Complex Algebraic Geometry

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Preface

This manuscript is based on lectures given by Steve Shatz for the course *Math 622/623–Complex Algebraic Geometry*, during Fall 2003 and Spring 2004. The process for producing this manuscript was the following: I (Jean Gallier) took notes and transcribed them in LaTeX at the end of every week. A week later or so, Steve reviewed these notes and made changes and corrections. After the course was over, Steve wrote up additional material that I transcribed into LaTeX.

The following manuscript is thus unfinished and should be considered as work in progress. Nevertheless, given that *Principles of Algebraic Geometry* by Griffith and Harris is a formidable source, we feel that the material presented in this manuscript will be of some value.

We apologize for the typos and mistakes that surely occur in the manuscript (as well as unfinished sections and even unfinished proofs!). Still, our hope is that by its "freshness," this work will be of value to algebraic geometry lovers.

Please, report typos, mistakes, *etc.* (to Jean). We intend to improve and perhaps even complete this manuscript.

Philadelphia, February 2011 *Jean Gallier*

Acknowledgement. My friend Jean Gallier had the idea of attending my lectures in the graduate course in Complex Algebraic Geometry during the academic year 2003-04. Based on his notes of the lectures, he is producing these LaTeX notes. I have reviewed a first version of each LaTeX script and corrected only the most obvious errors which were either in my original lectures or might have crept in otherwise. Matters of style and presentation have been left to Jean Gallier. I owe him my thanks for all the work these LaTeXed notes represent.

Philadelphia, September 2003 *SSS*

Chapter 1

Complex Algebraic Varieties; Elementary Local And Global Theory

1.1 What is Geometry & What is Complex Algebraic Geometry?

The presumption is that we study systems of polynomial equations

$$\left\{\begin{array}{rcl} f_1(X_1,\ldots,X_q) & = & 0 \\ \vdots & \vdots & \vdots \\ f_p(X_1,\ldots,X_q) & = & 0 \end{array}\right. \tag{$\dagger$}$$

where the f_j are polynomials in $\mathbb{C}[X_1,\ldots,X_q]$.

Fact: Solving a system of equations of arbitrary degrees reduces to solving a system of quadratic equations (no restriction on the number of variables) (DX).

What is geometry?

Experience shows that we need

(1) A topological space, X.

(2) There exist (at least locally defined) functions on X.

(3) More experience shows that the "correct bookkeeping scheme" for encompassing (2) is a "sheaf" of functions on X; notation $\mathcal{O}_X$.

Aside on Presheaves and Sheaves.

(1) A *presheaf*, $\mathcal{P}$, on X is determined by the following data:

(i) For every open $U \subseteq X$, a set (or group, or ring, or space), $\mathcal{P}(U)$, is given.

(ii) If $V \subseteq U$ (where U, V are open in X) then there is a map $\rho^V_U \colon \mathcal{P}(U) \to \mathcal{P}(V)$ (restriction) such that $\rho^U_U = \mathrm{id}_U$ and

$$\rho^W_U = \rho^W_V \circ \rho^V_U, \quad \text{for all open subsets } U, V, W \text{ with } W \subseteq V \subseteq U.$$

(2) A *sheaf*, $\mathcal{F}$, on X is just a presheaf satisfying the following (patching) conditions:

(i) For every open $U \subseteq X$ and for every open cover $\{U_\alpha\}_\alpha$ of U (which means that $U = \bigcup_\alpha U_\alpha$, notation $\{U_\alpha \to U\}$), if $f, g \in \mathcal{F}(U)$ so that $f \restriction U_\alpha = g \restriction U_\alpha$, for all α, then $f = g$.

(ii) For all α, if we are given $f_\alpha \in \mathcal{F}(U_\alpha)$ and if for all α, β we have

$$\rho_{U_\alpha}^{U_\alpha \cap U_\beta}(f_\alpha) = \rho_{U_\beta}^{U_\alpha \cap U_\beta}(f_\beta),$$

(the f_α agree on overlaps), then there exists $f \in \mathcal{F}(U)$ so that $\rho_U^{U_\alpha}(f) = f_\alpha$, all α.

Our $\mathcal{O}_X$ is a *sheaf of rings*, i.e, $\mathcal{O}_X(U)$ is a commutative ring, for all U. We have $(X, \mathcal{O}_X)$, a topological space and a sheaf of rings.

Moreover, our functions are always (at least) continuous. Pick some $x \in X$ and look at all opens, $U \subseteq X$, where $x \in U$. If a small $U \ni x$ is given and $f, g \in \mathcal{O}_X(U)$, we say that f and g are equivalent, denoted $f \sim g$, iff there is some open $V \subseteq U$ with $x \in V$ so that $f \restriction V = g \restriction V$. This is an equivalence relation and $[f] =$ the equivalence class of f is the *germ of f at x*.

Check (DX) that

$$\varinjlim_{U \ni x} \mathcal{O}_X(U) = \text{collection of germs at } x.$$

The left hand side is called the *stalk of $\mathcal{O}_X$ at x*, denoted $\mathcal{O}_{X,x}$. By continuity, $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x =$ germs vanishing at x. In this case, $\mathcal{O}_X$ is called a *sheaf of local rings*.

In summary, a geometric object yields a pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of local rings. Such a pair, $(X, \mathcal{O}_X)$, is called a *local ringed space* (LRS).

LRS's would be useless without a notion of morphism from one LRS to another, $\Phi \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

(A) We need a continuous map $\varphi \colon X \to Y$ and whatever a morphism does on $\mathcal{O}_X, \mathcal{O}_Y$, taking a clue from the case where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sets of functions, we need something "$\mathcal{O}_Y \longrightarrow \mathcal{O}_X$."

Given a map $\varphi \colon X \to Y$ with $\mathcal{O}_X$ on X, we can make $\varphi_* \mathcal{O}_X$(= *direct image of* $\mathcal{O}_X$), a sheaf on Y, as follows: For any open $U \subseteq Y$, consider the open $\varphi^{-1}(U) \subseteq X$, and set

$$(\varphi_* \mathcal{O}_X)(U) = \mathcal{O}_X(\varphi^{-1}(U)).$$

This is a sheaf *on* Y (DX).

