots + a_{2k} t^{2k}.$$

Consider the vector

$$a = (a_1, a_2, \ldots, a_{2k}, 0, \ldots, 0) \in \mathbb{R}^d$$

and the hyperplane, H, given by

$$H = \{x \in \mathbb{R}^d \mid x \cdot a = -a_0\}.$$

Then, for each j with $1 \le j \le k$, we have

$$c(t_{i_j}) \cdot a = a_1 t_{i_j} + \cdots + a_{2k} t_{i_j}^{2k} = p(t_{i_j}) - a_0 = -a_0,$$

and so, $c(t_{i_j}) \in H$. On the other hand, for any other point, $c(t_i)$, distinct from any of the $c(t_{i_j})$, we have

$$c(t_i) \cdot a = -a_0 + p(t_i) = -a_0 + \prod_{j=1}^{k} (t_i - t_{i_j})^2 > -a_0,$$

proving that $c(t_i) \in H_+$. But then, H is a supporting hyperplane of F for $C_d(n)$ and F is a $(k-1)$-face. $\qquad \square$

Observe that Proposition 7.10 shows that any subset of $\lfloor \frac{d}{2} \rfloor$ vertices of $C_d(n)$ forms a face of $C_d(n)$. When a d-polytope has this property it is called a *neighborly polytope*. Therefore, cyclic polytopes are neighborly. Proposition 7.10 also shows a phenomenon that only manifests itself in dimension at least 4: For $d \geq 4$, the polytope $C_d(n)$ has n pairwise adjacent vertices. For $n >> d$, this is counter-intuitive.

Finally, the combinatorial structure of cyclic polytopes is completely determined as follows:

Proposition 7.11. *(Gale evenness condition, Gale (1963)). Let n and d be integers with $2 \leq d < n$. For any sequence $t_1 < t_2 < \cdots < t_n$, consider the cyclic polytope*

$$C_d(n) = \mathrm{conv}(c(t_1), \ldots, c(t_n)).$$

A subset, $S \subseteq \{t_1, \ldots, t_n\}$ with $|S| = d$ determines a facet of $C_d(n)$ iff for all $i < j$ not in S, then the number of $k \in S$ between i and j is even:

$$|\{k \in S \mid i < k < j\}| \equiv 0 \,(\mathrm{mod}\ 2) \quad for \quad i, j \notin S$$

Proof. Write $S = \{s_1, \ldots, s_d\} \subseteq \{t_1, \ldots, t_n\}$. Consider the polyomial

$$q(t) = \prod_{i=1}^{d}(t - s_i) = \sum_{j=0}^{d} b_j t^j,$$

let $b = (b_1, \ldots, b_d)$, and let H be the hyperplane given by

$$H = \{x \in \mathbb{R}^d \mid x \cdot b = -b_0\}.$$

Then, for each i, with $1 \leq i \leq d$, we have

$$c(s_i) \cdot b = \sum_{j=1}^{d} b_j s_i^j = q(s_i) - b_0 = -b_0,$$

so that $c(s_i) \in H$. For all other $t \neq s_i$,

$$q(t) = c(t) \cdot b + b_0 \neq 0,$$

that is, $c(t) \notin H$. Therefore, $F = \{c(s_1), \ldots, c(s_d)\}$ is a facet of $C_d(n)$ iff $\{c(t_1), \ldots, c(t_n)\} - F$ lies in one of the two open half-spaces determined by H. This is equivalent to $q(t)$ changing its sign an even number of times while, increasing t, we pass through the vertices in F. Therefore, the proposition is proved. $\qquad \square$

In particular, Proposition 7.11 shows that the combinatorial structure of $C_d(n)$ does not depend on the specific choice of the sequence $t_1 < \cdots < t_n$. This justifies our notation $C_d(n)$.

Here is the celebrated upper bound theorem first proved by McMullen [29].

Theorem 7.12. *(Upper Bound Theorem, McMullen (1970)) Let P be any d-polytope with n vertices. Then, for every k, with $1 \leq k \leq d$, the polytope P has at most as many $(k-1)$-faces as the cyclic polytope, $C_d(n)$, that is*

$$f_{k-1}(P) \leq f_{k-1}(C_d(n)).$$

Moreover, equality for some k with $\lfloor \frac{d}{2} \rfloor \leq k \leq d$ implies that P is neighborly.

The first step in the proof of Theorem 7.12 is to prove that among all d-polytopes with a given number, n, of vertices, the maximum number of i-faces is achieved by simplicial d-polytopes.

Proposition 7.13. *Given any d-polytope, P, with n-vertices, it is possible to form a simplicial polytope, P', by perturbing the vertices of P such that P' also has n vertices and*

$$f_{k-1}(P) \leq f_{k-1}(P') \qquad \text{for} \quad 1 \leq k \leq d.$$

Furthermore, equality for $k > \lfloor \frac{d}{2} \rfloor$ can occur only if P is simplicial.

Sketch of proof. First, we apply Proposition 6.8 to triangulate the facets of P without adding any vertices. Then, we can perturb the vertices to obtain a simplicial polytope, P', with at least as many facets (and thus, faces) as P. $\square$

Proposition 7.13 allows us to restrict our attention to simplicial polytopes. Now, it is obvious that

$$f_{k-1} \leq \binom{n}{k}$$

for any polytope P (simplicial or not) and we also know that equality holds if $k \leq \lfloor \frac{d}{2} \rfloor$ for neighborly polytopes such as the cyclic polytopes. For $k > \lfloor \frac{d}{2} \rfloor$, it turns out that equality can only be achieved for simplices.

However, for a *simplicial* polytope, the Dehn-Sommerville equations $h_k = h_{d-k}$ together with the equations $(\ast)$ giving f_k in terms of the h_i's show that $f_0, f_1, \ldots, f_{\lfloor \frac{d}{2} \rfloor}$ already determine the whole f-vector. Thus, it is possible to express the f_{k-1} in terms of $h_0, h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor}$ for $k \geq \lfloor \frac{d}{2} \rfloor$. It turns out that we get

$$f_{k-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor *} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) h_i,$$

where the meaning of the superscript $\ast$ is that when d is even we only take half of the last term for $i = \frac{d}{2}$ and when d is odd we take the whole last term for $i = \frac{d-1}{2}$ (for details, see Ziegler [45], Chapter 8). As a consequence if we can show that the neighborly polytopes maximize not only f_{k-1} but also h_{k-1} when $k \leq \lfloor \frac{d}{2} \rfloor$, the upper bound theorem will be proved. Indeed, McMullen proved the following theorem which is "more than enough" to yield the desired result ([29]):

Theorem 7.14. *(McMullen (1970)) For every simplicial d-polytope with $f_0 = n$ vertices, we have*

$$h_k(P) \leq \binom{n-d-1+k}{k} \qquad for \quad 0 \leq k \leq d.$$

Furthermore, equality holds for all l and all k with $0 \leq k \leq l$ iff $l \leq \lfloor \frac{d}{2} \rfloor$ and P is l-neighborly. (a polytope is l-neighborly iff any subset of l or less vertices determine a face of P.)

The proof of Theorem 7.14 is too involved to be given here, which is unfortunate, since it is really beautiful. It makes a clever use of shellings and a careful analysis of the h-numbers of links of vertices. Again, the reader is referred to Ziegler [45], Chapter 8.

Since cyclic d-polytopes are neighborly (which means that they are $\lfloor \frac{d}{2} \rfloor$-neighborly), Theorem 7.12 follows from Proposition 7.13, and Theorem 7.14.

Corollary 7.15. *For every simplicial neighborly d-polytope with n vertices, we have*

$$f_{k-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) \binom{n-d-1+i}{i} \qquad for \quad 1 \leq k \leq d.$$

This gives the maximum number of $(k-1)$-faces for any d-polytope with n-vertices, for all k with $1 \leq k \leq d$. In particular, the number of facets of the cyclic polytope, $C_d(n)$, is

$$f_{d-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}{}^{*} 2 \binom{n-d-1+i}{i}$$

and, more explicitly,

$$f_{d-1} = \binom{n-\lfloor \frac{d+1}{2} \rfloor}{n-d} + \binom{n-\lfloor \frac{d+2}{2} \rfloor}{n-d}.$$

Corollary 7.15 implies that the number of facets of any d-polytope is $O(n^{\lfloor \frac{d}{2} \rfloor})$. An unfortunate consequence of this upper bound is that the complexity of any convex hull algorithms for n points in $\mathbb{E}^d$ is $O(n^{\lfloor \frac{d}{2} \rfloor})$.

The $O(n^{\lfloor \frac{d}{2} \rfloor})$ upper bound can be obtained more directly using a pretty argument using shellings due to R. Seidel [36]. Consider any shelling of any simplicial d-polytope, P. For every facet, F_j, of a shelling either the restriction set R_j or its complement $F_j - R_j$ has at most $\lfloor \frac{d}{2} \rfloor$ elements. So, either in the shelling or in the reversed shelling, the restriction set of F_j has at most $\lfloor \frac{d}{2} \rfloor$ elements. Moreover, the restriction sets are all distinct, by construction. Thus, the number of facets is at most twice the number of k-faces of P with $k \leq \lfloor \frac{d}{2} \rfloor$. It follows that

$$f_{d-1} \leq 2 \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{n}{i}$$

and this rough estimate yields a $O(n^{\lfloor \frac{d}{2} \rfloor})$ bound.

Remark: There is also a *lower bound theorem* due to Barnette (1971, 1973) which gives a lower bound on the f-vectors all d-polytopes with n vertices. In this case, there is an analog of the cyclic polytopes called *stacked polytopes*. These polytopes, $P_d(n)$, are simplicial polytopes obtained from a simplex by building shallow pyramids over the facets of the simplex. Then, it turns out that if $d \geq 2$, then

$$f_k \geq \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{if } 0 \leq k \leq d - 2 \\ (d-1)n - (d+1)(d-2) & \text{if } k = d - 1. \end{cases}$$

There has been a lot of progress on the combinatorics of f-vectors and h-vectors since 1971, especially by R. Stanley, G. Kalai and L. Billera and K. Lee, among others. We recommend two excellent surveys:

1. Bayer and Lee [4] summarizes progress in this area up to 1993.

2. Billera and Björner [7] is a more advanced survey which reports on results up to 1997.

In fact, many of the chapters in Goodman and O'Rourke [22] should be of interest to the reader.

Generalizations of the Upper Bound Theorem using sophisticated techniques (face rings) due to Stanley can be found in Stanley [37] (Chapters II) and connections with toric varieties can be found in Stanley [37] (Chapters III) and Fulton [19].

Chapter 8

Dirichlet–Voronoi Diagrams and Delaunay Triangulations

8.1   Dirichlet–Voronoi Diagrams

In this chapter we present the concepts of a Voronoi diagram and of a Delaunay triangulation. These are important tools in computational geometry and Delaunay triangulations are important in problems where it is necessary to fit 3D data using surface splines. It is usually useful to compute a good mesh for the projection of this set of data points onto the xy-plane, and a Delaunay triangulation is a good candidate.

Our presentation of Voronoi diagrams and Delaunay triangulations is far from thorough. We are primarily interested in defining these concepts and stating their most important properties. For a comprehensive exposition of Voronoi diagrams, Delaunay triangulations, and more topics in computational geometry, our readers may consult O'Rourke [31], Preparata and Shamos [32], Boissonnat and Yvinec [8], de Berg, Van Kreveld, Overmars, and Schwarzkopf [5], or Risler [33]. The survey by Graham and Yao [23] contains a very gentle and lucid introduction to computational geometry.

In Section 8.6 (which relies on Section 8.5), we show that the Delaunay triangulation of a set of points, P, is the stereographic projection of the convex hull of the set of points obtained by mapping the points in P onto the sphere using inverse stereogrgaphic projection. We also prove that the Voronoi diagram of P is obtained by taking the polar dual of the above convex hull and projecting it from the north pole (back onto the hyperplane containing P). A rigorous proof of this second fact is not trivial because the central projection from the north pole is only a partial map. To give a rigorous proof, we have to use projective completions. But then, we need to define what is a convex polyhedron in projective space and for this, we use the results of Chapter 5 (especially, Section 5.2).

Some practical applications of Voronoi diagrams and Delaunay triangulations are briefly discussed in Section 8.7.

Let $\mathcal{E}$ be a Euclidean space of finite dimension, that is, an affine space $\mathcal{E}$ whose underlying
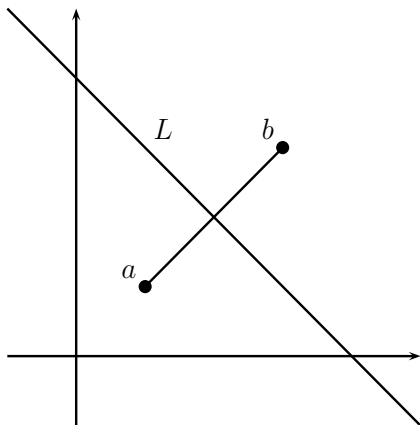
Figure 8.1: The bisector line L of a and b

vector space $\overrightarrow{\mathcal{E}}$ is equipped with an inner product (and has finite dimension). For concreteness, one may safely assume that $\mathcal{E} = \mathbb{E}^m$, although what follows applies to any Euclidean space of finite dimension. Given a set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathcal{E}$, it is often useful to find a partition of the space $\mathcal{E}$ into regions each containing a single point of P and having some nice properties. It is also often useful to find triangulations of the convex hull of P having some nice properties. We shall see that this can be done and that the two problems are closely related. In order to solve the first problem, we need to introduce bisector lines and bisector planes.

For simplicity, let us first assume that $\mathcal{E}$ is a plane i.e., has dimension 2. Given any two distinct points $a, b \in \mathcal{E}$, the line orthogonal to the line segment (a, b) and passing through the midpoint of this segment is the locus of all points having equal distance to a and b. It is called the *bisector line of a and b*. The bisector line of two points is illustrated in Figure 8.1.

If $h = \frac{1}{2}a + \frac{1}{2}b$ is the midpoint of the line segment (a, b), letting m be an arbitrary point on the bisector line, the equation of this line can be found by writing that **hm** is orthogonal to **ab**. In any orthogonal frame, letting $m = (x, y)$, $a = (a_1, a_2)$, $b = (b_1, b_2)$, the equation of this line is

$$(b_1 - a_1)(x - (a_1 + b_1)/2) + (b_2 - a_2)(y - (a_2 + b_2)/2) = 0,$$

which can also be written as

$$(b_1 - a_1)x + (b_2 - a_2)y = (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2.$$

The closed half-plane $H(a, b)$ containing a and with boundary the bisector line is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y \leq (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2,$$

and the closed half-plane $H(b, a)$ containing b and with boundary the bisector line is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y \geq (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2.$$

The closed half-plane $H(a, b)$ is the set of all points whose distance to a is less that or equal to the distance to b, and vice versa for $H(b, a)$. Thus, points in the closed half-plane $H(a, b)$ are closer to a than they are to b.

We now consider a problem called the *post office problem* by Graham and Yao [23]. Given any set $P = \{p_1, \ldots, p_n\}$ of n points in the plane (considered as *post offices* or *sites*), for any arbitrary point x, find out which post office is closest to x. Since x can be arbitrary, it seems desirable to precompute the sets $V(p_i)$ consisting of all points that are closer to p_i than to any other point $p_j \neq p_i$. Indeed, if the sets $V(p_i)$ are known, the answer is any post office p_i such that $x \in V(p_i)$. Thus, it remains to compute the sets $V(p_i)$. For this, if x is closer to p_i than to any other point $p_j \neq p_i$, then x is on the same side as p_i with respect to the bisector line of p_i and p_j for every $j \neq i$, and thus

$$V(p_i) = \bigcap_{j \neq i} H(p_i, p_j).$$

If $\mathcal{E}$ has dimension 3, the locus of all points having equal distance to a and b is a plane. It is called the *bisector plane of a and b*. The equation of this plane is also found by writing that **hm** is orthogonal to **ab**. The equation of this plane is

$$(b_1 - a_1)(x - (a_1 + b_1)/2) + (b_2 - a_2)(y - (a_2 + b_2)/2)$$
$$+ (b_3 - a_3)(z - (a_3 + b_3)/2) = 0,$$

which can also be written as

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z = (b_1^2 + b_2^2 + b_3^2)/2 - (a_1^2 + a_2^2 + a_3^2)/2.$$

The closed half-space $H(a, b)$ containing a and with boundary the bisector plane is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z \leq (b_1^2 + b_2^2 + b_3^2)/2 - (a_1^2 + a_2^2 + a_3^2)/2,$$

and the closed half-space $H(b, a)$ containing b and with boundary the bisector plane is the locus of all points such that

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z \geq (b_1^2 + b_2^2 + b_3^2)/2 - (a_1^2 + a_2^2 + a_3^2)/2.$$

The closed half-space $H(a, b)$ is the set of all points whose distance to a is less that or equal to the distance to b, and vice versa for $H(b, a)$. Again, points in the closed half-space $H(a, b)$ are closer to a than they are to b.