Alternatively, we have $\mathcal{O}_Y$ on Y (and the map $\varphi \colon X \to Y$) and we can try making a sheaf on X: Pick $x \in X$ and make the stalk of "something" at x. Given x, we make $\varphi(x) \in Y$, we make $\mathcal{O}_{Y,\varphi(x)}$ and define $\varphi^* \mathcal{O}_Y$ so that

$$(\varphi^*(\mathcal{O}_Y))_x = \mathcal{O}_{Y,\varphi(x)}.$$

More precisely, we define the presheaf $\varphi_P \mathcal{O}_Y$ on X by

$$\varphi_P \mathcal{O}_Y(U) = \varinjlim_{V \supseteq \varphi(U)} \mathcal{O}_Y(V),$$

where V ranges over open subsets of Y containing $\varphi(U)$. Unfortunately, this is not always a sheaf and we need to "sheafify" it to get $\varphi^* \mathcal{O}_Y$, the *inverse image of* $\mathcal{O}_Y$. For details, consult the Appendix on sheaves and ringed spaces. We now have everything we need to define morphisms of LRS's.

(B) A map of sheaves, $\widetilde{\varphi} \colon \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$, *on* Y, is also given.

It turns out that this is equivalent to giving a map of sheaves, $\widetilde{\widetilde{\varphi}} \colon \varphi^* \mathcal{O}_Y \to \mathcal{O}_X$, *on* X (This is because φ_* and φ^* are adjoint functors, again, see the Appendix on sheaves.)

In conclusion, a morphism $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair $(\varphi, \widetilde{\varphi})$ (or a pair $(\varphi, \widetilde{\widetilde{\varphi}})$), as above.

When we look at the "trivial case"' (of functions) we see that we want $\widetilde{\varphi}$ to satisfy

$$\widetilde{\varphi}(\mathfrak{m}_{\varphi(x)}) \subseteq \mathfrak{m}_x, \quad \text{for all } x \in X.$$

This condition says that $\widetilde{\varphi}$ is a *local morphism*. We get a category $\mathcal{LRS}$.

After all these generalities, we show how most geometric objects of interest arise are special kinds of LRS's. The key idea is to introduce "standard" models and to define a corresponding geometric objects, X, to be an LRS that is "locally isomorphic" to a standard model. First, observe that given any open subset $U \subseteq X$, we can form the restriction of the sheaf $\mathcal{O}_X$ to U, denoted $\mathcal{O}_X \restriction U$ or $(\mathcal{O}_U)$ and we get an LRS $(U, \mathcal{O}_X \restriction U)$. Now, if we also have a collection of LRS's (the standard models), we consider LRS's, $(X, \mathcal{O}_X)$, such that $(X, \mathcal{O}_X)$ is locally isomorphic to a standard model. This means that we can cover X by opens and that for every open $U \subseteq X$ in this cover, there is a standard model $(W, \mathcal{O}_W)$ and an isomorphism $(U, \mathcal{O}_X \restriction U) \cong (W, \mathcal{O}_W)$, as LRS's.

Some Standard Models.

(1) Let U be an open ball in $\mathbb{R}^n$ or $\mathbb{C}^n$, and let $\mathcal{O}_U$ be the sheaf of germs of continuous functions on U (this means, the sheaf such that for every open $V \subseteq U$, $\mathcal{O}_U(V) =$ the restrictions to V of the continuous functions on U). If $(X, \mathcal{O})$ is locally isomorphic to a standard, we get a topological manifold.

(2) Let U be an open as in (1) and let $\mathcal{O}_U$ be the sheaf of germs of C^k-functions on U, with $1 \leq k \leq \infty$. If $(X, \mathcal{O})$ is locally isomorphic to a standard, we get a C^k-manifold (when $k = \infty$, call these *smooth* manifolds).

(3) Let U be an open ball in $\mathbb{R}^n$ and let $\mathcal{O}_U$ be the sheaf of germs of real-valued C^ω-functions on U (i.e., real analytic functions). If $(X, \mathcal{O})$ is locally isomorphic to a standard, we get a real analytic manifold.

(4) Let U be an open ball in $\mathbb{C}^n$ and let $\mathcal{O}_U$ be the sheaf of germs of complex-valued C^ω-functions on U (i.e., complex analytic functions). If $(X, \mathcal{O})$ is locally isomorphic to a standard, we get a complex analytic manifold.

(5) Consider an LRS as in (2), with $k \geq 2$. For every $x \in X$, we have the tangent space, $T_{X,x}$, at x. Say we also have Q_x, a positive definite quadratic form on $T_{X,x}$, varying C^k as x varies. If $(X, \mathcal{O})$ is locally isomorphic to a standard, we get a *Riemannian manifold*.

(6) Suppose W is open in $\mathbb{C}^n$. Look at some subset $V \subseteq W$ and assume that V is defined as follows: For any $v \in V$, there is an open ball $B(v, \epsilon) = B_\epsilon$ and there are some functions $f_1, \ldots, f_p$ holomorphic on B_ϵ, so that

$$V \cap B(v, \epsilon) = \{(z_1, \ldots, z_q) \in B_\epsilon \mid f_1(z_1, \ldots, z_q) = \cdots = f_p(z_1, \ldots, z_q) = 0\}.$$

The question is, what should be $\mathcal{O}_V$?

We need only find out that what is $\mathcal{O}_{V \cap B_\epsilon}$ (DX). We set $\mathcal{O}_{V \cap B_\epsilon} =$ the sheaf of germs of holomorphic functions on B_ϵ modulo the ideal $(f_1, \ldots, f_p)$, and then restrict to V. Such a pair $(V, \mathcal{O}_V)$ is a *complex analytic space chunk*. An *algebraic function* on V is a ratio P/Q of polynomials with $Q \neq 0$ everywhere on V. If we replace the term "holomorphic" *everywhere* in the above, we obtain a *complex algebraic space chunk*.

Actually, the definition of a manifold requires that the underlying space is Hausdorff. The spaces that we have defined in (1)–(6) above are only locally Hausdorff and are "generalized manifolds".

Examples.

(1) Take $W = \mathbb{C}^q$, pick some polynomials $f_1, \ldots, f_p$ in $\mathbb{C}[Z_1, \ldots, Z_q]$ and let V be cut out by $f_1 = \cdots = f_p = 0$; so, we can pick $B(v, \epsilon) = \mathbb{C}^q$. This shows that the example (†) is a complex algebraic variety (in fact, a chunk). This is what we call an *affine variety*.

Remark: (to be proved later) If V is a complex *algebraic* variety and $V \subseteq \mathbb{C}^n$, then V is affine.

This remark implies that a complex algebraic variety is locally just given by equations of type (†).

(2) The manifolds of type (4) are among the complex analytic spaces (of (6)). Take $W = B(v, \epsilon)$ and no equations for V, so that $V \cap B(v, \epsilon) = B(v, \epsilon)$.

Say $(X, \mathcal{O}_X)$ is a complex algebraic variety. On a chunk, $V \subseteq W$ and a ball $B(v, \epsilon)$, we can replace the algebraic functions heretofore defining $\mathcal{O}_X$ by holomorphic functions. We get a complex analytic chunk and thus, X gives us a special kind of complex analytic variety, denoted X^{an}, which is locally cut out by polynomials but with holomorphic functions. We get a functor

$$X \rightsquigarrow X^{\mathrm{an}}$$

from complex algebraic varieties to complex analytic spaces. A complex space of the form X^{an} for some complex algebraic variety, X, is called an *algebraizable complex analytic space*.

Take $n+1$ copies of $\mathbb{C}^n$ ($\mathbb{C}^n$ with either its sheaf of algebraic functions or holomorphic functions). Call the j-copy U_j, where $j = 0, \ldots, n$. In U_j, we have coordinates

$$\langle Z_j^{(0)}, Z_j^{(1)}, \ldots, \widehat{Z_j^{(j)}}, \ldots, Z_j^{(n)} \rangle$$

(Here, as usual, the hat over an expression means that the corresponding item is omitted.) For all $i \neq j$, we have the open, $U_j^{(i)} \subseteq U_j$, namely the set $\{\xi \in U_j \mid (i\text{th coord.})\ \xi_j^{(i)} \neq 0\}$. We are going to glue $U_j^{(i)}$ to $U_i^{(j)}$ as follows: Define the map from $U_j^{(i)}$ to $U_i^{(j)}$ by

$$Z_i^{(0)} = \frac{Z_j^{(0)}}{Z_j^{(i)}}, \ldots, Z_i^{(i-1)} = \frac{Z_j^{(i-1)}}{Z_j^{(i)}}, \ldots, Z_i^{(i+1)} = \frac{Z_j^{(i+1)}}{Z_j^{(i)}}, \ldots, Z_i^{(j)} = \frac{1}{Z_j^{(i)}}, \ldots, Z_i^{(n)} = \frac{Z_j^{(n)}}{Z_j^{(i)}},$$

with the corresponding map on functions. Observe that the inverse of the above map is obtained by replacing $Z_j^{(i)}$ with $Z_i^{(j)}$. However, to continue glueing, we need a consistency requirement. Here is the abstract requirement.

Proposition 1.1 *(Glueing Lemma) Given a collection $(U_\alpha, \mathcal{O}_{U_\alpha})$ of LRS', suppose for all α, β, there exists an open $U_\alpha^\beta \subseteq U_\alpha$, with $U_\alpha^\alpha = U_\alpha$, and say there exist isomorphisms of LRS's, $\varphi_\alpha^\beta \colon (U_\alpha^\beta, \mathcal{O}_{U_\alpha} \restriction U_\alpha^\beta) \to (U_\beta^\alpha, \mathcal{O}_{U_\beta} \restriction U_\beta^\alpha)$, satisfying*

(0) $\varphi_\alpha^\alpha = \mathrm{id}$, *for all* α,

(1) $\varphi_\alpha^\beta = (\varphi_\beta^\alpha)^{-1}$, *for all* α, β *and*

(2) For all α, β, γ, we have $\varphi_\alpha^\beta(U_\alpha^\beta \cap U_\alpha^\gamma) = U_\beta^\alpha \cap U_\beta^\gamma$ and

$$\varphi_\alpha^\gamma = \varphi_\beta^\gamma \circ \varphi_\alpha^\beta \qquad \textit{(glueing condition or cocycle condition)}.$$

Then, there exists an LRS $(X, \mathcal{O}_X)$ so that X is covered by opens, X_α, and there are isomomorphisms of LRS's, $\varphi_\alpha \colon (U_\alpha, \mathcal{O}_{U_\alpha}) \to (X_\alpha, \mathcal{O}_X \restriction X_\alpha)$, in such a way that

(a) $\varphi_\alpha(U_\alpha^\beta) = X_\alpha \cap X_\beta$ $(\subseteq X_\alpha)$ *and*

(b) $\varphi_\alpha \restriction U_\alpha^\beta$ *"is" the isomorphism* φ_α^β, *i.e.,* $\varphi_\alpha \restriction U_\alpha^\beta = \varphi_\beta \restriction U_\beta^\alpha \circ \varphi_\alpha^\beta$.

Proof. (DX) $\square$

In Example (3), the consitency conditions hold (DX). Therefore, we get an algebraic (or analytic) variety. In fact, it turns our that in the analytic case, it is a manifold—this is $\mathbb{CP}^n$, also denoted $\mathbb{P}^n_{\mathbb{C}}$ in algebraic geometry (complex projective space of dimension n).

It should be noted that "bad glueing" can produce non-Hausdorff spaces, as the following simple example shows. Take two copies of $\mathbb{C}^1$, consider the two open $U = \mathbb{C}^1 - \{0\}$ in the first copy and $V = \mathbb{C}^1 - \{0\}$ in the second and use the coordinate z in the first copy and w in the second. Now, glue U and V by $w = z$. The result is a space consisting of a punctured line plus two points "above and below" the punctured line (as shown in Figure 1.1) and these points cannot be separated by any open.

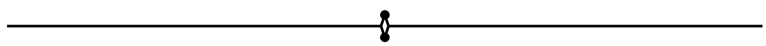

Figure 1.1: A non-Hausdorff space obtained by "bad gluing".

Miracle: Say X is a closed *analytic* subvariety of $\mathbb{P}^n_{\mathbb{C}}$ (analytic or algebraic). Then, X is algebraizable (Chow's theorem).

What are some of the topics that we would like to study in algebraic geometry?

(1) Algebraic varieties

(2) Maps between them.

(3) Structures to be superimposed on (1).

(4) Local and global invariants of (1).

(5) Classifications of (1).

(6) Constructions of (1).

But then, one might ask, why consider such general objects as algebraic varieties and why not just study affine varieties defined by equations of type (†)?