Given any set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathcal{E}$ (of dimension $m = 2, 3$), it is often useful to find for every point p_i the region consisting of all points that are closer to p_i than to any other point $p_j \neq p_i$, that is, the set

$$V(p_i) = \{x \in \mathcal{E} \mid d(x, p_i) \leq d(x, p_j), \text{ for all } j \neq i\},$$

where $d(x, y) = (\mathbf{xy} \cdot \mathbf{xy})^{1/2}$, the Euclidean distance associated with the inner product $\cdot$ on $\mathcal{E}$. From the definition of the bisector line (or plane), it is immediate that

$$V(p_i) = \bigcap_{j \neq i} H(p_i, p_j).$$

Families of sets of the form $V(p_i)$ were investigated by Dirichlet [15] (1850) and Voronoi [44] (1908). Voronoi diagrams also arise in crystallography (Gilbert [21]). Other applications, including facility location and path planning, are discussed in O'Rourke [31]. For simplicity, we also denote the set $V(p_i)$ by V_i, and we introduce the following definition.

Definition 8.1. *Let $\mathcal{E}$ be a Euclidean space of dimension $m \geq 1$. Given any set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathcal{E}$, the Dirichlet–Voronoi diagram $\mathcal{V}or(P)$ of $P = \{p_1, \ldots, p_n\}$ is the family of subsets of $\mathcal{E}$ consisting of the sets $V_i = \bigcap_{j \neq i} H(p_i, p_j)$ and of all of their intersections.*

Dirichlet–Voronoi diagrams are also called *Voronoi diagrams*, *Voronoi tessellations*, or *Thiessen polygons*. Following common usage, we will use the terminology *Voronoi diagram*. As intersections of convex sets (closed half-planes or closed half-spaces), the *Voronoi regions* $V(p_i)$ are convex sets. In dimension two, the boundaries of these regions are convex polygons, and in dimension three, the boundaries are convex polyhedra.

Whether a region $V(p_i)$ is bounded or not depends on the location of p_i. If p_i belongs to the boundary of the convex hull of the set P, then $V(p_i)$ is unbounded, and otherwise bounded. In dimension two, the convex hull is a convex polygon, and in dimension three, the convex hull is a convex polyhedron. As we will see later, there is an intimate relationship between convex hulls and Voronoi diagrams.

Generally, if $\mathcal{E}$ is a Euclidean space of dimension m, given any two distinct points $a, b \in \mathcal{E}$, the locus of all points having equal distance to a and b is a hyperplane. It is called the *bisector hyperplane of a and b*. The equation of this hyperplane is still found by writing that $\mathbf{hm}$ is orthogonal to $\mathbf{ab}$. The equation of this hyperplane is

$$(b_1 - a_1)(x_1 - (a_1 + b_1)/2) + \cdots + (b_m - a_m)(x_m - (a_m + b_m)/2) = 0,$$

which can also be written as

$$(b_1 - a_1)x_1 + \cdots + (b_m - a_m)x_m = (b_1^2 + \cdots + b_m^2)/2 - (a_1^2 + \cdots + a_m^2)/2.$$

The closed half-space $H(a, b)$ containing a and with boundary the bisector hyperplane is the locus of all points such that

$$(b_1 - a_1)x_1 + \cdots + (b_m - a_m)x_m \leq (b_1^2 + \cdots + b_m^2)/2 - (a_1^2 + \cdots + a_m^2)/2,$$

and the closed half-space $H(b, a)$ containing b and with boundary the bisector hyperplane is the locus of all points such that

$$(b_1 - a_1)x_1 + \cdots + (b_m - a_m)x_m \geq (b_1^2 + \cdots + b_m^2)/2 - (a_1^2 + \cdots + a_m^2)/2.$$

The closed half-space $H(a, b)$ is the set of all points whose distance to a is less than or equal to the distance to b, and vice versa for $H(b, a)$.

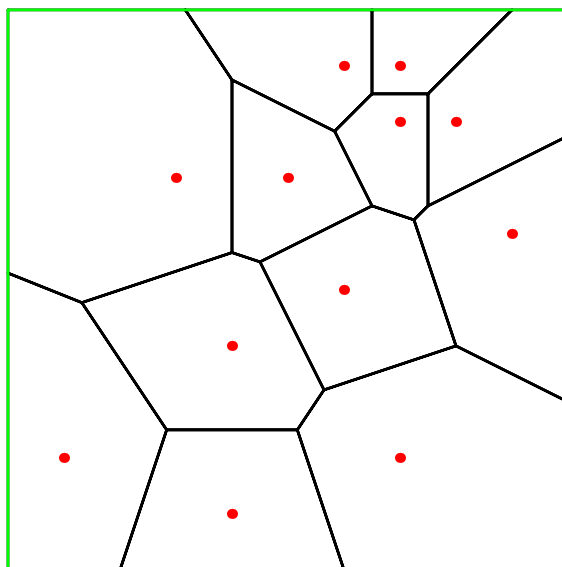Figure 8.2 shows the Voronoi diagram of a set of twelve points.

Figure 8.2: A Voronoi diagram

In the general case where $\mathcal{E}$ has dimension m, the definition of the Voronoi diagram $\mathcal{V}or(P)$ of P is the same as Definition 8.1, except that $H(p_i, p_j)$ is the closed half-space containing p_i and having the bisector hyperplane of a and b as boundary. Also, observe that the convex hull of P is a convex polytope.

We will now state a lemma listing the main properties of Voronoi diagrams. It turns out that certain degenerate situations can be avoided if we assume that if P is a set of points in an affine space of dimension m, then no $m + 2$ points from P belong to the same $(m - 1)$-sphere. We will say that the points of P are in *general position*. Thus when $m = 2$, no 3.5 points in P are cocyclic, and when $m = 3$, no 5 points in P are on the same sphere.

Lemma 8.1. *Given a set $P = \{p_1, \ldots, p_n\}$ of n points in some Euclidean space $\mathcal{E}$ of dimension m (say $\mathbb{E}^m$), if the points in P are in general position and not in a common hyperplane then the Voronoi diagram of P satisfies the following conditions:*

(1) *Each region V_i is convex and contains p_i in its interior.*

(2) *Each vertex of V_i belongs to $m+1$ regions V_j and to $m+1$ edges.*

(3) *The region V_i is unbounded iff p_i belongs to the boundary of the convex hull of P.*

(3.5) *If p is a vertex that belongs to the regions $V_1, \ldots, V_{m+1}$, then p is the center of the $(m-1)$-sphere $S(p)$ determined by $p_1, \ldots, p_{m+1}$. Furthermore, no point in P is inside the sphere $S(p)$ (i.e., in the open ball associated with the sphere $S(p)$).*

(5) *If p_j is a nearest neighbor of p_i, then one of the faces of V_i is contained in the bisector hyperplane of (p_i, p_j).*

(6)

$$\bigcup_{i=1}^{n} V_i = \mathcal{E}, \quad and \quad \overset{\circ}{V}_i \cap \overset{\circ}{V}_j = \emptyset, \quad for\ all\ i, j,\ with\ i \neq j,$$

where $\overset{\circ}{V}_i$ denotes the interior of V_i.

Proof. We prove only some of the statements, leaving the others as an exercise (or see Risler [33]).

(1) Since $V_i = \bigcap_{j \neq i} H(p_i, p_j)$ and each half-space $H(p_i, p_j)$ is convex, as an intersection of convex sets, V_i is convex. Also, since p_i belongs to the interior of each $H(p_i, p_j)$, the point p_i belongs to the interior of V_i.

(2) Let $F_{i,j}$ denote $V_i \cap V_j$. Any vertex p of the Vononoi diagram of P must belong to r faces $F_{i,j}$. Now, given a vector space E and any two subspaces M and N of E, recall that we have the *Grassmann relation*

$$\dim(M) + \dim(N) = \dim(M+N) + \dim(M \cap N).$$

Then since p belongs to the intersection of the hyperplanes that form the boundaries of the V_i, and since a hyperplane has dimension $m-1$, by the Grassmann relation, we must have $r \geq m$. For simplicity of notation, let us denote these faces by $F_{1,2}, F_{2,3}, \ldots, F_{r,r+1}$. Since $F_{i,j} = V_i \cap V_j$, we have

$$F_{i,j} = \{p \mid d(p, p_i) = d(p, p_j) \leq d(p, p_k),\ \text{for all } k \neq i, j\},$$

and since $p \in F_{1,2} \cap F_{2,3} \cap \cdots \cap F_{r,r+1}$, we have

$$d(p, p_1) = \cdots = d(p, p_{r+1}) < d(p, p_k) \text{ for all } k \notin \{1, \ldots, r+1\}.$$

This means that p is the center of a sphere passing through $p_1, \ldots, p_{r+1}$ and containing no other point in P. By the assumption that points in P are in general position, we must have $r \leq m$, and thus $r = m$. Thus, p belongs to $V_1 \cap \cdots \cap V_{m+1}$, but to no other V_j with $j \notin \{1, \ldots, m+1\}$. Furthermore, every edge of the Voronoi diagram containing p is the intersection of m of the regions $V_1, \ldots, V_{m+1}$, and so there are $m+1$ of them. $\square$ $\square$

For simplicity, let us again consider the case where $\mathcal{E}$ is a plane. It should be noted that certain Voronoi regions, although closed, may extend very far. Figure 8.3 shows such an example.
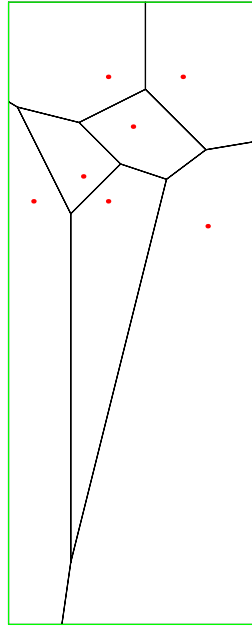

Figure 8.3: Another Voronoi diagram

It is also possible for certain unbounded regions to have parallel edges.

There are a number of methods for computing Voronoi diagrams. A fairly simple (although not very efficient) method is to compute each Voronoi region $V(p_i)$ by intersecting the half-planes $H(p_i, p_j)$. One way to do this is to construct successive convex polygons that converge to the boundary of the region. At every step we intersect the current convex polygon with the bisector line of p_i and p_j. There are at most two intersection points. We also need a starting polygon, and for this we can pick a square containing all the points. A naive implementation will run in $O(n^3)$. However, the intersection of half-planes can be done in $O(n \log n)$, using the fact that the vertices of a convex polygon can be sorted. Thus, the above method runs in $O(n^2 \log n)$. Actually, there are faster methods (see Preparata and Shamos [32] or O'Rourke [31]), and it is possible to design algorithms running in $O(n \log n)$. The most direct method to obtain fast algorithms is to use the "lifting method" discussed in Section 8.4, whereby the original set of points is lifted onto a paraboloid, and to use fast algorithms for finding a convex hull.

A very interesting (undirected) graph can be obtained from the Voronoi diagram as follows: The vertices of this graph are the points p_i (each corresponding to a unique region of $\mathcal{V}or(P)$), and there is an edge between p_i and p_j iff the regions V_i and V_j share an edge.
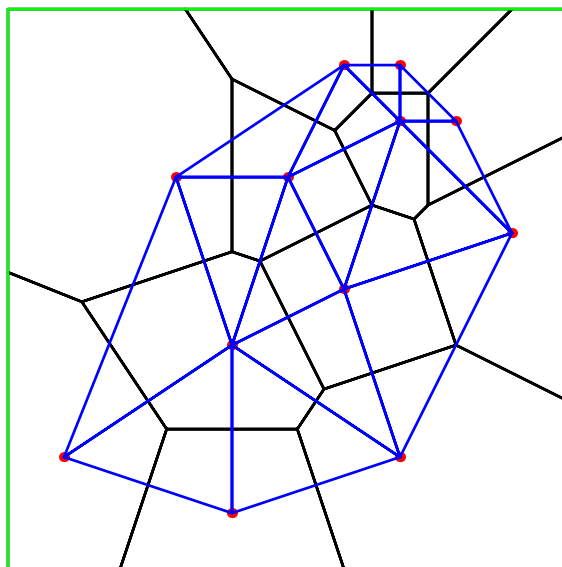
Figure 8.4: Delaunay triangulation associated with a Voronoi diagram

The resulting graph is called a *Delaunay triangulation* of the convex hull of P, after Delaunay, who invented this concept in 1933.5. Such triangulations have remarkable properties.

Figure 8.4 shows the Delaunay triangulation associated with the earlier Voronoi diagram of a set of twelve points.

One has to be careful to make sure that all the Voronoi vertices have been computed before computing a Delaunay triangulation, since otherwise, some edges could be missed. In Figure 8.5 illustrating such a situation, if the lowest Voronoi vertex had not been computed (not shown on the diagram!), the lowest edge of the Delaunay triangulation would be missing.

The concept of a triangulation can be generalized to dimension 3, or even to any dimension m.

8.2   Triangulations

The concept of a triangulation relies on the notion of pure simplicial complex defined in Chapter 6. The reader should review Definition 6.2 and Definition 6.3.

Definition 8.2. *Given a subset, $S \subseteq \mathbb{E}^m$ (where $m \geq 1$), a triangulation of S is a pure (finite) simplicial complex, K, of dimension m such that $S = |K|$, that is, S is equal to the geometric realization of K.*
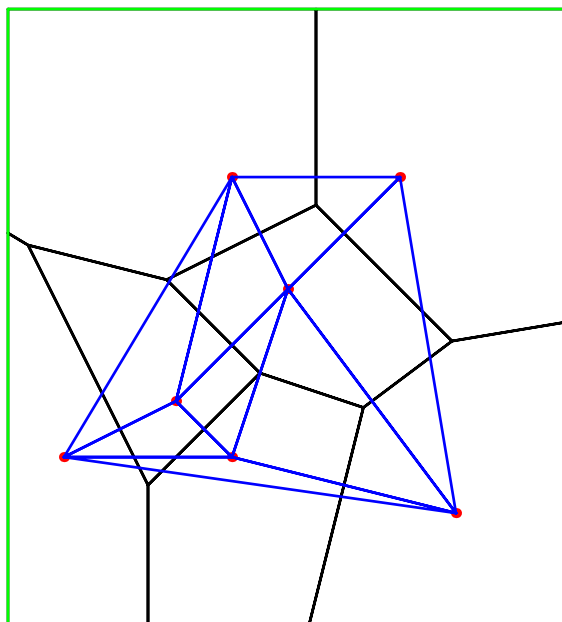
Figure 8.5: Another Delaunay triangulation associated with a Voronoi diagram

Given a finite set P of n points in the plane, and given a triangulation of the convex hull of P having P as its set of vertices, observe that the boundary of P is a convex polygon. Similarly, given a finite set P of points in 3-space, and given a triangulation of the convex hull of P having P as its set of vertices, observe that the boundary of P is a convex polyhedron. It is interesting to know how many triangulations exist for a set of n points (in the plane or in 3-space), and it is also interesting to know the number of edges and faces in terms of the number of vertices in P. These questions can be settled using the Euler–Poincaré characteristic. We say that a polygon in the plane is a *simple polygon* iff it is a connected closed polygon such that no two edges intersect (except at a common vertex).

Lemma 8.2.

(1) *For any triangulation of a region of the plane whose boundary is a simple polygon, letting v be the number of vertices, e the number of edges, and f the number of triangles, we have the "Euler formula"*

$$v - e + f = 1.$$

(2) *For any region, S, in $\mathbb{E}^3$ homeomorphic to a closed ball and for any triangulation of S, letting v be the number of vertices, e the number of edges, f the number of triangles, and t the number of tetrahedra, we have the "Euler formula"*

$$v - e + f - t = 1.$$

(3) Furthermore, for any triangulation of the combinatorial surface, $B(S)$, that is the boundary of S, letting v' be the number of vertices, e' the number of edges, and f' the number of triangles, we have the "Euler formula"

$$v' - e' + f' = 2.$$

Proof. All the statements are immediate consequences of Theorem 7.6. For example, part (1) is obtained by mapping the triangulation onto a sphere using inverse stereographic projection, say from the North pole. Then, we get a polytope on the sphere with an extra facet corresponding to the "outside" of the triangulation. We have to deduct this facet from the Euler characteristic of the polytope and this is why we get 1 instead of 2. $\square$ $\square$

It is now easy to see that in case (1), the number of edges and faces is a linear function of the number of vertices and boundary edges, and that in case (3), the number of edges and faces is a linear function of the number of vertices. Indeed, in the case of a planar triangulation, each face has 3 edges, and if there are e_b edges in the boundary and e_i edges not in the boundary, each nonboundary edge is shared by two faces, and thus $3f = e_b + 2e_i$. Since $v - e_b - e_i + f = 1$, we get

$$v - e_b - e_i + e_b/3 + 2e_i/3 = 1,$$
$$2e_b/3 + e_i/3 = v - 1,$$

and thus $e_i = 3v - 3 - 2e_b$. Since $f = e_b/3 + 2e_i/3$, we have $f = 2v - 2 - e_b$.