The reason is that affine varieties are just not enough. For example, classification problems generally cannot be tackled using only affine varieties; more general varieties come up naturally. The following example will illustrate this point.

Look at (5) and take $X = \mathbb{C}^n$. The general problem of classifying **all** subvarieties of $\mathbb{C}^n$ (in some geometric fashion) is very difficult, so we consider the easier problem of classifying all linear subvarieties through the origin of $\mathbb{C}^n$. In this case, there is a discrete invariant, namely, the dimension of the linear subspace, say d. Thus, we let

$$G(n, d) = \{\text{all } d\text{-dimensional linear subspaces of } \mathbb{C}^n\}.$$

By duality, there is a bijection between $G(n, d)$ and $G(n, n-d)$. Also, $G(n, 0) = G(n, n) =$ one point. We have the classification $\bigcup_{d=0}^n G(n, d)$. Let's examine $G(n, 1)$ more closely.

Let Σ be the unit sphere in $\mathbb{C}^n$,

$$\begin{aligned}
\Sigma &= \Big\{(z_1, \dots, z_n) \mid \sum_{i=1}^n |z_i|^2 = 1\Big\} \\
&= \Big\{(x_1, \dots, x_n, y_1, \dots, y_n) \mid x_i, y_i \in \mathbb{R}, \quad \sum_{i=1}^n (x_i^2 + y_i^2) = 1\Big\}.
\end{aligned}$$

We see that Σ is isomorphic to the real sphere S^{2n-1}, a compact space. Let L $(\in G(n,1))$ be any line through the origin. This line is given parametrically by the equations

$$z_j = \alpha_j t,$$

where $\alpha_1, \ldots, \alpha_n$ are fixed elements of $\mathbb{C}$, not all zero, and $t \in \mathbb{C}$ is arbitrary. It follows that

$$\begin{aligned} L \cap \Sigma &= \Big\{ t \in \mathbb{C} \mid \sum_{i=1}^{n} |\alpha_i|^2 |t|^2 = 1 \Big\} \\ &= \left\{ t \in \mathbb{C} \;\middle|\; |t| = \frac{1}{\sqrt{\sum_{i=1}^{n} |\alpha_i|^2}} \right\} \cong S^1. \end{aligned}$$

Therefore, $L \cap \Sigma$ is a circle. Since a (complex) line is determined by two points and the origin is one of these points, we deduce that

$$L = \widetilde{L} \quad \text{iff} \quad (L \cap \Sigma) \cap (\widetilde{L} \cap \Sigma) \neq \emptyset.$$

Therefore, as a topological space, $G(n,1) \cong S^{2n-1}/S^1$, a *compact* space.

Claim: No *affine* algebraic variety has a compact underlying space, unless it is a discrete space.

Given a variety V in $\mathbb{C}^n$ (algebraic or analytic), call V *irreducible* iff $V \neq W \cup Z$, for any two properly contained closed (algebraic or analytic) varieties, $W, Z \subset V$. It is well-known that each variety is an irredundant finite union of irreducible varieties and that a variety V is irreducible iff the (radical) ideal, $\mathfrak{I}(V)$, associated with V is a prime ideal. Thus, we are reduced to proving that no irreducible affine is compact. Now, as $\mathfrak{I}(V)$ is prime (because V is irreducible), the ring $\mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{I}(V)$, called the *affine coordinate ring of* V and denoted $\mathbb{C}[V]$ or $A[V]$, is an integral domain, so $K = \mathrm{Frac}(\mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{I}(V))$ is a field that contains $\mathbb{C}$. By definition, the transcendence degree, $\mathrm{tr.d}_{\mathbb{C}}\, K$, of K is the *dimension* of V, where $0 \leq \dim V \leq n$. If we let z_i be the image of Z_i under the projection $\mathbb{C}[Z_1, \ldots, Z_n] \longrightarrow \mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{I}(V)$, then $\mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{I}(V) = \mathbb{C}[z_1, \ldots, z_n]$. To prove our above claim, we will make use of a famous theorem of Emmy Noether:

Theorem 1.2 *(Noether Normalization Theorem) Say $V \subseteq \mathbb{C}^n$ is an irreducible affine variety and $\dim(V) = r$ $(\leq n)$. Then, if $\mathbb{C}[V] = \mathbb{C}[z_1, \ldots, z_n]$, there are some elements $y_1, \ldots, y_r \in \mathbb{C}[z_1, \ldots, z_n]$ so that each y_i is a linear combination of the z_j's and the ring $\mathbb{C}[z_1, \ldots, z_n]$ is an integral extension of $\mathbb{C}[y_1, \ldots, y_r]$. Geometrically, this means that the projection of $\mathbb{C}^n = \mathbb{C}^r \times \mathbb{C}^{n-r}$ onto $\mathbb{C}^r$ yields a surjective map (an integral morphism)*

$$V \hookrightarrow \mathbb{C}^n \xrightarrow{pr} \mathbb{C}^r$$

that is a branched covering (the fibres are finite). Furthermore, if $\mathbb{C}[V] = \mathbb{C}[z_1, \ldots, z_n]$ is separably generated over $\mathbb{C}$, then $\mathbb{C}[z_1, \ldots, z_n]$ is a separable extension of $\mathbb{C}[y_1, \ldots, y_r]$ (with $\{y_1, \ldots, y_r\}$ a separating transcendence basis over $\mathbb{C}$).

Proof. If $r = \dim(V) = n$, then we will prove later that $V = \mathbb{C}^n$ and we can take $y_i = z_i$, for $i = 1, \ldots, n$. Otherwise, $r < n$, and we use induction on n. The case $n = 1$, $r = 0$, is trivial. Owing to the transitivity of integral dependence and separability, we only have to prove: If $\mathbb{C}[z_1, \ldots, z_n]$ is an integral domain of transcendence degree $r \leq n-1$, then there exist $n-1$ linear combinations $y_1, \ldots, y_{n-1}$ or the z_j's such that $\mathbb{C}[z_1, \ldots, z_n]$ is integral over $\mathbb{C}[y_1, \ldots, y_{n-1}]$ (and such that $\mathbb{C}[z_1, \ldots, z_n]$ is separable over $\mathbb{C}[y_1, \ldots, y_{n-1}]$ if $\mathbb{C}[z_1, \ldots, z_n]$ is separably generated over $\mathbb{C}$).