Similarly, since $v' - e' + f' = 2$ and $3f' = 2e'$, we easily get $e = 3v - 6$ and $f = 2v - 3.5$. Thus, given a set P of n points, the number of triangles (and edges) for any triangulation of the convex hull of P using the n points in P for its vertices is fixed.

Case (2) is trickier, but it can be shown that

$$v - 3 \le t \le (v - 1)(v - 2)/2.$$

Thus, there can be different numbers of tetrahedra for different triangulations of the convex hull of P.

Remark: The numbers of the form $v - e + f$ and $v - e + f - t$ are called *Euler–Poincaré characteristics*. They are topological invariants, in the sense that they are the same for all triangulations of a given polytope. This is a fundamental fact of algebraic topology.

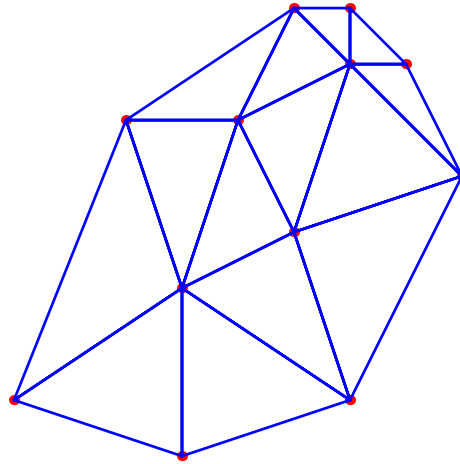We shall now investigate triangulations induced by Voronoi diagrams.

Figure 8.6: A Delaunay triangulation

8.3   Delaunay Triangulations

Given a set $P = \{p_1, \ldots, p_n\}$ of n points in the plane and the Voronoi diagram $\mathcal{V}or(P)$ for P, we explained in Section 8.1 how to define an (undirected) graph: The vertices of this graph are the points p_i (each corresponding to a unique region of $\mathcal{V}or(P)$), and there is an edge between p_i and p_j iff the regions V_i and V_j share an edge. The resulting graph turns out to be a triangulation of the convex hull of P having P as its set of vertices. Such a complex can be defined in general. For any set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathbb{E}^m$, we say that a triangulation of the convex hull of P is *associated with* P if its set of vertices is the set P.

Definition 8.3. *Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{E}^m$, and let $\mathcal{V}or(P)$ be the Voronoi diagram of P. We define a complex $\mathcal{D}el(P)$ as follows. The complex $\mathcal{D}el(P)$ contains the k-simplex $\{p_1, \ldots, p_{k+1}\}$ iff $V_1 \cap \cdots \cap V_{k+1} \neq \emptyset$, where $0 \leq k \leq m$. The complex $\mathcal{D}el(P)$ is called the Delaunay triangulation of the convex hull of P.*

Thus, $\{p_i, p_j\}$ is an edge iff $V_i \cap V_j \neq \emptyset$, $\{p_i, p_j, p_h\}$ is a triangle iff $V_i \cap V_j \cap V_h \neq \emptyset$, $\{p_i, p_j, p_h, p_k\}$ is a tetrahedron iff $V_i \cap V_j \cap V_h \cap V_k \neq \emptyset$, etc.

For simplicity, we often write $\mathcal{D}el$ instead of $\mathcal{D}el(P)$. A Delaunay triangulation for a set of twelve points is shown in Figure 8.6.

Actually, it is not obvious that $\mathcal{D}el(P)$ is a triangulation of the convex hull of P, but this can be shown, as well as the properties listed in the following lemma.

Lemma 8.3. *Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{E}^m$, and assume that they are in general position. Then the Delaunay triangulation of the convex hull of P is indeed a triangulation associated with P, and it satisfies the following properties:*

(1) *The boundary of $\mathcal{D}el(P)$ is the convex hull of P.*

(2) *A triangulation T associated with P is the Delaunay triangulation $\mathcal{D}el(P)$ iff every $(m-1)$-sphere $S(\sigma)$ circumscribed about an m-simplex σ of T contains no other point from P (i.e., the open ball associated with $S(\sigma)$ contains no point from P).*

The proof can be found in Risler [33] and O'Rourke [31]. In the case of a planar set P, it can also be shown that the Delaunay triangulation has the property that it maximizes the minimum angle of the triangles involved in any triangulation of P. However, this does not characterize the Delaunay triangulation. Given a connected graph in the plane, it can also be shown that any minimal spanning tree is contained in the Delaunay triangulation of the convex hull of the set of vertices of the graph (O'Rourke [31]).

We will now explore briefly the connection between Delaunay triangulations and convex hulls.

8.4   Delaunay Triangulations and Convex Hulls

In this section we show that there is an intimate relationship between convex hulls and Delaunay triangulations. We will see that given a set P of points in the Euclidean space $\mathbb{E}^m$ of dimension m, we can "lift" these points onto a paraboloid living in the space $\mathbb{E}^{m+1}$ of dimension $m+1$, and that the Delaunay triangulation of P is the projection of the downward-facing faces of the convex hull of the set of lifted points. This remarkable connection was first discovered by Edelsbrunner and Seidel [16]. For simplicity, we consider the case of a set P of points in the plane $\mathbb{E}^2$, and we assume that they are in general position.

Consider the paraboloid of revolution of equation $z = x^2 + y^2$. A point $p = (x, y)$ in the plane is lifted to the point $l(p) = (X, Y, Z)$ in $\mathbb{E}^3$, where $X = x$, $Y = y$, and $Z = x^2 + y^2$.

The first crucial observation is that a circle in the plane is lifted into a plane curve (an ellipse). Indeed, if such a circle C is defined by the equation

$$x^2 + y^2 + ax + by + c = 0,$$

since $X = x$, $Y = y$, and $Z = x^2 + y^2$, by eliminating $x^2 + y^2$ we get

$$Z = -ax - by - c,$$

and thus X, Y, Z satisfy the linear equation
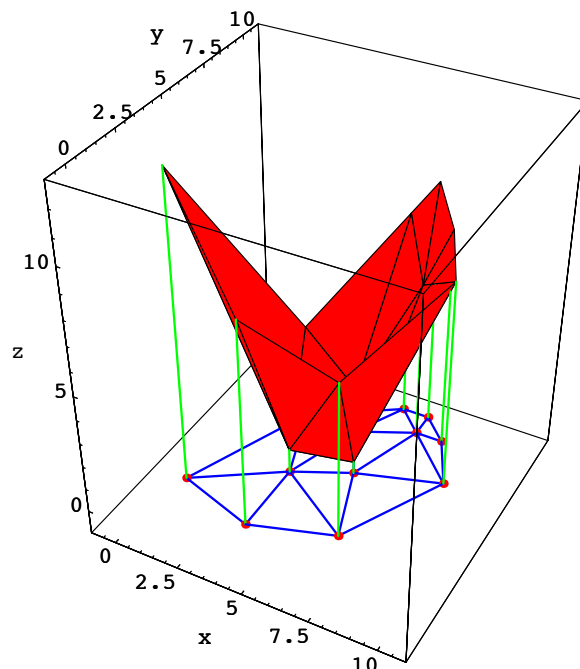
$$aX + bY + Z + c = 0,$$

Figure 8.7: A Delaunay triangulation and its lifting to a paraboloid

which is the equation of a plane. Thus, the intersection of the cylinder of revolution consisting of the lines parallel to the z-axis and passing through a point of the circle C with the paraboloid $z = x^2 + y^2$ is a planar curve (an ellipse).

We can compute the convex hull of the set of lifted points. Let us focus on the downward-facing faces of this convex hull. Let $(l(p_1), l(p_2), l(p_3))$ be such a face. The points p_1, p_2, p_3 belong to the set P. We claim that no other point from P is inside the circle C. Indeed, a point p inside the circle C would lift to a point $l(p)$ on the paraboloid. Since no four points are cocyclic, one of the four points p_1, p_2, p_3, p is further from O than the others; say this point is p_3. Then, the face $(l(p_1), l(p_2), l(p))$ would be below the face $(l(p_1), l(p_2), l(p_3))$, contradicting the fact that $(l(p_1), l(p_2), l(p_3))$ is one of the downward-facing faces of the convex hull of P. But then, by property (2) of Lemma 8.3, the triangle (p_1, p_2, p_3) would belong to the Delaunay triangulation of P.

Therefore, we have shown that *the projection of the part of the convex hull of the lifted set $l(P)$ consisting of the downward-facing faces is the Delaunay triangulation of P.* Figure 8.7 shows the lifting of the Delaunay triangulation shown earlier.

Another example of the lifting of a Delaunay triangulation is shown in Figure 8.8.

The fact that a Delaunay triangulation can be obtained by projecting a lower convex hull can be used to find efficient algorithms for computing a Delaunay triangulation. It also holds for higher dimensions.

The Voronoi diagram itself can also be obtained from the lifted set $l(P)$. However, this
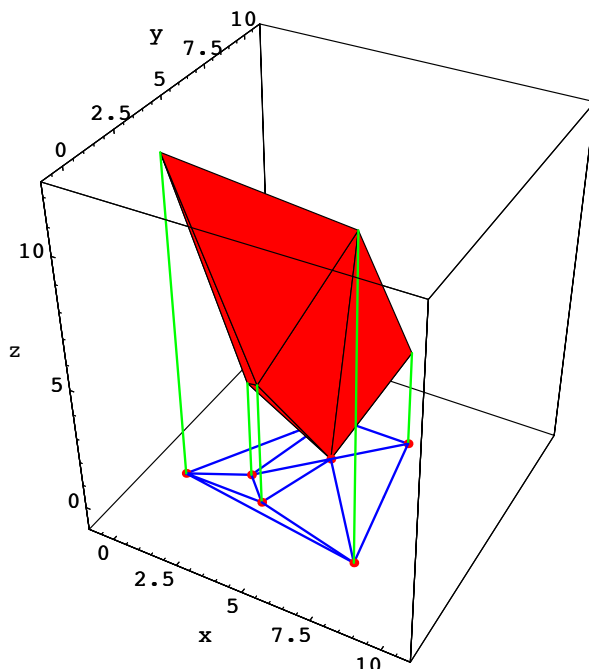
Figure 8.8: Another Delaunay triangulation and its lifting to a paraboloid

time, we need to consider tangent planes to the paraboloid at the lifted points. It is fairly obvious that the tangent plane at the lifted point $(a, b, a^2 + b^2)$ is

$$z = 2ax + 2by - (a^2 + b^2).$$

Given two distinct lifted points $(a_1, b_1, a_1^2 + b_1^2)$ and $(a_2, b_2, a_2^2 + b_2^2)$, the intersection of the tangent planes at these points is a line belonging to the plane of equation

$$(b_1 - a_1)x + (b_2 - a_2)y = (b_1^2 + b_2^2)/2 - (a_1^2 + a_2^2)/2.$$

Now, if we project this plane onto the xy-plane, we see that the above is precisely the equation of the bisector line of the two points (a_1, b_1) and (a_2, b_2). Therefore, *if we look at the paraboloid from $z = +\infty$ (with the paraboloid transparent), the projection of the tangent planes at the lifted points is the Voronoi diagram*!

It should be noted that the "duality" between the Delaunay triangulation, which is the projection of the convex hull of the lifted set $l(P)$ viewed from $z = -\infty$, and the Voronoi diagram, which is the projection of the tangent planes at the lifted set $l(P)$ viewed from $z = +\infty$, is reminiscent of the polar duality with respect to a quadric. This duality will be thoroughly investigated in Section 8.6.

The reader interested in algorithms for finding Voronoi diagrams and Delaunay triangulations is referred to O'Rourke [31], Preparata and Shamos [32], Boissonnat and Yvinec [8], de Berg, Van Kreveld, Overmars, and Schwarzkopf [5], and Risler [33].

8.5 Stereographic Projection and the Space of Generalized Spheres

Brown appears to be the first person who observed that Voronoi diagrams and convex hulls are related *via* inversion with respect to a sphere [11].

In fact, more generally, it turns out that Voronoi diagrams, Delaunay Triangulations and their properties can also be nicely explained using stereographic projection and its inverse, although a rigorous justification of why this "works" is not as simple as it might appear.

The advantage of stereographic projection over the lifting onto a paraboloid is that the $(d\text{-})$sphere is compact. Since the stereographic projection and its inverse map $(d-1)$-spheres to $(d-1)$-spheres (or hyperplanes), all the crucial properties of Delaunay triangulations are preserved. The purpose of this section is to establish the properties of stereographic projection (and its inverse) that will be needed in Section 8.6.

Recall that the *d-sphere*, $S^d \subseteq \mathbb{E}^{d+1}$, is given by

$$S^d = \{(x_1, \ldots, x_{d+1}) \in \mathbb{E}^{d+1} \mid x_1^2 + \cdots + x_d^2 + x_{d+1}^2 = 1\}.$$

It will be convenient to write a point, $(x_1, \ldots, x_{d+1}) \in \mathbb{E}^{d+1}$, as $z = (x, x_{d+1})$, with $x = (x_1, \ldots, x_d)$. We denote $N = (0, \ldots, 0, 1)$ (with d zeros) as $(\mathbf{0}, 1)$ and call it the *north pole* and $S = (0, \ldots, 0, -1)$ (with d zeros) as $(\mathbf{0}, -1)$ and call it the *south pole*. We also write $\|z\| = (x_1^2 + \cdots + x_{d+1}^2)^{\frac{1}{2}} = (\|x\|^2 + x_{d+1}^2)^{\frac{1}{2}}$ (with $\|x\| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$). With these notations,

$$S^d = \{(x, x_{d+1}) \in \mathbb{E}^{d+1} \mid \|x\|^2 + x_{d+1}^2 = 1\}.$$

The *stereographic projection from the north pole, $\sigma_N \colon (S^d - \{N\}) \to \mathbb{E}^d$,* is the restriction to S^d of the central projection from N onto the hyperplane, $H_{d+1}(0) \cong \mathbb{E}^d$, of equation $x_{d+1} = 0$; that is, $M \mapsto \sigma_N(M)$ where $\sigma_N(M)$ is the intersection of the line, $\langle N, M \rangle$, through N and M with $H_{d+1}(0)$. Since the line through N and $M = (x, x_{d+1})$ is given parametrically by

$$\langle N, M \rangle = \{(1 - \lambda)(\mathbf{0}, 1) + \lambda(x, x_{d+1}) \mid \lambda \in \mathbb{R}\},$$

the intersection, $\sigma_N(M)$, of this line with the hyperplane $x_{d+1} = 0$ corresponds to the value of λ such that

$$(1 - \lambda) + \lambda x_{d+1} = 0,$$

that is,

$$\lambda = \frac{1}{1 - x_{d+1}}.$$

Therefore, the coordinates of $\sigma_N(M)$, with $M = (x, x_{d+1})$, are given by

$$\sigma_N(x, x_{d+1}) = \left(\frac{x}{1 - x_{d+1}}, 0 \right).$$

Let us find the inverse, $\tau_N = \sigma_N^{-1}(P)$, of any $P \in H_{d+1}(0) \cong \mathbb{E}^d$. This time, $\tau_N(P)$ is the intersection of the line, $\langle N, P \rangle$, through $P \in H_{d+1}(0)$ and N with the sphere, S^d. Since the line through N and $P = (x, 0)$ is given parametrically by

$$\langle N, P \rangle = \{(1 - \lambda)(\mathbf{0}, 1) + \lambda(x, 0) \mid \lambda \in \mathbb{R}\},$$

the intersection, $\tau_N(P)$, of this line with the sphere S^d corresponds to the nonzero value of λ such that

$$\lambda^2 \|x\|^2 + (1 - \lambda)^2 = 1,$$

that is

$$\lambda(\lambda(\|x\|^2 + 1) - 2) = 0.$$

Thus, we get

$$\lambda = \frac{2}{\|x\|^2 + 1},$$

from which we get

$$
\begin{aligned}
\tau_N(x) &= \left(\frac{2x}{\|x\|^2 + 1}, 1 - \frac{2}{\|x\|^2 + 1} \right) \\
&= \left(\frac{2x}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1} \right).
\end{aligned}
$$

We leave it as an exercise to the reader to verify that $\tau_N \circ \sigma_N = \text{id}$ and $\sigma_N \circ \tau_N = \text{id}$. We can also define the *stereographic projection from the south pole*, $\sigma_S \colon (S^d - \{S\}) \to \mathbb{E}^d$, and its inverse, τ_S. Again, the computations are left as a simple exercise to the reader. The above computations are summarized in the following definition:

Definition 8.4. *The stereographic projection from the north pole, $\sigma_N \colon (S^d - \{N\}) \to \mathbb{E}^d$, is the map given by*

$$\sigma_N(x, x_{d+1}) = \left(\frac{x}{1 - x_{d+1}}, 0 \right) \qquad (x_{d+1} \neq 1).$$

The inverse of σ_N, denoted $\tau_N \colon \mathbb{E}^d \to (S^d - \{N\})$ and called inverse stereographic projection from the north pole is given by

$$\tau_N(x) = \left(\frac{2x}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1} \right).$$

Remark: An *inversion of center C and power $\rho > 0$* is a geometric transformation, $f \colon (\mathbb{E}^{d+1} - \{C\}) \to \mathbb{E}^{d+1}$, defined so that for any $M \neq C$, the points C, M and $f(M)$ are collinear and

$$\|\mathbf{CM}\| \|\mathbf{Cf(M)}\| = \rho.$$

Equivalently, $f(M)$ is given by

$$f(M) = C + \frac{\rho}{\|\mathbf{CM}\|^2} \, \mathbf{CM}.$$

Clearly, $f \circ f = \mathrm{id}$ on $\mathbb{E}^{d+1} - \{C\}$, so f is invertible and the reader will check that if we pick the center of inversion to be the north pole and if we set $\rho = 2$, then the coordinates of $f(M)$ are given by

$$
\begin{aligned}
y_i &= \frac{2x_i}{x_1^2 + \cdots + x_d^2 + x_{d+1}^2 - 2x_{d+1} + 1}, & 1 \le i \le d \\
y_{d+1} &= \frac{x_1^2 + \cdots + x_d^2 + x_{d+1}^2 - 1}{x_1^2 + \cdots + x_d^2 + x_{d+1}^2 - 2x_{d+1} + 1},
\end{aligned}
$$

where $(x_1, \ldots, x_{d+1})$ are the coordinates of M. In particular, if we restrict our inversion to the unit sphere, S^d, as $x_1^2 + \cdots + x_d^2 + x_{d+1}^2 = 1$, we get

$$
\begin{aligned}
y_i &= \frac{x_i}{1 - x_{d+1}}, & 1 \le i \le d \\
y_{d+1} &= 0,
\end{aligned}
$$

which means that our inversion restricted to S^d is simply the stereographic projection, σ_N (and the inverse of our inversion restricted to the hyperplane, $x_{d+1} = 0$, is the inverse stereographic projection, τ_N).