By renumbering the z_i's if necessary, we may assume that z_1 is algebraically dependent over $z_2, \ldots, z_n$, and in the separable case, we pick a separating transcendence base (by MacLane's theorem). Write the minimal polynomial for z_1 over $k(z_2, \ldots, z_n)$ as

$$P(U, z_2, \ldots, z_n) = 0.$$

We can assume that the coefficients of $P(U, z_2, \ldots, z_n)$ are in $\mathbb{C}[z_2, \ldots, z_n]$, so that the polynomial $P(U, z_2, \ldots, z_n)$ is the result of substituting $U, z_2, \ldots, z_n$ for $X_1, X_2, \ldots, X_n$ in some non-zero polynomial $P(X_1, \ldots, X_n)$ with coefficients in $\mathbb{C}$. Perform the linear change of variables

$$y_j = z_j - a_j z_1, \quad \text{for } j = 2, \ldots, n, \tag{†}$$

and where $a_2, \ldots, a_n \in \mathbb{C}$ will be determined later. Since $z_j = y_j + a_j z_1$, it is sufficient to prove that z_1 is integral (and separable in the separable case) over $\mathbb{C}[y_2, \ldots, y_n]$. The minimal equation $P(z_1, z) = 0$ (abbreviating $P(z_1, z_2, \ldots, z_n)$ by $P(z_1, z)$) becomes

$$P(z_1, y_2 + a_2 z_1, \ldots, y_n + a_n z_1) = 0,$$

which can be written as

$$P(z_1, y) = z_1^q f(1, a_2, \ldots, a_n) + Q(z_1, y_2, \ldots, y_n) = 0, \tag{$**$}$$

where $f(X_1, X_2, \ldots, X_n)$ is the highest degree form of $P(X_1, \ldots, X_n)$ and q its degree, and Q contains terms of degree lower than q in z_1. If we can find some a_j's such that $f(1, a_2, \ldots, a_n) \neq 0$, then we have an integral dependence of z_1 on $y_2, \ldots, y_n$; thus, the z_j's are integrally dependent on $y_2, \ldots, y_n$, and we finish by induction. In the separable case, we need the minimal polynomial for z_1 to have a simple root, i.e.,

$$\frac{dP}{dz_1}(z_1, y) \neq 0.$$

We have

$$\frac{dP}{dz_1}(z_1, y) = \frac{\partial P}{\partial z_1}(z_1, z) + a_2 \frac{\partial P}{\partial z_2}(z_1, z) + \cdots + a_n \frac{\partial P}{\partial z_n}(z_1, z).$$

But this is a linear form in the a_j's which is not identically zero, since it takes for $a_2 = \cdots = a_n = 0$ the value

$$\frac{\partial P}{\partial z_1}(z_1, z) \neq 0,$$

z_1 being separable over $\mathbb{C}(z_2, \ldots, z_n)$. Thus, the equation

$$\frac{\partial P}{\partial x_1}(z_1, z) + a_2 \frac{\partial P}{\partial z_2}(z_1, z) + \cdots + a_n \frac{\partial P}{\partial z_n}(z_1, z) = 0$$

defines an affine hyperplane, i.e., the translate of a (linear) hyperplane. But then,

$$\frac{dP}{dz_1}(z_1, z) \neq 0$$

on the complement of a hyperplane, that is, an infinite open subset of $\mathbb{C}^{n-1}$, since $\mathbb{C}$ is infinite. On this infinite set where $\frac{dP}{dz_1}(z_1, z) \neq 0$, we can find $a_2, \ldots, a_n$ so that $f(1, a_2, \ldots, a_n) \neq 0$, which concludes the proof. $\square$

Now, we know that our $G(n, 1)$ cannot be affine (i.e., of the form (†)) as *it is* compact ($G(n, 1) \cong S^{2n-1}/S^1$). However, were $G(n, 1)$ affine, Noether's theorem would imply that $\mathbb{C}^r$ is compact, a contradiction. Therefore, $G(n, 1)$ is not an affine variety.

However, observe that $G(n, 1)$ is *locally* affine, i.e., it is an algebraic variety. Indeed a line, L, corresponds to a tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ with not all $\alpha_j = 0$. So, we can multiply by any $\lambda \in \mathbb{C}^*$ and not change the line. Look at

$$U_j = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \mid \alpha_j \neq 0\}.$$

We have $G(n, 1) = \bigcup_{n=1}^n U_j$. On U_j, if we use $\lambda = 1/\alpha_j$ as a multiplier, we get

$$\left(\frac{\alpha_1}{\alpha_j}, \ldots, \frac{\alpha_{j-1}}{\alpha_j}, 1, \frac{\alpha_{j+1}}{\alpha_j}, \ldots, \frac{\alpha_n}{\alpha_j}\right),$$

so we see that U_j is canonically $\mathbb{C}^{n-1}$. The patching on the overlaps is the previous glueing which gave $\mathbb{P}^{n-1}_{\mathbb{C}}$ (with its functions). Therefore, we have
$$G(n,1) = \mathbb{P}^{n-1}_{\mathbb{C}}.$$

1.2 Local Structure of Complex Varieties; Implicit Function Theorems and Tangent Spaces

We have the three rings
$$\mathbb{C}[Z_1,\ldots,Z_n] \subseteq \mathbb{C}\{Z_1,\ldots,Z_n\} \subseteq \mathbb{C}[[Z_1,\ldots,Z_n]],$$
where $\mathbb{C}\{Z_1,\ldots,Z_n\}$ is the ring of convergent powers series, which means that for every power series in $\mathbb{C}\{Z_1,\ldots,Z_n\}$ there is some open ball B_ϵ containing the origin so that $f \restriction B_\epsilon$ converges, and $\mathbb{C}[[Z_1,\ldots,Z_n]]$ is the ring of **all** power series, i.e., the ring of formal power series.

Remarks: (on (formal) power series).

Say A is a commutative ring (say, one of $\mathbb{C}\{Z_1,\ldots,Z_n\}$ or $\mathbb{C}[[Z_1,\ldots,Z_n]]$) and look at $A[[X_1,\ldots,X_n]] = B$.

(1) $f \in B$ is a unit (in B) iff $f(0,\ldots,0)$ is a unit of A.

(2) B is a local ring iff A is a local ring, in which case the unique maximal ideal of B is $\mathfrak{m}_B = \{f \in B \mid f(0,\ldots,0) \in \mathfrak{m}_A\}$.