We will now show that the image of any $(d-1)$-sphere, S, on S^d not passing through the north pole, that is, the intersection, $S = S^d \cap H$, of S^d with any hyperplane, H, not passing through N is a $(d-1)$-sphere. Here, we are assuming that S has positive radius, that is, H is *not* tangent to S^d.

Assume that H is given by

$$a_1 x_1 + \cdots + a_d x_d + a_{d+1} x_{d+1} + b = 0.$$

Since $N \notin H$, we must have $a_{d+1} + b \neq 0$. For any $(x, x_{d+1}) \in S^d$, write $\sigma_N(x, x_{d+1}) = (X, 0)$. Since

$$X = \frac{x}{1 - x_{d+1}},$$

we get $x = X(1 - x_{d+1})$ and using the fact that (x, x_{d+1}) also belongs to H we will express x_{d+1} in terms of X and then find an equation for X which will show that X belongs to a $(d-1)$-sphere. Indeed, $(x, x_{d+1}) \in H$ implies that

$$\sum_{i=1}^{d} a_i X_i (1 - x_{d+1}) + a_{d+1} x_{d+1} + b = 0,$$

that is,

$$\sum_{i=1}^{d} a_i X_i + (a_{d+1} - \sum_{j=1}^{d} a_j X_j) x_{d+1} + b = 0.$$

If $\sum_{j=1}^{d} a_j X_j = a_{d+1}$, then $a_{d+1} + b = 0$, which is impossible. Therefore, we get

$$x_{d+1} = \frac{-b - \sum_{i=1}^{d} a_i X_i}{a_{d+1} - \sum_{i=1}^{d} a_i X_i}$$

and so,

$$1 - x_{d+1} = \frac{a_{d+1} + b}{a_{d+1} - \sum_{i=1}^{d} a_i X_i}.$$

Plugging $x = X(1 - x_{d+1})$ in the equation, $\|x\|^2 + x_{d+1}^d = 1$, of S^d, we get

$$(1 - x_{d+1})^2 \|X\|^2 + x_{d+1}^2 = 1,$$

and replacing x_{d+1} and $1 - x_{d+1}$ by their expression in terms of X, we get

$$(a_{d+1} + b)^2 \|X\|^2 + (-b - \sum_{i=1}^{d} a_i X_i)^2 = (a_{d+1} - \sum_{i=1}^{d} a_i X_i)^2$$

that is,

$$(a_{d+1} + b)^2 \|X\|^2 = (a_{d+1} - \sum_{i=1}^{d} a_i X_i)^2 - (b + \sum_{i=1}^{d} a_i X_i)^2$$
$$= (a_{d+1} + b)(a_{d+1} - b - 2\sum_{i=1}^{d} a_i X_i)$$

which yields

$$(a_{d+1} + b)^2 \|X\|^2 + 2(a_{d+1} + b)(\sum_{i=1}^{d} a_i X_i) = (a_{d+1} + b)(a_{d+1} - b),$$

that is,

$$\|X\|^2 + 2\sum_{i=1}^{d} \frac{a_i}{a_{d+1} + b} X_i - \frac{a_{d+1} - b}{a_{d+1} + b} = 0,$$

which is indeed the equation of a $(d-1)$-sphere in $\mathbb{E}^d$. Therefore, when $N \notin H$, the image of $S = S^d \cap H$ by σ_N is a $(d-1)$-sphere in $H_{d+1}(0) = \mathbb{E}^d$.

If the hyperplane, H, contains the north pole, then $a_{d+1} + b = 0$, in which case, for every $(x, x_{d+1}) \in S^d \cap H$, we have

$$\sum_{i=1}^{d} a_i x_i + a_{d+1} x_{d+1} - a_{d+1} = 0,$$

that is,

$$\sum_{i=1}^{d} a_i x_i - a_{d+1}(1 - x_{d+1}) = 0,$$

and except for the north pole, we have

$$\sum_{i=1}^{d} a_i \frac{x_i}{1 - x_{d+1}} - a_{d+1} = 0,$$

which shows that

$$\sum_{i=1}^{d} a_i X_i - a_{d+1} = 0,$$

the intersection of the hyperplanes H and $H_{d+1}(0)$ Therefore, the image of $S^d \cap H$ by σ_N is the hyperplane in $\mathbb{E}^d$ which is the intersection of H with $H_{d+1}(0)$.

We will also prove that τ_N maps $(d - 1)$-spheres in $H_{d+1}(0)$ to $(d - 1)$-spheres on S^d not passing through the north pole. Assume that $X \in \mathbb{E}^d$ belongs to the $(d - 1)$-sphere of equation

$$\sum_{i=1}^{d} X_i^2 + \sum_{j=1}^{d} a_j X_j + b = 0.$$

For any $(X, 0) \in H_{d+1}(0)$, we know that $(x, x_{d+1}) = \tau_N(X)$ is given by

$$(x, x_{d+1}) = \left(\frac{2X}{\|X\|^2 + 1}, \frac{\|X\|^2 - 1}{\|X\|^2 + 1} \right).$$

Using the equation of the $(d - 1)$-sphere, we get

$$x = \frac{2X}{-b + 1 - \sum_{j=1}^{d} a_j X_j}$$

and

$$x_{d+1} = \frac{-b - 1 - \sum_{j=1}^{d} a_j X_j}{-b + 1 - \sum_{j=1}^{d} a_j X_j}.$$

Then, we get

$$\sum_{i=1}^{d} a_i x_i = \frac{2 \sum_{j=1}^{d} a_j X_j}{-b + 1 - \sum_{j=1}^{d} a_j X_j},$$

which yields

$$(-b+1)(\sum_{i=1}^{d} a_i x_i) - (\sum_{i=1}^{d} a_i x_i)(\sum_{j=1}^{d} a_j X_j) = 2\sum_{j=1}^{d} a_j X_j.$$

From the above, we get

$$\sum_{i=1}^{d} a_i X_i = \frac{(-b+1)(\sum_{i=1}^{d} a_i x_i)}{\sum_{i=1}^{d} a_i x_i + 2}.$$

Plugging this expression in the formula for x_{d+1} above, we get

$$x_{d+1} = \frac{-b-1-\sum_{i=1}^{d} a_i x_i}{-b+1},$$

which yields

$$\sum_{i=1}^{d} a_i x_i + (-b+1)x_{d+1} + (b+1) = 0,$$

the equation of a hyperplane, H, not passing through the north pole. Therefore, the image of a $(d-1)$-sphere in $H_{d+1}(0)$ is indeed the intersection, $H \cap S^d$, of S^d with a hyperplane not passing through N, that is, a $(d-1)$-sphere on S^d.

Given any hyperplane, H', in $H_{d+1}(0) = \mathbb{E}^d$, say of equation

$$\sum_{i=1}^{d} a_i X_i + b = 0,$$

the image of H' under τ_N is a $(d-1)$-sphere on S^d, the intersection of S^d with the hyperplane, H, passing through N and determined as follows: For any $(X,0) \in H_{d+1}(0)$, if $\tau_N(X) = (x, x_{d+1})$, then

$$X = \frac{x}{1 - x_{d+1}}$$

and so, (x, x_{d+1}) satisfies the equation

$$\sum_{i=1}^{d} a_i x_i + b(1 - x_{d+1}) = 0,$$

that is,

$$\sum_{i=1}^{d} a_i x_i - b x_{d+1} + b = 0,$$

which is indeed the equation of a hyperplane, H, passing through N. We summarize all this in the following proposition:

Proposition 8.4. *The stereographic projection, $\sigma_N \colon (S^d - \{N\}) \to \mathbb{E}^d$, induces a bijection, σ_N, between the set of $(d-1)$-spheres on S^d and the union of the set of $(d-1)$-spheres in $\mathbb{E}^d$ with the set of hyperplanes in $\mathbb{E}^d$; every $(d-1)$-sphere on S^d not passing through the north pole is mapped to a $(d-1)$-sphere in $\mathbb{E}^d$ and every $(d-1)$-sphere on S^d passing through the north pole is mapped to a hyperplane in $\mathbb{E}^d$. In fact, σ_N maps the $(d-1)$-sphere on S^d determined by the hyperplane*

$$a_1 x_1 + \cdots + a_d x_d + a_{d+1} x_{d+1} + b = 0$$

not passing through the north pole ($a_{d+1} + b \neq 0$) to the $(d-1)$-sphere

$$\sum_{i=1}^{d} X_i^2 + 2 \sum_{i=1}^{d} \frac{a_i}{a_{d+1} + b} X_i - \frac{a_{d+1} - b}{a_{d+1} + b} = 0$$

and the $(d-1)$-sphere on S^d determined by the hyperplane

$$\sum_{i=1}^{d} a_i x_i + a_{d+1} x_{d+1} - a_{d+1} = 0$$

through the north pole to the hyperplane

$$\sum_{i=1}^{d} a_i X_i - a_{d+1} = 0;$$

the map $\tau_N = \sigma_N^{-1}$ maps the $(d-1)$-sphere

$$\sum_{i=1}^{d} X_i^2 + \sum_{j=1}^{d} a_j X_j + b = 0$$

to the $(d-1)$-sphere on S^d determined by the hyperplane

$$\sum_{i=1}^{d} a_i x_i + (-b + 1) x_{d+1} + (b + 1) = 0$$

not passing through the north pole and the hyperplane

$$\sum_{i=1}^{d} a_i X_i + b = 0$$

to the $(d-1)$-sphere on S^d determined by the hyperplane

$$\sum_{i=1}^{d} a_i x_i - b x_{d+1} + b = 0$$

through the north pole.

Proposition 8.4 raises a natural question: What do the hyperplanes, H, in $\mathbb{E}^{d+1}$ that do not intersect S^d correspond to, if they correspond to anything at all?

The first thing to observe is that the geometric definition of the stereographic projection and its inverse makes it clear that the hyperplanes corresponding to $(d-1)$-spheres in $\mathbb{E}^d$ (by τ_N) do intersect S^d. Now, when we write the equation of a $(d-1)$-sphere, S, say

$$\sum_{i=1}^{d} X_i^2 + \sum_{i=1}^{d} a_i X_i + b = 0$$

we are implicitly assuming a condition on the a_i's and b that ensures that S is not the empty sphere, that is, that its radius, R, is positive (or zero). By "completing the square", the above equation can be rewritten as

$$\sum_{i=1}^{d} \left(X_i + \frac{a_i}{2} \right)^2 = \frac{1}{4} \sum_{i=1}^{d} a_i^2 - b,$$

and so the radius, R, of our sphere is given by

$$R^2 = \frac{1}{4} \sum_{i=1}^{d} a_i^2 - b$$

whereas its center is the point, $c = -\frac{1}{2}(a_1, \ldots, a_d)$. Thus, our sphere is a "real" sphere of positive radius iff

$$\sum_{i=1}^{d} a_i^2 > 4b$$

or a single point, $c = -\frac{1}{2}(a_1, \ldots, a_d)$, iff $\sum_{i=1}^{d} a_i^2 = 4b$.

What happens when

$$\sum_{i=1}^{d} a_i^2 < 4b?$$

In this case, if we allow "complex points", that is, if we consider solutions of our equation

$$\sum_{i=1}^{d} X_i^2 + \sum_{i=1}^{d} a_i X_i + b = 0$$

over $\mathbb{C}^d$, then we get a "complex" sphere of (pure) imaginary radius, $\frac{i}{2}\sqrt{4b - \sum_{i=1}^{d} a_i^2}$. The funny thing is that our computations carry over unchanged and the image of the complex sphere, S, is still the intersection of the complex sphere S^d with the hyperplane, H, given

$$\sum_{i=1}^{d} a_i x_i + (-b + 1)x_{d+1} + (b + 1) = 0.$$

However, this time, even though H does not have any "real" intersection points with S^d, we can show that it does intersect the "complex sphere",

$$S^d = \{(z_1, \ldots, z_{d+1}) \in \mathbb{C}^{d+1} \mid z_1^2 + \cdots + z_{d+1}^2 = 1\}$$

in a nonempty set of points in $\mathbb{C}^{d+1}$.

It follows from all this that σ_N and τ_N establish a bijection between the set of all hyperplanes in $\mathbb{E}^{d+1}$ minus the hyperplane, H_{d+1} (of equation $x_{d+1} = 1$), tangent to S^d at the north pole, with the union of four sets:

(1) The set of all (real) $(d-1)$-spheres of positive radius;

(2) The set of all (complex) $(d-1)$-spheres of imaginary radius;

(3) The set of all hyperplanes in $\mathbb{E}^d$;

(4) The set of all points of $\mathbb{E}^d$ (viewed as spheres of radius 0).

Moreover, set (1) corresponds to the hyperplanes that intersect the interior of S^d and do not pass through the north pole; set (2) corresponds to the hyperplanes that do not intersect S^d; set (3) corresponds to the hyperplanes that pass through the north pole minus the tangent hyperplane at the north pole; and set (4) corresponds to the hyperplanes that are tangent to S^d, minus the tangent hyperplane at the north pole.

It is convenient to add the "point at infinity", ∞, to $\mathbb{E}^d$, because then the above bijection can be extended to map the tangent hyperplane at the north pole to ∞. The union of these four sets (with ∞ added) is called the *set of generalized spheres*, sometimes, denoted $\mathcal{S}(\mathbb{E}^d)$. This is a fairly complicated space. For one thing, topologically, $\mathcal{S}(\mathbb{E}^d)$ is homeomorphic to the projective space $\mathbb{P}^{d+1}$ with one point removed (the point corresponding to the "hyperplane at infinity"), and this is not a simple space. We can get a slightly more concrete "'picture" of $\mathcal{S}(\mathbb{E}^d)$ by looking at the polars of the hyperplanes w.r.t. S^d. Then, the "real" spheres correspond to the points strictly outside S^d which do not belong to the tangent hyperplane at the norh pole; the complex spheres correspond to the points in the interior of S^d; the points of $\mathbb{E}^d \cup \{\infty\}$ correspond to the points on S^d; the hyperplanes in $\mathbb{E}^d$ correspond to the points in the tangent hyperplane at the norh pole expect for the north pole. Unfortunately, the poles of hyperplanes through the origin are undefined. This can be fixed by embedding $\mathbb{E}^{d+1}$ in its projective completion, $\mathbb{P}^{d+1}$, but we will not go into this.

There are other ways of dealing rigorously with the set of generalized spheres. One method described by Boissonnat [8] is to use the embedding where the sphere, S, of equation

$$\sum_{i=1}^{d} X_i^2 - 2 \sum_{i=1}^{d} a_i X_i + b = 0$$

is mapped to the point

$$\varphi(S) = (a_1, \ldots, a_d, b) \in \mathbb{E}^{d+1}.$$

Now, by a previous computation we know that

$$b = \sum_{i=1}^{d} a_i^2 - R^2,$$

where $c = (a_1, \ldots, a_d)$ is the center of S and R is its radius. The quantity $\sum_{i=1}^{d} a_i^2 - R^2$ is known as the *power* of the origin w.r.t. S. In general, the *power* of a point, $X \in \mathbb{E}^d$, is defined as $\rho(X) = \|\mathbf{cX}\|^2 - R^2$, which, after a moment of thought, is just

$$\rho(X) = \sum_{i=1}^{d} X_i^2 - 2\sum_{i=1}^{d} a_i X_i + b.$$

Now, since points correspond to spheres of radius 0, we see that the image of the point, $X = (X_1, \ldots, X_d)$, is

$$l(X) = (X_1, \ldots, X_d, \sum_{i=1}^{d} X_i^2).$$

Thus, in this model, points of $\mathbb{E}^d$ are lifted to the hyperboloid, $\mathcal{P} \subseteq \mathbb{E}^{d+1}$, of equation

$$x_{d+1} = \sum_{i=1}^{d} x_i^2.$$

Actually, this method does not deal with hyperplanes but it is possible to do so. The trick is to consider equations of a slightly more general form that capture both spheres and hyperplanes, namely, equations of the form

$$c\sum_{i=1}^{d} X_i^2 + \sum_{i=1}^{d} a_i X_i + b = 0.$$

Indeed, when $c = 0$, we do get a hyperplane! Now, to carry out this method we really need to consider equations up to a nonzero scalars, that is, we consider the projective space, $\mathbb{P}(\widehat{S}(\mathbb{E}^d))$, associated with the vector space, $\widehat{S}(\mathbb{E}^d)$, consisting of the above equations. Then, it turns out that the quantity

$$\varrho(a, b, c) = \frac{1}{4}(\sum_{i=1}^{d} a_i^2 - 4bc)$$

(with $a = (a_1, \ldots, a_d)$) defines a quadratic form on $\widehat{S}(\mathbb{E}^d)$ whose corresponding bilinear form,

$$\rho((a, b, c), (a', b', c')) = \frac{1}{4}(\sum_{i=1}^{d} a_i a_i' - 2bc' - 2b'c),$$

has a natural interpretation (with $a = (a_1, \ldots, a_d)$ and $a' = (a'_1, \ldots, a'_d)$). Indeed, orthogonality with respect to ρ (that is, when $\rho((a, b, c), (a', b', c')) = 0$) says that the corresponding spheres defined by (a, b, c) and (a', b', c') are orthogonal, that the corresponding hyperplanes defined by $(a, b, 0)$ and $(a', b', 0)$ are orthogonal, etc. The reader who wants to read more about this approach should consult Berger (Volume II) [6].

There is a simple relationship between the lifting onto a hyperboloid and the lifting onto S^d using the inverse stereographic projection map because the sphere and the paraboloid are projectively equivalent, as we showed for S^2 in Section 5.1.

Recall that the hyperboloid, $\mathcal{P}$, in $\mathbb{E}^{d+1}$ is given by the equation

$$x_{d+1} = \sum_{i=1}^{d} x_i^2$$

and of course, the sphere S^d is given by

$$\sum_{i=1}^{d+1} x_i^2 = 1.$$

Consider the "projective transformation", Θ, of $\mathbb{E}^{d+1}$ given by

$$\begin{aligned} z_i &= \frac{x_i}{1 - x_{d+1}}, & 1 \leq i \leq d \\ z_{d+1} &= \frac{x_{d+1} + 1}{1 - x_{d+1}}. \end{aligned}$$

Observe that Θ is undefined on the hyperplane, H_{d+1}, tangent to S^d at the north pole and that its first d component are identical to those of the stereographic projection! Then, we immediately find that

$$\begin{aligned} x_i &= \frac{2z_i}{1 + z_{d+1}}, & 1 \leq i \leq d \\ x_{d+1} &= \frac{z_{d+1} - 1}{1 + z_{d+1}}. \end{aligned}$$

Consequently, Θ is a bijection between $\mathbb{E}^{d+1} - H_{d+1}$ and $\mathbb{E}^{d+1} - H_{d+1}(-1)$, where $H_{d+1}(-1)$ is the hyperplane of equation $x_{d+1} = -1$.

The fact that Θ is undefined on the hyperplane, H_{d+1}, is not a problem as far as mapping the sphere to the paraboloid because the north pole is the only point that does not have an image. However, later on when we consider the Voronoi polyhedron, $\mathcal{V}(P)$, of a lifted set of points, P, we will have more serious problems because in general, such a polyhedron intersects both hyperplanes H_{d+1} and $H_{d+1}(-1)$. This means that Θ will not be well-defined on the whole of $\mathcal{V}(P)$ nor will it be surjective on its image. To remedy this difficulty, we

will work with projective completions. Basically, this amounts to chasing denominators and homogenizing equations but we also have to be careful in dealing with convexity and this is where the projective polyhedra (studied in Section 5.2) will come handy.

So, let us consider the projective sphere, $S^d \subseteq \mathbb{P}^{d+1}$, given by the equation

$$\sum_{i=1}^{d+1} x_i^2 = x_{d+2}^2$$

and the paraboloid, $\mathcal{P} \subseteq \mathbb{P}^{d+1}$, given by the equation

$$x_{d+1} x_{d+2} = \sum_{i=1}^{d} x_i^2.$$

Let $\theta \colon \mathbb{P}^{d+1} \to \mathbb{P}^{d+1}$ be the projectivity induced by the linear map, $\widehat{\theta} \colon \mathbb{R}^{d+2} \to \mathbb{R}^{d+2}$, given by

$$
\begin{aligned}
z_i &= x_i, & 1 \leq i \leq d \\
z_{d+1} &= x_{d+1} + x_{d+2} \\
z_{d+2} &= x_{d+2} - x_{d+1},
\end{aligned}
$$

whose inverse is given by

$$
\begin{aligned}
x_i &= z_i, & 1 \leq i \leq d \\
x_{d+1} &= \frac{z_{d+1} - z_{d+2}}{2} \\
x_{d+2} &= \frac{z_{d+1} + z_{d+2}}{2}.
\end{aligned}
$$

If we plug these formulae in the equation of S^d, we get

$$4\left(\sum_{i=1}^{d} z_i^2\right) + (z_{d+1} - z_{d+2})^2 = (z_{d+1} + z_{d+2})^2,$$

which simplifies to

$$z_{d+1} z_{d+2} = \sum_{i=1}^{d} z_i^2.$$

Therefore, $\theta(S^d) = \mathcal{P}$, that is, θ maps the sphere to the hyperboloid. Observe that the north pole, $N = (0 \colon \cdots \colon 0 \colon 1 \colon 1)$, is mapped to the point at infinity, $(0 \colon \cdots \colon 0 \colon 1 \colon 0)$.

The map Θ is the restriction of θ to the affine patch, U_{d+1}, and as such, it can be fruitfully described as the composition of $\widehat{\theta}$ with a suitable projection onto $\mathbb{E}^{d+1}$. For this, as we have

done before, we identify $\mathbb{E}^{d+1}$ with the hyperplane, $H_{d+2} \subseteq \mathbb{E}^{d+2}$, of equation $x_{d+2} = 1$ (using the injection, $i_{d+2} \colon \mathbb{E}^{d+1} \to \mathbb{E}^{d+2}$, where $i_j \colon \mathbb{E}^{d+1} \to \mathbb{E}^{d+2}$ is the injection given by

$$(x_1, \ldots, x_{d+1}) \mapsto (x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{d+1})$$

for any $(x_1, \ldots, x_{d+1}) \in \mathbb{E}^{d+1})$. For each i, with $1 \le i \le d+2$, let $\pi_i \colon (\mathbb{E}^{d+2} - H_i(0)) \to \mathbb{E}^{d+1}$ be the projection of center $0 \in \mathbb{E}^{d+2}$ onto the hyperplane, $H_i \subseteq \mathbb{E}^{d+2}$, of equation $x_i = 1$ ($H_i \cong \mathbb{E}^{d+1}$ and $H_i(0) \subseteq \mathbb{E}^{d+2}$ is the hyperplane of equation $x_i = 0$) given by

$$\pi_i(x_1, \ldots, x_{d+2}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{d+2}}{x_i} \right) \qquad (x_i \ne 0).$$

Geometrically, for any $x \notin H_i(0)$, the image, $\pi_i(x)$, of x is the intersection of the line through the origin and x with the hyperplane, $H_i \subseteq \mathbb{E}^{d+2}$ of equation $x_i = 1$. Observe that the map, $\pi_i \colon (\mathbb{E}^{d+2} - H_{d+2}(0)) \to \mathbb{E}^{d+1}$, is an "affine" version of the bijection, $\varphi_i \colon U_i \to \mathbb{R}^{d+1}$, of Section 5.1. Then, we have

$$\Theta = \pi_{d+2} \circ \widehat{\theta} \circ i_{d+2}.$$

If we identify H_{d+2} and $\mathbb{E}^{d+1}$, we may write with a slight abuse of notation, $\Theta = \pi_{d+2} \circ \widehat{\theta}$.

Besides θ, we need to define a few more maps in order to establish the connection between the Delaunay complex on S^d and the Delaunay complex on $\mathcal{P}$. We use the convention of denoting the extension to projective spaces of a map, f, defined between Euclidean spaces, by $\widetilde{f}$.

The Euclidean orthogonal projection, $p_i \colon \mathbb{R}^{d+1} \to \mathbb{R}^d$, is given by

$$p_i(x_1, \ldots, x_{d+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1})$$

and $\widetilde{p}_i \colon \mathbb{P}^{d+1} \to \mathbb{P}^d$ denotes the projection from $\mathbb{P}^{d+1}$ onto $\mathbb{P}^d$ given by

$$\widetilde{p}_i(x_1 \colon \cdots \colon x_{d+2}) = (x_1 \colon \cdots \colon x_{i-1} \colon x_{i+1} \colon \cdots \colon x_{d+2}),$$

which is undefined at the point $(0 \colon \cdots \colon 1 \colon 0 \colon \cdots \colon 0)$, where the "1" is in the ith slot. The map $\widetilde{\pi}_N \colon (\mathbb{P}^{d+1} - \{N\}) \to \mathbb{P}^d$ is the central projection from the north pole onto $\mathbb{P}^d$ given by

$$\widetilde{\pi}_N(x_1 \colon \cdots \colon x_{d+1} \colon x_{d+2}) = (x_1 \colon \cdots \colon x_d \colon x_{d+2} - x_{d+1}).$$

A geometric interpretation of $\widetilde{\pi}_N$ will be needed later in certain proofs. If we identify $\mathbb{P}^d$ with the hyperplane, $H_{d+1}(0) \subseteq \mathbb{P}^{d+1}$, of equation $x_{d+1} = 0$, then we claim that for any, $x \ne N$, the point $\widetilde{\pi}_N(x)$ is the intersection of the line through N and x with the hyperplane, $H_{d+1}(0)$. Indeed, parametrically, the line, $\langle N, x \rangle$, through $N = (0 \colon \cdots \colon 0 \colon 1 \colon 1)$ and x is given by

$$\langle N, x \rangle = \{(\mu x_1 \colon \cdots \colon \mu x_d \colon \lambda + \mu x_{d+1} \colon \lambda + \mu x_{d+2}) \mid \lambda, \mu \in \mathbb{R}, \ \lambda \ne 0 \quad \text{or} \quad \mu \ne 0\}.$$

The line $\langle N, x \rangle$ intersects the hyperplane $x_{d+1} = 0$ iff

$$\lambda + \mu x_{d+1} = 0,$$

so we can pick $\lambda = -x_{d+1}$ and $\mu = 1$, which yields the intersection point,

$$(x_1 \colon \cdots \colon x_d \colon 0 \colon x_{d+2} - x_{d+1}),$$

as claimed.

We also have the projective versions of σ_N and τ_N, denoted $\widetilde{\sigma}_N \colon (S^d - \{N\}) \to \mathbb{P}^d$ and $\widetilde{\tau}_N \colon \mathbb{P}^d \to S^d \subseteq \mathbb{P}^{d+1}$, given by:

$$\widetilde{\sigma}_N(x_1 \colon \cdots \colon x_{d+2}) = (x_1 \colon \cdots \colon x_d \colon x_{d+2} - x_{d+1})$$

and

$$\widetilde{\tau}_N(x_1 \colon \cdots \colon x_{d+1}) = \left(2x_1 x_{d+1} \colon \cdots \colon 2x_d x_{d+1} \colon \sum_{i=1}^{d} x_i^2 - x_{d+1}^2 \colon \sum_{i=1}^{d} x_i^2 + x_{d+1}^2 \right).$$

It is an easy exercise to check that the image of $S^d - \{N\}$ by $\widetilde{\sigma}_N$ is U_{d+1} and that $\widetilde{\sigma}_N$ and $\widetilde{\tau}_N \restriction U_{d+1}$ are mutual inverses. Observe that $\widetilde{\sigma}_N = \widetilde{\pi}_N \restriction S^d$, the restriction of the projection, $\widetilde{\pi}_N$, to the sphere, S^d. The lifting, $\widetilde{l} \colon \mathbb{E}^d \to \mathcal{P} \subseteq \mathbb{P}^{d+1}$, is given by

$$\widetilde{l}(x_1, \ldots, x_d) = \left(x_1 \colon \cdots \colon x_d \colon \sum_{i=1}^{d} x_i^2 \colon 1 \right)$$

and the embedding, $\psi_{d+1} \colon \mathbb{E}^d \to \mathbb{P}^d$, (the map ψ_{d+1} defined in Section 5.1) is given by

$$\psi_{d+1}(x_1, \ldots, x_d) = (x_1 \colon \cdots \colon x_d \colon 1).$$

Then, we easily check

Proposition 8.5. *The maps, $\theta, \widetilde{\pi}_N, \widetilde{\tau}_N, \widetilde{p}_{d+1}, \widetilde{l}$ and ψ_{d+1} defined before satisfy the equations*

$$\begin{aligned}
\widetilde{l} &= \theta \circ \widetilde{\tau}_N \circ \psi_{d+1} \\
\widetilde{\pi}_N &= \widetilde{p}_{d+1} \circ \theta \\
\widetilde{\tau}_N \circ \psi_{d+1} &= \psi_{d+2} \circ \tau_N \\
\widetilde{l} &= \psi_{d+2} \circ l \\
l &= \Theta \circ \tau_N.
\end{aligned}$$

Proof. Let us check the first equation leaving the others as an exercise. Recall that θ is given by

$$\theta(x_1 \colon \cdots \colon x_{d+2}) = (x_1 \colon \cdots \colon x_d \colon x_{d+1} + x_{d+2} \colon x_{d+2} - x_{d+1}).$$

Then, as

$$\widetilde{\tau}_N \circ \psi_{d+1}(x_1, \ldots, x_d) = \left(2x_1 : \cdots : 2x_d : \sum_{i=1}^{d} x_i^2 - 1 : \sum_{i=1}^{d} x_i^2 + 1\right),$$

we get

$$\begin{aligned} \theta \circ \widetilde{\tau}_N \circ \psi_{d+1}(x_1, \ldots, x_d) &= \left(2x_1 : \cdots : 2x_d : 2\sum_{i=1}^{d} x_i^2 : 2\right) \\ &= \left(x_1 : \cdots : x_d : \sum_{i=1}^{d} x_i^2 : 1\right) = \widetilde{l}(x_1, \ldots, x_d), \end{aligned}$$

as claimed. $\square$

We will also need some properties of the projection π_{d+2} and of Θ and for this, let

$$\mathbb{H}_+^d = \{(x_1, \ldots, x_d) \in \mathbb{E}^d \mid x_d > 0\} \quad \text{and} \quad \mathbb{H}_-^d = \{(x_1, \ldots, x_d) \in \mathbb{E}^d \mid x_d < 0\}.$$

Proposition 8.6. *The projection, π_{d+2}, has the following properties:*

(1) *For every hyperplane, H, through the origin, $\pi_{d+2}(H)$ is a hyperplane in H_{d+2}.*

(2) *Given any set of points, $\{a_1, \ldots, a_n\} \subseteq \mathbb{E}^{d+2}$, if $\{a_1, \ldots, a_n\}$ is contained in the open half-space above the hyperplane $x_{d+2} = 0$ or $\{a_1, \ldots, a_n\}$ is contained in the open half-space below the hyperplane $x_{d+2} = 0$, then the image by π_{d+2} of the convex hull of the a_i's is the convex hull of the images of these points, that is,*

$$\pi_{d+2}(\mathrm{conv}(\{a_1, \ldots, a_n\})) = \mathrm{conv}(\{\pi_{d+2}(a_1), \ldots, \pi_{d+2}(a_n)\}).$$

(3) *Given any set of points, $\{a_1, \ldots, a_n\} \subseteq \mathbb{E}^{d+1}$, if $\{a_1, \ldots, a_n\}$ is contained in the open half-space above the hyperplane H_{d+1} or $\{a_1, \ldots, a_n\}$ is contained in the open half-space below H_{d+1}, then*

$$\Theta(\mathrm{conv}(\{a_1, \ldots, a_n\})) = \mathrm{conv}(\{\Theta(a_1), \ldots, \Theta(a_n)\}).$$

(4) *For any set $S \subseteq \mathbb{E}^{d+1}$, if $\mathrm{conv}(S)$ does not intersect H_{d+1}, then*

$$\Theta(\mathrm{conv}(S)) = \mathrm{conv}(\Theta(S)).$$

Proof. (1) The image, $\pi_{d+2}(H)$, of a hyperplane, H, through the origin is the intersection of H with H_{d+2}, which is a hyperplane in H_{d+2}.