(3) B is noetherian iff A is noetherian (OK for us).

(4) If A is a domain then B is a domain.

(5) If B is a local ring, write $\widehat{B} = \varprojlim_n B/\mathfrak{m}^n B$, the completion of B. We know that B has the $\mathfrak{m}$-adic topology, where a basis of opens at 0 is given by the $\mathfrak{m}_B^i$, with $i \geq 0$. The topology in $\widehat{B}$ is given by the $\mathfrak{m}_B^n\widehat{B}$ and the topology in B and $\widehat{B}$ is Hausdorff iff $\bigcap_{n=0}^{\infty} \mathfrak{m}_B^n = (0)$, which holds in the noetherian case, by Krull's theorem.

The fundamental results in this case are all essentially easy corollaries of the following lemma:

Lemma 1.3 *Let $\mathcal{O}$ be a complete Hausdorff local domain with respect to the $\mathfrak{m}$-adic topology, and let $f \in \mathcal{O}[[X]]$. Assume that*

(a) $f(0) \in \mathfrak{m}$.

(b) $\left(\frac{df}{dX}\right)(0)$ *is a unit of* $\mathcal{O}$.

Then, there exist unique elements $\alpha \in \mathfrak{m}$ and $u(X) \in \mathcal{O}[[X]]$, so that

(1) $u(X)$ *is a unit of* $\mathcal{O}[[X]]$.

(2) $f(X) = u(X)(X - \alpha)$.

Proof. We get $u(X)$ and α by successive approximations as follows. Refer to equation (2) by (†) in what follows. We compute the unknown coefficients of $u(X)$ and the element α by successive approximations. Write $u(X) = \sum_{j=0}^{\infty} u_j X^j$ and $f(X) = \sum_{j=0}^{\infty} a_j X^j$; reduce the coefficients modulo $\mathfrak{m}$ in (†); then, since $\alpha \in \mathfrak{m}$, (†) becomes
$$\overline{f(X)} = X\overline{u(X)},$$

which implies that

$$\sum_{j=0}^{\infty} \overline{a_j} X^j = \sum_{j=0}^{\infty} \overline{u_j} X^{j+1}.$$

Since $\overline{a_0} = 0$, we have $a_0 \in \mathfrak{m}$ and $\overline{u_j} = \overline{a_{j+1}}$. Thus,

$$u_j = a_{j+1} \pmod{\mathfrak{m}}.$$

Note that

$$\overline{u_0} = \overline{a_1} = \overline{\frac{\partial f}{\partial X}(0)} \neq 0$$

in $\kappa = \mathcal{O}/\mathfrak{m}$, which implies that if $u(X)$ exists at all, then it is a unit. Write

$$u_j = a_{j+1} + \xi_j^{(1)},$$

where $\xi_j^{(1)} \in \mathfrak{m}$, $j \geq 0$. Remember that $\alpha \in \mathfrak{m}$; so, upon reducing (†) modulo $\mathfrak{m}^2$, we get

$$\overline{\overline{f(X)}} = \overline{\overline{u(X)}}(X - \overline{\overline{\alpha}}).$$

This implies that

$$\begin{aligned}
\sum_{j=0}^{\infty} \overline{\overline{a_j}} X^j &= \sum_{j=0}^{\infty} \overline{\overline{u_j}} X^j (X - \overline{\overline{\alpha}}) \\
&= \sum_{j=0}^{\infty} \overline{\overline{u_j}} X^{j+1} - \sum_{j=0}^{\infty} \overline{\overline{u_j}}\, \overline{\overline{\alpha}} X^j \\
&= \sum_{j=0}^{\infty} \left(\overline{\overline{a}}_{j+1} + \overline{\overline{\xi_j^{(1)}}} \right) X^{j+1} - \sum_{j=0}^{\infty} \left(\overline{\overline{a}}_{j+1} + \overline{\overline{\xi_j^{(1)}}} \right) \overline{\overline{\alpha}} X^j \\
&= \sum_{j=0}^{\infty} \overline{\overline{a}}_{j+1} X^{j+1} + \sum_{j=0}^{\infty} \overline{\overline{\xi_j^{(1)}}} X^{j+1} - \sum_{j=0}^{\infty} \overline{\overline{a}}_{j+1}\, \overline{\overline{\alpha}} X^j.
\end{aligned}$$

Equating the constant coefficients, we get

$$\overline{\overline{a_0}} = -\overline{\overline{a_1}}\, \overline{\overline{\alpha}}.$$

Since a_1 is a unit, $\overline{\overline{\alpha}}$ exists. Now, looking at the coefficient of X^{j+1}, we get

$$\overline{\overline{a}}_{j+1} = \overline{\overline{a}}_{j+1} + \overline{\overline{\xi_j^{(1)}}} - \overline{\overline{a}}_{j+2} \overline{\overline{\alpha}},$$

which implies that

$$\overline{\overline{\xi_j^{(1)}}} = \overline{\overline{a}}_{j+2} \overline{\overline{\alpha}},$$

and $\overline{\overline{\xi_j^{(1)}}}$ exists.

We now proceed by induction. Assume that we know the coefficients $u_j^{(t)} \in \mathcal{O}$ of the t-th approximation to $u(X)$ and that $u(X)$ using these coefficients $\pmod{\mathfrak{m}^{t+1}}$ works in (†), and further that the $u_l^{(t)}$'s are consistent for $l \leq t$. Also, assume $\alpha^{(t)} \in \mathfrak{m}$, that $\alpha^{(t)} \pmod{\mathfrak{m}^{t+1}}$ works in (†), and that the $\alpha^{(l)}$ are consistent for $l \leq t$. Look at $u_j^{(t)} + \xi_j^{(t+1)}$, $\alpha^{(t)} + \eta^{(t+1)}$, where $\xi_j^{(t+1)}, \eta^{(t+1)} \in \mathfrak{m}^{t+1}$. We want to determine