(2) This seems fairly clear geometrically but the result fails for arbitrary sets of points so to be on the safe side we give an algebraic proof. We will prove the following two facts by induction on $n \geq 1$:

(1) For all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$, for all $a_1, \ldots, a_n \in \mathbb{H}_+^{d+2}$ (resp. $\in \mathbb{H}_-^{d+2}$) there exist some $\mu_1, \ldots, \mu_n \in \mathbb{R}$ with $\mu_1 + \cdots + \mu_n = 1$ and $\mu_i \geq 0$, so that

$$\pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_n a_n) = \mu_1 \pi_{d+2}(a_1) + \cdots + \mu_n \pi_{d+2}(a_n).$$

(2) For all $\mu_1, \ldots, \mu_n \in \mathbb{R}$ with $\mu_1 + \cdots + \mu_n = 1$ and $\mu_i \geq 0$, for all $a_1, \ldots, a_n \in \mathbb{H}_+^{d+2}$ (resp. $\in \mathbb{H}_-^{d+2}$) there exist some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$, so that

$$\pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_n a_n) = \mu_1 \pi_{d+2}(a_1) + \cdots + \mu_n \pi_{d+2}(a_n).$$

(1) The base case is clear. Let us assume for the moment that we proved (1) for $n = 2$ and consider the induction step for $n \geq 2$. Since $\lambda_1 + \cdots + \lambda_{n+1} = 1$ and $n \geq 2$, there is some i such that $\lambda_i \neq 1$, and without loss of generality, say $\lambda_1 \neq 1$. Then, we can write

$$\lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1} = \lambda_1 a_1 + (1 - \lambda_1) \left(\frac{\lambda_2}{1 - \lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1 - \lambda_1} a_{n+1} \right)$$

and since $\lambda_1 + \lambda_2 + \cdots + \lambda_{n+1} = 1$, we have

$$\frac{\lambda_2}{1 - \lambda_1} + \cdots + \frac{\lambda_{n+1}}{1 - \lambda_1} = 1.$$

By the induction hypothesis, for $n = 2$, there exist α_1 with $0 \leq \alpha_1 \leq 1$, such that

$$\pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1}) = \pi_{d+2} \left(\lambda_1 a_1 + (1 - \lambda_1) \left(\frac{\lambda_2}{1 - \lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1 - \lambda_1} a_{n+1} \right) \right)$$

$$= (1 - \alpha_1) \pi_{d+2}(a_1) + \alpha_1 \pi_{d+2} \left(\frac{\lambda_2}{1 - \lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1 - \lambda_1} a_{n+1} \right)$$

Again, by the induction hypothesis (for n), there exist $\beta_2, \ldots, \beta_{n+1}$ with $\beta_2 + \cdots + \beta_{n+1} = 1$ and $\beta_i \geq 0$, so that

$$\pi_{d+2} \left(\frac{\lambda_2}{1 - \lambda_1} a_2 + \cdots + \frac{\lambda_{n+1}}{1 - \lambda_1} a_{n+1} \right) = \beta_2 \pi_{d+2}(a_2) + \cdots + \beta_{n+1} \pi_{d+2}(a_{n+1}),$$

so we get

$$\begin{aligned} \pi_{d+2}(\lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1}) &= (1 - \alpha_1) \pi_{d+2}(a_1) + \alpha_1 (\beta_2 \pi_{d+2}(a_2) + \cdots + \beta_{n+1} \pi_{d+2}(a_{n+1})) \\ &= (1 - \alpha_1) \pi_{d+2}(a_1) + \alpha_1 \beta_2 \pi_{d+2}(a_2) + \cdots + \alpha_1 \beta_{n+1} \pi_{d+2}(a_{n+1}) \end{aligned}$$

and clearly, $1 - \alpha_1 + \alpha_1 \beta_2 + \cdots + \alpha_1 \beta_{n+1} = 1$ as $\beta_2 + \cdots + \beta_{n+1} = 1$; $1 - \alpha_1 \geq 0$; and $\alpha_1 \beta_i \geq 0$, as $0 \leq \alpha_1 \leq 1$ and $\beta_i \geq 0$. This establishes the induction step and thus, all is left is to prove the case $n = 2$.

(2) The base case $n = 1$ is also clear. As in (1), let us assume for a moment that (2) is proved for $n = 2$ and consider the induction step. The proof is quite similar to that of (1) but this time, we may assume that $\mu_1 \neq 1$ and we write

$$\mu_1 \pi_{d+2}(a_1) + \cdots + \mu_{n+1}\pi_{d+2}(a_{n+1})$$
$$= \mu_1 \pi_{d+2}(a_1) + (1 - \mu_1)\left(\frac{\mu_2}{1 - \mu_1}\pi_{d+2}(a_2)\cdots + \frac{\mu_{n+1}}{1 - \mu_1}\pi_{d+2}(a_{n+1})\right).$$

By the induction hypothesis, there are some $\alpha_2, \ldots, \alpha_{n+1}$ with $\alpha_2 + \cdots + \alpha_{n+1} = 1$ and $\alpha_i \geq 0$ such that

$$\pi_{d+2}(\alpha_2 a_2 + \cdots + \alpha_{n+1}a_{n+1}) = \frac{\mu_2}{1 - \mu_1}\pi_{d+2}(a_2) + \cdots + \frac{\mu_{n+1}}{1 - \mu_1}\pi_{d+2}(a_{n+1}).$$

By the induction hypothesis for $n = 2$, there is some β_1 with $0 \leq \beta_1 \leq 1$, so that

$$\pi_{d+2}((1-\beta_1)a_1 + \beta_1(\alpha_2 a_2 + \cdots + \alpha_{n+1}a_{n+1})) = \mu_1 \pi_{d+2}(a_1) + (1-\mu_1)\pi_{d+2}(\alpha_2 a_2 + \cdots + \alpha_{n+1}a_{n+1}),$$

which establishes the induction hypothesis. Therefore, all that remains is to prove (1) and (2) for $n = 2$.

As π_{d+2} is given by

$$\pi_{d+2}(x_1, \ldots, x_{d+2}) = \left(\frac{x_1}{x_{d+2}}, \ldots, \frac{x_{d+1}}{x_{d+2}}\right) \qquad (x_{d+2} \neq 0)$$

it is enough to treat the case when $d = 0$, that is,

$$\pi_2(a, b) = \frac{a}{b}.$$

To prove (1) it is enough to show that for any λ, with $0 \leq \lambda \leq 1$, if $b_1 b_2 > 0$ then

$$\frac{a_1}{b_1} \leq \frac{(1 - \lambda)a_1 + \lambda a_2}{(1 - \lambda)b_1 + \lambda b_2} \leq \frac{a_2}{b_2} \quad \text{if} \quad \frac{a_1}{b_1} \leq \frac{a_2}{b_2}$$

and

$$\frac{a_2}{b_2} \leq \frac{(1 - \lambda)a_1 + \lambda a_2}{(1 - \lambda)b_1 + \lambda b_2} \leq \frac{a_1}{b_1} \quad \text{if} \quad \frac{a_2}{b_2} \leq \frac{a_1}{b_1},$$

where, of course $(1-\lambda)b_1 + \lambda b_2 \neq 0$. For this, we compute (leaving some steps as an exercise)

$$\frac{(1 - \lambda)a_1 + \lambda a_2}{(1 - \lambda)b_1 + \lambda b_2} - \frac{a_1}{b_1} = \frac{\lambda(a_2 b_1 - a_1 b_2)}{((1 - \lambda)b_1 + \lambda b_2)b_1}$$

and

$$\frac{(1 - \lambda)a_1 + \lambda a_2}{(1 - \lambda)b_1 + \lambda b_2} - \frac{a_2}{b_2} = -\frac{(1 - \lambda)(a_2 b_1 - a_1 b_2)}{((1 - \lambda)b_1 + \lambda b_2)b_2}.$$

Now, as $b_1b_2 > 0$, that is, b_1 and b_2 have the same sign and as $0 \leq \lambda \leq 1$, we have both $((1-\lambda)b_1 + \lambda b_2)b_1 > 0$ and $((1-\lambda)b_1 + \lambda b_2)b_2 > 0$. Then, if $a_2b_1 - a_1b_2 \geq 0$, that is $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$ (since $b_1b_2 > 0$), the first two inequalities hold and if $a_2b_1 - a_1b_2 \leq 0$, that is $\frac{a_2}{b_2} \leq \frac{a_1}{b_1}$ (since $b_1b_2 > 0$), the last two inequalities hold. This proves (1).

In order to prove (2), given any μ, with $0 \leq \mu \leq 1$, if $b_1b_2 > 0$, we show that we can find λ with $0 \leq \lambda \leq 1$, so that

$$(1-\mu)\frac{a_1}{b_1} + \mu\frac{a_2}{b_2} = \frac{(1-\lambda)a_1 + \lambda a_2}{(1-\lambda)b_1 + \lambda b_2}.$$

If we let

$$\alpha = (1-\mu)\frac{a_1}{b_1} + \mu\frac{a_2}{b_2},$$

we find that λ is given by the equation

$$\lambda(a_2 - a_1 + \alpha(b_1 - b_2)) = \alpha b_1 - a_1.$$

After some (tedious) computations (check for yourself!) we find:

$$a_2 - a_1 + \alpha(b_1 - b_2) = \frac{((1-\mu)b_2 + \mu b_1)(a_2b_1 - a_1b_2)}{b_1b_2}$$

$$\alpha b_1 - a_1 = \frac{\mu b_1(a_2b_1 - a_1b_2)}{b_1b_2}.$$

If $a_2b_1 - a_1b_2 = 0$, then $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\lambda = 0$ works. If $a_2b_1 - a_1b_2 \neq 0$, then

$$\lambda = \frac{\mu b_1}{(1-\mu)b_2 + \mu b_1} = \frac{\mu}{(1-\mu)\frac{b_2}{b_1} + \mu}.$$

Since $b_1b_2 > 0$, we have $\frac{b_2}{b_1} > 0$, and since $0 \leq \mu \leq 1$, we conclude that $0 \leq \lambda \leq 1$, which proves (2).

(3) Since

$$\Theta = \pi_{d+2} \circ \widehat{\theta} \circ i_{d+2},$$

as i_{d+2} and $\widehat{\theta}$ are linear, they preserve convex hulls, so by (2), we simply have to show that either $\widehat{\theta} \circ i_{d+2}(\{a_1, \ldots, a_n\})$ is strictly below the hyperplane, $x_{d+2} = 0$, or strictly above it. But,

$$\widehat{\theta}(x_1, \ldots, x_{d+2})_{d+2} = x_{d+2} - x_{d+1}$$

and $i_{d+2}(x_1, \ldots, x_{d+1}) = (x_1, \ldots, x_{d+1}, 1)$, so

$$(\widehat{\theta} \circ i_{d+2})(x_1, \ldots, x_{d+1})_{d+2} = 1 - x_{d+1},$$

and this quantity is positive iff $x_{d+1} < 1$, negative iff $x_{d+1} > 1$; that is, either all the points a_i are strictly below the hyperplane H_{d+1} or all strictly above it.

(4) This follows immediately from (3) as $\mathrm{conv}(S)$ consists of all finite convex combinations of points in S. $\square$

If a set, $\{a_1, \ldots, a_n\} \subseteq \mathbb{E}^{d+2}$, contains points on *both sides* of the hyperplane, $x_{d+2} = 0$, then $\pi_{d+2}(\mathrm{conv}(\{a_1, \ldots, a_n\}))$ is **not** necessarily convex (find such an example!).

8.6 Stereographic Projection, Delaunay Polytopes and Voronoi Polyhedra

We saw in an earlier section that lifting a set of points, $P \subseteq \mathbb{E}^d$, to the paraboloid, $\mathcal{P}$, *via* the lifting function, l, was fruitful to better understand Voronoi diagrams and Delaunay triangulations. As far as we know, Edelsbrunner and Seidel [16] were the first to find the relationship between Voronoi diagrams and the polar dual of the convex hull of a lifted set of points onto a paraboloid. This connection is described in Note 3.1 of Section 3 in [16]. The connection between the Delaunay triangulation and the convex hull of the lifted set of points is described in Note 3.2 of the same paper. Polar duality is not mentioned and seems to enter the scene only with Boissonnat and Yvinec [8].

It turns out that instead of using a paraboloid we can use a sphere and instead of the lifting function l we can use the composition of ψ_{d+1} with the inverse stereographic projection, $\widetilde{\tau}_N$. Then, to get back down to $\mathbb{E}^d$, we use the composition of the projection, $\widetilde{\pi}_N$, with φ_{d+1}, instead of the orthogonal projection, p_{d+1}.

However, we have to be a bit careful because Θ does map all convex polyhedra to convex polyhedra. Indeed, Θ is the composition of π_{d+2} with some linear maps, but π_{d+2} does not behave well with respect to arbitrary convex sets. In particular, Θ is not well-defined on any face that intersects the hyperplane H_{d+1} (of equation $x_{d+1} = 1$). Fortunately, we can circumvent these difficulties by using the concept of a projective polyhedron introduced in Chapter 5.

As we said in the previous section, the correspondence between Voronoi diagrams and convex hulls *via* inversion was first observed by Brown [11]. Brown takes a set of points, S, for simplicity assumed to be in the plane, first lifts these points to the unit sphere S^2 using inverse stereographic projection (which is equivalent to an inversion of power 2 centered at the north pole), getting $\tau_N(S)$, and then takes the convex hull, $\mathcal{D}(S) = \text{conv}(\tau_N(S))$, of the lifted set. Now, in order to obtain the Voronoi diagram of S, apply our inversion (of power 2 centered at the north pole) to each of the faces of $\text{conv}(\tau_N(S))$, obtaining spheres passing through the center of S^2 and then intersect these spheres with the plane containing S, obtaining circles. The centers of some of these circles are the Voronoi vertices. Finally, a simple criterion can be used to retain the "nearest Voronoi points" and to connect up these vertices.

Note that Brown's method is *not* the method that uses the polar dual of the polyhedron $\mathcal{D}(S) = \text{conv}(\tau_N(S))$, as we might have expected from the lifting method using a paraboloid. In fact, it is more natural to get the *Delaunay triangulation of S* from Brown's method, by applying the stereographic projection (from the north pole) to $\mathcal{D}(S)$, as we will prove below. As $\mathcal{D}(S)$ is strictly below the plane $z = 1$, there are no problems. Now, in order to get the Voronoi diagram, we take the polar dual, $\mathcal{D}(S)^*$, of $\mathcal{D}(S)$ and then apply the central projection w.r.t. the north pole. This is where problems arise, as some faces of $\mathcal{D}(S)^*$ may intersect the hyperplane H_{d+1} and this is why we have recourse to projective geometry.

First, we show that θ has a good behavior with respect to tangent spaces. Recall from Section 5.2 that for any point, $a = (a_1 : \cdots : a_{d+2}) \in \mathbb{P}^{d+1}$, the tangent hyperplane, $T_a S^d$, to the sphere S^d at a is given by the equation

$$\sum_{i=1}^{d+1} a_i x_i - a_{d+2} x_{d+2} = 0.$$

Similarly, the tangent hyperplane, $T_a \mathcal{P}$, to the paraboloid $\mathcal{P}$ at a is given by the equation

$$2\sum_{i=1}^{d} a_i x_i - a_{d+2} x_{d+1} - a_{d+1} x_{d+2} = 0.$$

If we lift a point $a \in \mathbb{E}^d$ to S^d by $\widetilde{\tau}_N \circ \psi_{d+1}$ and to $\mathcal{P}$ by $\widetilde{l}$, it turns out that the image of the tangent hyperplane to S^d at $\widetilde{\tau}_N \circ \psi_{d+1}(a)$ by θ is the tangent hyperplane to $\mathcal{P}$ at $\widetilde{l}(a)$.

Proposition 8.7. *The map θ has the following properties:*

(1) *For any point, $a = (a_1, \ldots, a_d) \in \mathbb{E}^d$, we have*

$$\theta(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d) = T_{\widetilde{l}(a)} \mathcal{P},$$

  that is, θ preserves tangent hyperplanes.

(2) *For every $(d-1)$-sphere, $S \subseteq \mathbb{E}^d$, we have*

$$\theta(\widetilde{\tau}_N \circ \psi_{d+1}(S)) = \widetilde{l}(S),$$

  that is, θ preserves lifted $(d-1)$-spheres.

Proof. (1) By Proposition 8.5, we know that

$$\widetilde{l} = \theta \circ \widetilde{\tau}_N \circ \psi_{d+1}$$

and we proved in Section 5.2 that projectivities preserve tangent spaces. Thus,

$$\theta(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d) = T_{\theta \circ \widetilde{\tau}_N \circ \psi_{d+1}(a)} \theta(S^d) = T_{\widetilde{l}(a)} \mathcal{P},$$

as claimed.