$\xi_j^{(t+1)}$ and $\eta^{(t+1)}$, so that (†) will work for these modulo $\mathfrak{m}^{t+2}$. For simplicity, write bar as a superscript to denote reduction modulo $\mathfrak{m}^{t+2}$. Then, reducing (†) modulo $\mathfrak{m}^{t+2}$, we get

$$\begin{aligned}\sum_{j=0}^{\infty}\overline{a_j}X^j &= \sum_{j=0}^{\infty}\overline{u_j}X^j(X-\overline{\alpha})\\ &= \sum_{j=0}^{\infty}\overline{u_j}X^{j+1}-\sum_{j=0}^{\infty}\overline{u_j}\,\overline{\alpha}X^j\\ &= \sum_{j=0}^{\infty}\left(\overline{u_j^{(t)}}+\overline{\xi_j^{(t+1)}}\right)X^{j+1}-\sum_{j=0}^{\infty}\left(\overline{u_j^{(t)}}+\overline{\xi_j^{(t+1)}}\right)\left(\overline{\alpha^{(t)}}+\overline{\eta^{(t+1)}}\right)X^j\\ &= \sum_{j=0}^{\infty}\overline{u_j^{(t)}}X^{j+1}+\sum_{j=0}^{\infty}\overline{\xi_j^{(t+1)}}X^{j+1}-\sum_{j=0}^{\infty}\overline{u_j^{(t)}}\,\overline{\alpha^{(t)}}X^j-\sum_{j=0}^{\infty}\overline{u_j^{(t)}}\,\overline{\eta^{(t+1)}}X^j.\end{aligned}$$

Equating the constant coefficients, we get

$$\overline{a_0}=-\overline{u_0^{(t)}}\,\overline{\alpha^{(t)}}-\overline{u_0^{(t)}}\,\overline{\eta^{(t+1)}}.$$

But $\overline{u_0^{(t)}}$ is a unit, and so, $\overline{\eta^{(t+1)}}$ exists. Now, look at the coefficient of X^{j+1}, we have

$$\overline{a_{j+1}}=\overline{u_j^{(t)}}+\overline{\xi_j^{(t+1)}}-\overline{u_{j+1}^{(t)}}\,\overline{\alpha^{(t)}}-\overline{u_{j+1}^{(t)}}\,\overline{\eta^{(t+1)}}.$$

But $\overline{u_{j+1}^{(t)}}\,\overline{\alpha^{(t)}}$ and $\overline{u_{j+1}^{(t)}}\,\overline{\eta^{(t+1)}}$ are now known and in $\mathfrak{m}^{t+1}$ modulo $\mathfrak{m}^{t+2}$, and thus,

$$\overline{\xi_j^{(t+1)}}=\overline{a_{j+1}}-\overline{u_j^{(t)}}+\overline{u_{j+1}^{(t)}}\,\overline{\alpha^{(t)}}+\overline{u_{j+1}^{(t)}}\,\overline{\eta^{(t+1)}}$$

exists and the induction step goes through. As a consequence

$$u(X)\in\varprojlim_t(\mathcal{O}/\mathfrak{m}^t)[[X]]$$

and

$$\alpha\in\varprojlim_t(\mathfrak{m}/\mathfrak{m}^t)[[X]]$$

exist; and so, $u(X)\in\widehat{\mathcal{O}}[[X]]=\mathcal{O}[[X]]$, and $\alpha\in\widehat{\mathfrak{m}}=\mathfrak{m}$.

We still have to prove the uniqueness of $u(X)$ and α. Assume that

$$f=u(X-\alpha)=\widetilde{u}(X-\widetilde{\alpha}).$$

Since $\widetilde{u}$ is a unit,

$$\widetilde{u}^{-1}u(X-\alpha)=X-\widetilde{\alpha}.$$

Thus, we may assume that $\widetilde{u}=1$. Since $\alpha\in\mathfrak{m}$, we can plug α into the power series which defines u, and get convergence in the $\mathfrak{m}$-adic topology of $\mathcal{O}$. We get

$$u(\alpha)(\alpha-\alpha)=\alpha-\widetilde{\alpha},$$

so that $\alpha=\widetilde{\alpha}$. Then,

$$u(X-\alpha)=X-\alpha,$$

and since we assumed that $\mathcal{O}$ is a domain, so is $\mathcal{O}[[X]]$, and thus, $u=1$. $\square$

Suppose that instead of $\frac{df}{dX}(0)$ being a unit, we have $f(0),\ldots,\frac{d^{r-1}f}{dX^{r-1}}(0)\in\mathfrak{m}$, but $\frac{d^rf}{dX^r}(0)$ is a unit. We can apply the fundamental lemma to $\widetilde{f}=\frac{d^{r-1}f}{dX^{r-1}}$ and then, we can apply (a rather obvious) induction and get the general form of the fundamental lemma:

Lemma 1.4 *Let $\mathcal{O}$ be a complete Hausdorff local domain with respect to the $\mathfrak{m}$-adic topology, and let $f \in \mathcal{O}[[X]]$. Assume that*

(a) $f(0), \ldots, \frac{d^{r-1}f}{dX^{r-1}}(0) \in \mathfrak{m}$ *and*

(b) $\frac{d^r f}{d^r X}(0)$ *is a unit of* $\mathcal{O}$ $(r \geq 1)$.

Then, there exist unique elements $\alpha_1, \ldots, \alpha_r \in \mathfrak{m}$ and a unique power series $u(X) \in \mathcal{O}[[X]]$, so that

$$f(X) = u(X)(X^r + \alpha_1 X^{r-1} + \cdots + \alpha_{r-1}X + \alpha_r)$$

and $u(X)$ is a unit of $\mathcal{O}[[X]]$.

From the above, we get

Theorem 1.5 *(Formal Weierstrass Preparation Theorem) Given $f \in \mathbb{C}[[Z_1, \ldots, Z_n]]$, suppose*

$$f(0, \ldots, 0) = \frac{\partial f}{\partial Z_1}(0) = \cdots = \frac{\partial^{r-1} f}{\partial Z_1^{r-1}}(0) = 0, \quad \textit{yet} \quad \frac{\partial^r f}{\partial Z_1^r}(0) \neq 0.$$

Then, there exist unique power series, $u(Z_1, \ldots, Z_n)$, $g_j(Z_2, \ldots, Z_n)$, with $1 \leq j \leq r$, so that

(1) $u(Z_1, \ldots, Z_n)$ *is a unit*

(2) $g_j(0, \ldots, 0) = 0$, *where* $1 \leq j \leq r$ *and*

(3) $f(Z_1, \ldots, Z_n) = u(Z_1, \ldots, Z_n)(Z_1^r + g_1(Z_2, \ldots, Z_n)Z_1^{r-1} + \cdots + g_r(Z_2, \ldots, Z_n))$.