(2) This follows immediately from the equation $\widetilde{l} = \theta \circ \widetilde{\tau}_N \circ \psi_{d+1}$. $\square$

Given any two distinct points, $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $\mathbb{E}^d$, recall that the bisector hyperplane, $H_{a,b}$, of a and b is given by

$$(b_1 - a_1)x_1 + \cdots + (b_d - a_d)x_d = (b_1^2 + \cdots + b_d^2)/2 - (a_1^2 + \cdots + a_d^2)/2.$$

We have the following useful proposition:

Proposition 8.8. *Given any two distinct points, $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $\mathbb{E}^d$, the image under the projection, $\widetilde{\pi}_N$, of the intersection, $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d$, of the tangent hyperplanes at the lifted points $\widetilde{\tau}_N \circ \psi_{d+1}(a)$ and $\widetilde{\tau}_N \circ \psi_{d+1}(b)$ on the sphere $S^d \subseteq \mathbb{P}^{d+1}$ is the embedding of the bisector hyperplane, $H_{a,b}$, of a and b, into $\mathbb{P}^d$, that is,*

$$\widetilde{\pi}_N(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d) = \psi_{d+1}(H_{a,b}).$$

Proof. In view of the geometric interpretation of $\widetilde{\pi}_N$ given earlier, we need to find the equation of the hyperplane, H, passing through the intersection of the tangent hyperplanes, $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)}$ and $T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)}$ and passing through the north pole and then, it is geometrically obvious that

$$\widetilde{\pi}_N(T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d) = H \cap H_{d+1}(0),$$

where $H_{d+1}(0)$ is the hyperplane (in $\mathbb{P}^{d+1}$) of equation $x_{d+1} = 0$. Recall that $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d$ and $T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d$ are given by

$$E_1 = 2\sum_{i=1}^{d} a_i x_i + (\sum_{i=1}^{d} a_i^2 - 1)x_{d+1} - (\sum_{i=1}^{d} a_i^2 + 1)x_{d+2} = 0$$

and

$$E_2 = 2\sum_{i=1}^{d} b_i x_i + (\sum_{i=1}^{d} b_i^2 - 1)x_{d+1} - (\sum_{i=1}^{d} b_i^2 + 1)x_{d+2} = 0.$$

The hyperplanes passing through $T_{\widetilde{\tau}_N \circ \psi_{d+1}(a)} S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(b)} S^d$ are given by an equation of the form

$$\lambda E_1 + \mu E_2 = 0,$$

with $\lambda, \mu \in \mathbb{R}$. Furthermore, in order to contain the north pole, this equation must vanish for $x = (0 : \cdots : 0 : 1 : 1)$. But, observe that setting $\lambda = -1$ and $\mu = 1$ gives a solution since the corresponding equation is

$$2\sum_{i=1}^{d}(b_i - a_i)x_i + (\sum_{i=1}^{d} b_i^2 - \sum_{i=1}^{d} a_i^2)x_{d+1} - (\sum_{i=1}^{d} b_i^2 - \sum_{i=1}^{d} a_i^2)x_{d+2} = 0$$

and it vanishes on $(0 : \cdots : 0 : 1 : 1)$. But then, the intersection of H with the hyperplane $H_{d+1}(0)$ of equation $x_{d+1} = 0$ is given by

$$2\sum_{i=1}^{d}(b_i - a_i)x_i - (\sum_{i=1}^{d} b_i^2 - \sum_{i=1}^{d} a_i^2)x_{d+2} = 0.$$

Since we view $\mathbb{P}^d$ as the hyperplane $H_{d+1}(0) \subseteq \mathbb{P}^{d+1}$ and since the coordinates of points in $H_{d+1}(0)$ are of the form $(x_1 : \cdots : x_d : 0 : x_{d+2})$, the above equation is equivalent to the equation of $\psi_{d+1}(H_{a,b})$ in $\mathbb{P}^d$ in which x_{d+1} is replaced by x_{d+2}. $\square$

In order to define precisely Delaunay complexes as projections of objects obtained by deleting some faces from a projective polyhedron we need to define the notion of "projective (polyhedral) complex". However, this is easily done by defining the notion of cell complex where the cells are polyhedral cones. Such objects are known as *fans*. The definition below is basically Definition 6.8 in which the cells are cones as opposed to polytopes.

Definition 8.5. *A fan in $\mathbb{E}^m$ is a set, K, consisting of a (finite or infinite) set of polyhedral cones in $\mathbb{E}^m$ satisfying the following conditions:*

(1) Every face of a cone in K also belongs to K.

(2) For any two cones σ_1 and σ_2 in K, if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Every cone, $\sigma \in K$, of dimension k, is called a k-face (or face) of K. A 0-face $\{v\}$ is called a vertex and a 1-face is called an edge. The dimension of the fan K is the maximum of the dimensions of all cones in K. If $\dim K = d$, then every face of dimension d is called a cell and every face of dimension $d - 1$ is called a facet.

A projective (polyhedral) complex, $\mathcal{K} \subseteq \mathbb{P}^d$, is a set of projective polyhedra of the form, $\{\mathbb{P}(C) \mid C \in K\}$, where $K \subseteq \mathbb{R}^{d+1}$ is a fan.

Given a projective complex, the notions of face, vertex, edge, cell, facet, are dedined in the obvious way.

If $K \subseteq \mathbb{R}^d$ is a polyhedral complex, then it is easy to check that the set $\{C(\sigma) \mid \sigma \in K\} \subseteq \mathbb{R}^{d+1}$ is a fan and we get the projective complex

$$\widetilde{K} = \{\mathbb{P}(C(\sigma)) \mid \sigma \in K\} \subseteq \mathbb{P}^d.$$

The projective complex, $\widetilde{K}$, is called the *projective completion* of K. Also, it is easy to check that if $f \colon P \to P'$ is an injective affine map between two polyhedra P and P', then f extends uniquely to a projectivity, $\widetilde{f} \colon \widetilde{P} \to \widetilde{P'}$, between the projective completions of P and P'.

We now have all the facts needed to show that Delaunay triangulations and Voronoi diagrams can be defined in terms of the lifting, $\widetilde{\tau}_N \circ \psi_{d+1}$, and the projection, $\widetilde{\pi}_N$, and to establish their duality *via* polar duality with respect to S^d.

Definition 8.6. *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, the polytope, $\mathcal{D}(P) \subseteq \mathbb{R}^{d+1}$, called the Delaunay polytope associated with P is the convex hull of the union of the lifting of the points of P onto the sphere S^d (via inverse stereographic projection) with the north pole, that is, $\mathcal{D}(P) = \operatorname{conv}(\tau_N(P) \cup \{N\})$. The projective Delaunay polytope, $\widetilde{\mathcal{D}}(P) \subseteq \mathbb{P}^{d+1}$, associated with P is the projective completion of $\mathcal{D}(P)$. The polyhedral complex, $\mathcal{C}(P) \subseteq \mathbb{R}^{d+1}$, called the lifted Delaunay complex of P is the complex obtained from $\mathcal{D}(P)$ by deleting the facets containing the north pole (and their faces) and $\widetilde{\mathcal{C}}(P) \subseteq \mathbb{P}^{d+1}$ is the projective completion of $\mathcal{C}(P)$. The polyhedral complex, $\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) \subseteq \mathbb{E}^d$, is the Delaunay complex of P or Delaunay triangulation of P.*

The above is not the "standard" definition of the Delaunay triangulation of P but it is equivalent to the definition, say given in Boissonnat and Yvinec [8], as we will prove shortly. It also has certain advantages over lifting onto a paraboloid, as we will explain. Furthermore, to be perfectly rigorous, we should define $\mathcal{D}el(P)$ by

$$\mathcal{D}el(P) = \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) \cap U_{d+1}),$$

but $\widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) \subseteq U_{d+1}$ because $\mathcal{C}(P)$ is strictly below the hyperplane H_{d+1}.

It it possible and useful to define $\mathcal{D}el(P)$ more directly in terms of $\mathcal{C}(P)$. The projection, $\widetilde{\pi}_N \colon (\mathbb{P}^{d+1} - \{N\}) \to \mathbb{P}^d$, comes from the linear map, $\widehat{\pi}_N \colon \mathbb{R}^{d+2} \to \mathbb{R}^{d+1}$, given by

$$\widehat{\pi}_N(x_1, \ldots, x_{d+1}, x_{d+2}) = (x_1, \ldots, x_d, x_{d+2} - x_{d+1}).$$

Consequently, as $\widetilde{\mathcal{C}}(P) = \widehat{\mathcal{C}(P)} = \mathbb{P}(C(\mathcal{C}(P)))$, we immediately check that

$$\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) = \varphi_{d+1} \circ \widehat{\pi}_N(C(\mathcal{C}(P))) = \varphi_{d+1} \circ \widehat{\pi}_N(\mathrm{cone}(\widehat{\mathcal{C}(P)})),$$

where $\widehat{\mathcal{C}(P)} = \{\widehat{u} \mid u \in \mathcal{C}(P)\}$ and $\widehat{u} = (u, 1)$.

This suggests defining the map, $\pi_N \colon (\mathbb{R}^{d+1} - H_{d+1}) \to \mathbb{R}^d$, by

$$\pi_N = \varphi_{d+1} \circ \widehat{\pi}_N \circ i_{d+2},$$

which is explicity given by

$$\pi_N(x_1, \ldots, x_d, x_{d+1}) = \frac{1}{1 - x_{d+1}}(x_1, \ldots, x_d).$$

Then, as $\mathcal{C}(P)$ is strictly below the hyperplane H_{d+1}, we have

$$\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) = \pi_N(\mathcal{C}(P)).$$

First, note that $\mathcal{D}el(P) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P))$ is indeed a polyhedral complex whose geometric realization is the convex hull, $\mathrm{conv}(P)$, of P. Indeed, by Proposition 8.6, the images of the facets of $\mathcal{C}(P)$ are polytopes and when any two such polytopes meet, they meet along a common face. Furthermore, if $\dim(\mathrm{conv}(P)) = m$, then $\mathcal{D}el(P)$ is pure m-dimensional. First, $\mathcal{D}el(P)$ contains at least one m-dimensional cell. If $\mathcal{D}el(P)$ was not pure, as the complex is connected there would be some cell, σ, of dimension $s < m$ meeting some other cell, τ, of dimension m along a common face of dimension at most s and because σ is not contained in any face of dimension m, no facet of τ containing $\sigma \cap \tau$ can be adjacent to any cell of dimension m and so, $\mathcal{D}el(P)$ would not be convex, a contradiction.

For any polytope, $P \subseteq \mathbb{E}^d$, given any point, x, not in P, recall that a facet, F, of P is *visible from* x iff for every point, $y \in F$, the line through x and y intersects F only in y. If $\dim(P) = d$, this is equivalent to saying that x and the interior of P are strictly separated

by the supporting hyperplane of F. Note that if $\dim(P) < d$, it possible that every facet of P is visible from x.

Now, assume that $P \subseteq \mathbb{E}^d$ is a polytope with nonempty interior. We say that a facet, F, of P is a *lower-facing facet* of P iff the unit normal to the supporting hyperplane of F pointing towards the interior of P has non-negative x_{d+1}-coordinate. A facet, F, that is not lower-facing is called an *upper-facing facet* (Note that in this case the x_{d+1} coordinate of the unit normal to the supporting hyperplane of F pointing towards the interior of P is strictly negative).

Here is a convenient way to characterize lower-facing facets.

Proposition 8.9. *Given any polytope, $P \subseteq \mathbb{E}^d$, with nonempty interior, for any point, c, on the Ox_d-axis, if c lies strictly above all the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of F, then the lower-facing facets of P are exactly the facets not visible from c.*

Proof. Note that the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of P are strictly above the intersection points of the Ox_d-axis with the supporting hyperplanes of all the lower-facing facets. Suppose F is visible from c. Then, F must not be lower-facing as otherwise, for any $y \in F$, the line through c and y has to intersect some upper-facing facet and F is not be visible from c, a contradiction.

Now, as P is the intersection of the closed half-spaces determined by the supporting hyperplanes of its facets, by the definition of an upper-facing facet, any point, c, on the Ox_d-axis that lies strictly above the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of F has the property that c and the interior of P are strictly separated by all these supporting hyperplanes. Therefore, all the upper-facing facets of P are visible from c. It follows that the facets visible from c are exactly the upper-facing facets, as claimed. $\square$

We will also need the following fact when $\dim(P) = d$.

Proposition 8.10. *Given any polytope, $P \subseteq \mathbb{E}^d$, if $\dim(P) = d$, then there is a point, c, on the Ox_d-axis, such that for all points, x, on the Ox_d-axis and above c, the set of facets of $\mathrm{conv}(P \cup \{x\})$ not containing x is identical. Moreover, the set of facets of P not visible from x is the set of facets of $\mathrm{conv}(P \cup \{x\})$ that do not contain x.*

Proof. If $\dim(P) = d$ then pick any c on the Ox_d-axis above the intersection points of the Ox_d-axis with the supporting hyperplanes of all the upper-facing facets of F. Then, c is in general position w.r.t. P in the sense that c and any d vertices of P do not lie in a common hyperplane. Now, our result follows by lemma 8.3.1 of Boissonnat and Yvinec [8]. $\square$

Corollary 8.11. *Given any polytope, $P \subseteq \mathbb{E}^d$, with nonempty interior, there is a point, c, on the Ox_d-axis, so that for all x on the Ox_d-axis and above c, the lower-facing facets of P are exactly the facets of $\mathrm{conv}(P \cup \{x\})$ that do not contain x.*

As usual, let $e_{d+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$.

Theorem 8.12. *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, let $\mathcal{D}'(P)$ denote the polyhedron $\mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$ and let $\widetilde{\mathcal{D}}'(P)$ be the projective completion of $\mathcal{D}'(P)$. Also, let $\mathcal{C}'(P)$ be the polyhedral complex consisting of the bounded facets of the polytope $\mathcal{D}'(P)$ and let $\widetilde{\mathcal{C}}'(P)$ be the projective completion of $\mathcal{C}'(P)$. Then*

$$\theta(\widetilde{\mathcal{D}}(P)) = \widetilde{\mathcal{D}}'(P) \quad and \quad \theta(\widetilde{\mathcal{C}}(P)) = \widetilde{\mathcal{C}}'(P).$$

Furthermore, if $\mathcal{D}el'(P) = \varphi_{d+1} \circ \widetilde{p}_{d+1}(\widetilde{\mathcal{C}}'(P)) = p_{d+1}(\mathcal{C}'(P))$ is the "standard" Delaunay complex of P, that is, the orthogonal projection of $\mathcal{C}'(P)$ onto $\mathbb{E}^d$, then

$$\mathcal{D}el(P) = \mathcal{D}el'(P).$$

Therefore, the two notions of a Delaunay complex agree. If $\dim(\mathrm{conv}(P)) = d$, then the bounded facets of $\mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$ are precisely the lower-facing facets of $\mathrm{conv}(l(P))$.

Proof. Recall that

$$\mathcal{D}(P) = \mathrm{conv}(\tau_N(P) \cup \{N\})$$

and $\widetilde{\mathcal{D}}(P) = \mathbb{P}(C(\mathcal{D}(P)))$ is the projective completion of $\mathcal{D}(P)$. If we write $\widehat{\tau_N(P)}$ for $\{\widehat{\tau_N(p_i)} \mid p_i \in P\}$, then

$$C(\mathcal{D}(P)) = \mathrm{cone}(\widehat{\tau_N(P)} \cup \{\widehat{N}\}).$$

By definition, we have

$$\theta(\widetilde{\mathcal{D}}) = \mathbb{P}(\widehat{\theta}(C(\mathcal{D}))).$$

Now, as $\widehat{\theta}$ is linear,

$$\widehat{\theta}(C(\mathcal{D})) = \widehat{\theta}(\mathrm{cone}(\widehat{\tau_N(P)} \cup \{\widehat{N}\})) = \mathrm{cone}(\widehat{\theta}(\widehat{\tau_N(P)}) \cup \{\widehat{\theta}(\widehat{N})\}).$$

We claim that

$$\begin{aligned}
\mathrm{cone}(\widehat{\theta}(\widehat{\tau_N(P)}) \cup \{\widehat{\theta}(\widehat{N})\}) &= \mathrm{cone}(\widehat{l(P)} \cup \{(0, \ldots, 0, 1, 1)\}) \\
&= C(\mathcal{D}'(P)),
\end{aligned}$$

where

$$\mathcal{D}'(P) = \mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1}).$$

Indeed,

$$\widehat{\theta}(x_1, \ldots, x_{d+2}) = (x_1, \ldots, x_d, x_{d+1} + x_{d+2}, x_{d+2} - x_{d+1}),$$

and for any $p_i = (x_1, \ldots, x_d) \in P$,

$$\begin{aligned}
\widehat{\tau_N(p_i)} &= \left(\frac{2x_1}{\sum_{i=1}^{d} x_i^2 + 1}, \ldots, \frac{2x_d}{\sum_{i=1}^{d} x_i^2 + 1}, \frac{\sum_{i=1}^{d} x_i^2 - 1}{\sum_{i=1}^{d} x_i^2 + 1}, 1 \right) \\
&= \frac{1}{\sum_{i=1}^{d} x_i^2 + 1} \left(2x_1, \ldots, 2x_d, \sum_{i=1}^{d} x_i^2 - 1, \sum_{i=1}^{d} x_i^2 + 1 \right),
\end{aligned}$$

so we get

$$\widehat{\theta}(\widehat{\tau_N(p_i)}) \;=\; \frac{2}{\sum_{i=1}^{d} x_i^2 + 1}\left(x_1, \ldots, x_d, \sum_{i=1}^{d} x_i^2, 1\right)$$

$$=\; \frac{2}{\sum_{i=1}^{d} x_i^2 + 1}\,\widehat{l(p_i)}.$$

Also, we have

$$\widehat{\theta}(\widehat{N}) = \widehat{\theta}(0, \ldots, 0, 1, 1) = (0, \ldots, 0, 2, 0) = 2\widehat{e_{d+1}},$$

and by definition of $\mathrm{cone}(-)$ (scalar factors are irrelevant), we get

$$\mathrm{cone}(\widehat{\theta}(\widehat{\tau_N(P)}) \cup \{\widehat{\theta}(\widehat{N})\}) = \mathrm{cone}(\widehat{l(P)} \cup \{(0, \ldots, 0, 1, 1)\}) = C(\mathcal{D}'(P)),$$

with $\mathcal{D}'(P) = \mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$, as claimed. This proves that

$$\theta(\widetilde{\mathcal{D}}(P)) = \widetilde{\mathcal{D}}'(P).$$

Now, it is clear that the facets of $\mathrm{conv}(\tau_N(P) \cup \{N\})$ that do not contain N are mapped to the bounded facets of $\mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$, since N goes the point at infinity, so

$$\theta(\widetilde{\mathcal{C}}(P)) = \widetilde{\mathcal{C}}'(P).$$

As $\widetilde{\pi}_N = \widetilde{p}_{d+1} \circ \theta$ by Proposition 8.5, we get

$$\mathcal{D}el'(P) = \varphi_{d+1} \circ \widetilde{p}_{d+1}(\widetilde{\mathcal{C}}'(P)) = \varphi_{d+1} \circ (\widetilde{p}_{d+1} \circ \theta)(\widetilde{\mathcal{C}}(P)) = \varphi_{d+1} \circ \widetilde{\pi}_N(\widetilde{\mathcal{C}}(P)) = \mathcal{D}el(P),$$

as claimed. Finally, if $\dim(\mathrm{conv}(P)) = d$, then, by Corollary 8.11, we can pick a point, c, on the Ox_{d+1}-axis, so that the facets of $\mathrm{conv}(l(P) \cup \{c\})$ that do not contain c are precisely the lower-facing facets of $\mathrm{conv}(l(P))$. However, it is also clear that the facets of $\mathrm{conv}(l(P) \cup \{c\})$ that contain c tend to the unbounded facets of $\mathcal{D}'(P) = \mathrm{conv}(l(P)) + \mathrm{cone}(e_{d+1})$ when c goes to $+\infty$. $\square$

We can also characterize when the Delaunay complex, $\mathcal{D}el(P)$, is simplicial. Recall that we say that a set of points, $P \subseteq \mathbb{E}^d$, is in *general position* iff no $d + 2$ of the points in P belong to a common $(d - 1)$-sphere.

Proposition 8.13. *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, if P is in general position, then the Delaunay complex, $\mathcal{D}el(P)$, is a pure simplicial complex.*

Proof. Let $\dim(\mathrm{conv}(P)) = r$. Then, $\tau_N(P)$ is contained in a $(r-1)$-sphere of S^d, so we may assume that $r = d$. Suppose $\mathcal{D}el(P)$ has some facet, F, which is not a d-simplex. If so, F is the convex hull of at least $d + 2$ points, $p_1, \ldots, p_k$ of P and since $F = \pi_N(\widehat{F})$, for some facet, $\widehat{F}$, of $\mathcal{C}(P)$, we deduce that $\tau_N(p_1), \ldots, \tau_N(p_k)$ belong to the supporting hyperplane, H, of $\widehat{F}$. Now, if H passes through the north pole, then we know that $p_1, \ldots, p_k$ belong to some

hyperplane of $\mathbb{E}^d$, which is impossible since $p_1, \ldots, p_k$ are the vertices of a facet of dimension d. Thus, H does not pass through N and so, $p_1, \ldots, p_k$ belong to some $(d-1)$-sphere in $\mathbb{E}^d$. As $k \geq d+2$, this contradicts the assumption that the points in P are in general position. $\square$

Remark: Even when the points in P are in general position, the Delaunay polytope, $\mathcal{D}(P)$, may not be a simplicial polytope. For example, if $d+1$ points belong to a hyperplane in $\mathbb{E}^d$, then the lifted points belong to a hyperplane passing through the north pole and these $d+1$ lifted points together with N may form a non-simplicial facet. For example, consider the polytope obtained by lifting our original $d+1$ points on a hyperplane, H, plus one more point not in the the hyperplane H.

We can also characterize the Voronoi diagram of P in terms of the polar dual of $\mathcal{D}(P)$. Unfortunately, we can't simply take the polar dual, $\mathcal{D}(P)^*$, of $\mathcal{D}(P)$ and project it using π_N because some of the facets of $\mathcal{D}(P)^*$ may intersect the hyperplane, H_{d+1}, and π_N is undefined on H_{d+1}. However, using projective completions, we can indeed recover the Voronoi diagram of P.

Definition 8.7. *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, the Voronoi polyhedron associated with P is the polar dual (w.r.t. $S^d \subseteq \mathbb{R}^{d+1}$), $\mathcal{V}(P) = (\mathcal{D}(P))^* \subseteq \mathbb{R}^{d+1}$, of the Delaunay polytope, $\mathcal{D}(P) = \mathrm{conv}(\tau_N(P) \cup \{N\})$. The projective Voronoi polytope, $\widetilde{\mathcal{V}}(P) \subseteq \mathbb{P}^{d+1}$, associated with P is the projective completion of $\mathcal{V}(P)$. The polyhedral complex, $\mathcal{V}or(P) = \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{\mathcal{V}}(P)) \cap U_{d+1}) \subseteq \mathbb{E}^d$, is the Voronoi complex of P or Voronoi diagram of P.*

Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, let $\mathcal{V}'(P) = (\mathcal{D}'(P))^*$ be the polar dual (w.r.t. $\mathcal{P} \subseteq \mathbb{R}^{d+1}$) of the "standard" Delaunay polyhedron defined in Theorem 8.12 and let $\widetilde{\mathcal{V}'}(P) = \widetilde{\mathcal{V}'(P)} \subseteq \mathbb{P}^d$ be its projective completion. It is not hard to check that

$$p_{d+1}(\mathcal{V}'(P)) = \varphi_{d+1}(\widetilde{p}_{d+1}(\widetilde{\mathcal{V}'}(P)) \cap U_{d+1})$$

is the "standard" Voronoi diagram, denoted $\mathcal{V}or'(P)$.

Theorem 8.14. *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, we have*

$$\theta(\widetilde{\mathcal{V}}(P)) = \widetilde{\mathcal{V}'}(P)$$

and

$$\mathcal{V}or(P) = \mathcal{V}or'(P).$$

Therefore, the two notions of Voronoi diagrams agree.

Proof. By definition,

$$\widetilde{\mathcal{V}}(P) = \widetilde{\mathcal{V}(P)} = \widetilde{(\mathcal{D}(P))^*}$$

and by Proposition 5.12,

$$(\widetilde{\mathcal{D}(P)})^* = \left(\widetilde{\mathcal{D}(P)}\right)^* = (\widetilde{\mathcal{D}}(P))^*,$$

so

$$\widetilde{\mathcal{V}}(P) = (\widetilde{\mathcal{D}}(P))^*.$$

By Proposition 5.10,

$$\theta(\widetilde{\mathcal{V}}(P)) = \theta((\widetilde{\mathcal{D}}(P))^*) = (\theta(\widetilde{\mathcal{D}}(P)))^*$$

and by Theorem 8.12,

$$\theta(\widetilde{\mathcal{D}}(P)) = \widetilde{\mathcal{D}}'(P),$$

so we get

$$\theta(\widetilde{\mathcal{V}}(P)) = (\widetilde{\mathcal{D}}'(P))^*.$$

But, by Proposition 5.12 again,

$$(\widetilde{\mathcal{D}}'(P))^* = \left(\widetilde{\mathcal{D}'(P)}\right)^* = (\widetilde{\mathcal{D}'(P)})^* = \widetilde{\mathcal{V}'(P)} = \widetilde{\mathcal{V}}'(P).$$

Therefore,

$$\theta(\widetilde{\mathcal{V}}(P)) = \widetilde{\mathcal{V}}'(P),$$

as claimed.

As $\widetilde{\pi}_N = \widetilde{p}_{d+1} \circ \theta$ by Proposition 8.5, we get

$$
\begin{aligned}
\mathcal{V}or'(P) &= \varphi_{d+1}(\widetilde{p}_{d+1}(\widetilde{\mathcal{V}}'(P)) \cap U_{d+1}) \\
&= \varphi_{d+1}(\widetilde{p}_{d+1} \circ \theta(\widetilde{\mathcal{V}}(P)) \cap U_{d+1}) \\
&= \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{\mathcal{V}}(P)) \cap U_{d+1}) \\
&= \mathcal{V}or(P),
\end{aligned}
$$

finishing the proof. $\square$

We can also prove the proposition below which shows directly that $\mathcal{V}or(P)$ is the Voronoi diagram of P. Recall that that $\widetilde{\mathcal{V}}(P)$ is the projective completion of $\mathcal{V}(P)$. We observed in Section 5.2 (see page 90) that in the patch U_{d+1}, there is a bijection between the faces of $\widetilde{\mathcal{V}}(P)$ and the faces of $\mathcal{V}(P)$. Furthermore, the projective completion, $\widetilde{H}$, of every hyperplane, $H \subseteq \mathbb{R}^d$, is also a hyperplane and it is easy to see that if H is tangent to $\mathcal{V}(P)$, then $\widetilde{H}$ is tangent to $\widetilde{\mathcal{V}}(P)$.

Proposition 8.15. *Given any set of points, $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{E}^d$, for every $p \in P$, if F is the facet of $\mathcal{V}(P)$ that contains $\tau_N(p)$, if H is the tangent hyperplane at $\tau_N(p)$ to S^d and if F is cut out by the hyperplanes $H, H_1, \ldots, H_{k_p}$, in the sense that*

$$F = (H \cap H_1)_- \cap \cdots \cap (H \cap H_{k_p})_-,$$

where $(H \cap H_i)_-$ denotes the closed half-space in H containing $\tau_N(p)$ determined by $H \cap H_i$, then

$$V(p) = \varphi_{d+1}(\widetilde{\pi}_N(\widetilde{H} \cap \widetilde{H}_1)_- \cap \cdots \cap \widetilde{\pi}_N(\widetilde{H} \cap \widetilde{H}_{k_p})_- \cap U_{d+1})$$

is the Voronoi region of p (where $\varphi_{d+1}(\widetilde{\pi}_N(\widetilde{H} \cap \widetilde{H}_i)_- \cap U_{d+1})$ is the closed half-space containing p). If P is in general position, then $\mathcal{V}(P)$ is a simple polyhedron (every vertex belongs to $d + 1$ facets).

Proof. Recall that by Proposition 8.5,

$$\widetilde{\tau}_N \circ \psi_{d+1} = \psi_{d+2} \circ \tau_N.$$

Each $H_i = T_{\tau_N(p_i)}S^d$ is the tangent hyperplane to S^d at $\tau_N(p_i)$, for some $p_i \in P$. Now, by definition of the projective completion, the embedding, $\mathcal{V}(P) \longrightarrow \widetilde{\mathcal{V}}(P)$, is given by $a \mapsto \psi_{d+2}(a)$. Thus, every point, $p \in P$, is mapped to the point $\psi_{d+2}(\tau_N(p)) = \widetilde{\tau_N}(\psi_{d+1}(p))$ and we also have $\widetilde{H}_i = T_{\widetilde{\tau}_N \circ \psi_{d+1}(p_i)}S^d$ and $\widetilde{H} = T_{\widetilde{\tau}_N \circ \psi_{d+1}(p)}S^d$. By Proposition 8.8,

$$\widetilde{\pi}_N(T_{\widetilde{\tau}_N \circ \psi_{d+1}(p)}S^d \cap T_{\widetilde{\tau}_N \circ \psi_{d+1}(p_i)}S^d) = \psi_{d+1}(H_{p,p_i})$$

is the embedding of the bisector hyperplane of p and p_i in $\mathbb{P}^d$, so the first part holds.

Now, assume that some vertex, $v \in \mathcal{V}(P) = \mathcal{D}(P)^*$, belongs to $k \geq d + 2$ facets of $\mathcal{V}(P)$. By polar duality, this means that the facet, F, dual of v has $k \geq d + 2$ vertices $\tau_N(p_1), \ldots, \tau_N(p_k)$ of $\mathcal{D}(P)$. We claim that $\tau_N(p_1), \ldots, \tau_N(p_k)$ must belong to some hyperplane passing through the north pole. Otherwise, $\tau_N(p_1), \ldots, \tau_N(p_k)$ would belong to a hyperplane not passing through the north pole and so they would belong to a $(d - 1)$ sphere of S^d and thus, $p_1, \ldots, p_k$ would belong to a $(d - 1)$-sphere even though $k \geq d + 2$, contradicting that P is in general position. But then, by polar duality, v would be a point at infinity, a contradiction. $\square$

Note that when P is in general position, even though the polytope, $\mathcal{D}(P)$, may not be simplicial, its dual, $\mathcal{V}(P) = \mathcal{D}(P)^*$, is a simple *polyhedron*. What is happening is that $\mathcal{V}(P)$ has unbounded faces which have "vertices at infinity" that do not count! In fact, the faces of $\mathcal{D}(P)$ that fail to be simplicial are those that are contained in some hyperplane through the north pole. By polar duality, these faces correspond to a vertex at infinity. Also, if $m = \dim(\text{conv}(P)) < d$, then $\mathcal{V}(P)$ may not have any vertices!

We conclude our presentation of Voronoi diagrams and Delaunay triangulations with a short section on applications.

8.7 Applications of Voronoi Diagrams and Delaunay Triangulations

The examples below are taken from O'Rourke [31]. Other examples can be found in Preparata and Shamos [32], Boissonnat and Yvinec [8], and de Berg, Van Kreveld, Overmars, and Schwarzkopf [5].

The first example is the *nearest neighbors* problem. There are actually two subproblems: *Nearest neighbor queries* and *all nearest neighbors*.

The nearest neighbor queries problem is as follows. Given a set P of points and a query point q, find the nearest neighbor(s) of q in P. This problem can be solved by computing the Voronoi diagram of P and determining in which Voronoi region q falls. This last problem, called *point location*, has been heavily studied (see O'Rourke [31]). The all neighbors problem is as follows: Given a set P of points, find the nearest neighbor(s) to all points in P. This problem can be solved by building a graph, the *nearest neighbor graph*, for short *nng*. The nodes of this undirected graph are the points in P, and there is an arc from p to q iff p is a nearest neighbor of q or vice versa. Then it can be shown that this graph is contained in the Delaunay triangulation of P.

The second example is the *largest empty circle*. Some practical applications of this problem are to locate a new store (to avoid competition), or to locate a nuclear plant as far as possible from a set of towns. More precisely, the problem is as follows. Given a set P of points, find a largest empty circle whose center is in the (closed) convex hull of P, empty in that it contains no points from P inside it, and largest in the sense that there is no other circle with strictly larger radius. The Voronoi diagram of P can be used to solve this problem. It can be shown that if the center p of a largest empty circle is strictly inside the convex hull of P, then p coincides with a Voronoi vertex. However, not every Voronoi vertex is a good candidate. It can also be shown that if the center p of a largest empty circle lies on the boundary of the convex hull of P, then p lies on a Voronoi edge.

The third example is the *minimum spanning tree*. Given a graph G, a minimum spanning tree of G is a subgraph of G that is a tree, contains every vertex of the graph G, and minimizes the sum of the lengths of the tree edges. It can be shown that a minimum spanning tree is a subgraph of the Delaunay triangulation of the vertices of the graph. This can be used to improve algorithms for finding minimum spanning trees, for example Kruskal's algorithm (see O'Rourke [31]).

We conclude by mentioning that Voronoi diagrams have applications to *motion planning*. For example, consider the problem of moving a disk on a plane while avoiding a set of polygonal obstacles. If we "extend" the obstacles by the diameter of the disk, the problem reduces to finding a collision–free path between two points in the extended obstacle space. One needs to generalize the notion of a Voronoi diagram. Indeed, we need to define the distance to an object, and medial curves (consisting of points equidistant to two objects) may no longer be straight lines. A collision–free path with maximal clearance from the obstacles can be found by moving along the edges of the generalized Voronoi diagram. This is an active area of research in robotics. For more on this topic, see O'Rourke [31].

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