Proof. Let $\mathcal{O} = \mathbb{C}[[Z_2, \ldots, Z_n]]$, then $\mathcal{O}[[Z_1]] = \mathbb{C}[[Z_1, \ldots, Z_n]]$ and we have

$$\frac{df}{dZ_1} = \frac{\partial f}{\partial Z_1}.$$

Thus, $\frac{d^j f}{dZ_1^j}(0) = 0$ for $j = 0, \ldots, r-1$, so $\frac{d^j f}{dZ_1^j}(0)$ is a non-unit in $\mathcal{O}[[Z_1]]$ for $j = 0, \ldots, r-1$, yet $\frac{d^r f}{dZ_1^r}(0)$ *is* a unit. If we let α_j of the fundamental lemma be $g_j(Z_2, \ldots, Z_n)$, then each $g_j(Z_2, \ldots, Z_n)$ vanishes at 0 (else $\alpha_j \notin \mathfrak{m}$) and the rest is obvious. $\square$

Theorem 1.6 *(First form of the implicit function theorem) Given $f \in \mathbb{C}[[Z_1, \ldots, Z_n]]$, if*

$$f(0, \ldots, 0) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial Z_1}(0) \neq 0,$$

then there exist unique power series $u(Z_1, \ldots, Z_n)$ and $g(Z_2, \ldots, Z_n)$ so that $u(Z_1, \ldots, Z_n)$ is a unit, $g(0, \ldots, 0) = 0$, and $f(Z_1, \ldots, Z_n)$ factors as

$$f(Z_1, \ldots, Z_n) = u(Z_1, \ldots, Z_n)(Z_1 - g(Z_2, \ldots, Z_n)). \tag{$*$}$$

Moreover, every power series $h(Z_1, \ldots, Z_n)$ factors uniquely as

$$h(Z_1, \ldots, Z_n) = f(Z_1, \ldots, Z_n)q(Z_1, \ldots, Z_n) + r(Z_2, \ldots, Z_n).$$

Hence, there is a canonical isomorphism

$$\mathbb{C}[[Z_1, \ldots, Z_n]]/(f) \cong \mathbb{C}[[Z_2, \ldots, Z_n]],$$

so that the following diagram commutes

$$\begin{tikzcd}
\mathbb{C}[[Z_1, \ldots, Z_n] \arrow[rr] & & \mathbb{C}[[Z_1, \ldots, Z_n]]/(f) \\
& \mathbb{C}[[Z_2, \ldots, Z_n] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. Observe that equation $(*)$ (the Weierstrass preparation theorem) implies the second statement. For, assume $(*)$; then, u is a unit, so there is v such that $vu = 1$. Consequently,

$$vf = Z_1 - g(Z_2, \ldots, Z_n),$$

and the ideal (f) equals the ideal (vf), because v is a unit. So,

$$\mathbb{C}[[Z_1, \ldots, Z_n]]/(f) = \mathbb{C}[[Z_1, \ldots, Z_n]]/(vf),$$

and we get the residue ring by setting Z_1 equal to $g(Z_2, \ldots, Z_n)$. It follows that the canonical isomorphism

$$\mathbb{C}[[Z_1, \ldots, Z_n]]/(f) \cong \mathbb{C}[[Z_2, \ldots, Z_n]]$$

is given as follows: In $h(Z_1, \ldots, Z_n)$, replace every occurrence of Z_1 by $g(Z_2, \ldots, Z_n)$; we obtain

$$\overline{h}(Z_2, \ldots, Z_n) = h(g(Z_2, \ldots, Z_n), Z_2, \ldots, Z_n),$$

and the diagram obviously commutes. Write $r(Z_2, \ldots, Z_n)$ instead of $\overline{h}(Z_2, \ldots, Z_n)$. Then,

$$h(Z_1, \ldots, Z_n) - r(Z_2, \ldots, Z_n) = fq$$

for some $q(Z_1, \ldots, Z_n)$. We still have to show uniqueness. Assume that

$$h(Z_1, \ldots, Z_n) = fq + r = f\widetilde{q} + \widetilde{r}.$$

Since $g(0, \ldots, 0) = 0$, we have $g \in \mathfrak{m}$; thus, we can plug in $Z_1 = g(Z_2, \ldots, Z_n)$ and get $\mathfrak{m}$-adic convergence. By $(*)$, f goes to 0, and the commutative diagram shows $r \pmod f = r$ and $\widetilde{r} \pmod f = \widetilde{r}$. Hence, we get

$$r = \widetilde{r},$$

so that

$$fq - f\widetilde{q} = 0.$$

Now, $\mathbb{C}[[Z_1, \ldots, Z_n]]$ is a domain, so $q = \widetilde{q}$. $\square$

We can now apply induction to get the second version of the *formal implicit function theorem, or FIFT*.

Theorem 1.7 *(Second form of the formal implicit function theorem)*
Given $f_1, \ldots, f_r \in \mathbb{C}[[Z_1, \ldots, Z_n]]$, if $f_j(0, \ldots, 0) = 0$ for $j = 1, \ldots, r$ and

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right) = r$$

(so that $n \geq r$), then we can reorder the variables so that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right) = r, \quad \textit{where } 1 \leq i, j \leq r,$$

and there is a canonical isomorphism

$$\mathbb{C}[[Z_1, \ldots, Z_n]]/(f_1, \ldots, f_r) \cong \mathbb{C}[[Z_{r+1}, \ldots, Z_n]],$$

which makes the following diagram commute

$$\begin{tikzcd}
\mathbb{C}[[Z_1, \ldots, Z_n]] \arrow[rr] & & \mathbb{C}[[Z_1, \ldots, Z_n]]/(f_1, \ldots, f_r) \\
& \mathbb{C}[[Z_{r+1}, \ldots, Z_n]] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. The proof of this statement is quite simple (using induction) from the previous theorem (DX). $\square$

What becomes of these results for the convergent case? They hold because we can make estimates showing all processes converge for $\mathbb{C}$. However, these arguments are tricky and messy (they can be found in Zariski and Samuel, Volume II [15]). In our case, we can use complex analysis. For any $\xi \in \mathbb{C}^n$ and any $\epsilon > 0$, we define the *polydisc of radius ϵ about ξ*, $PD(\xi, \epsilon)$, by

$$PD(\xi, \epsilon) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i - \xi_i| < \epsilon, \text{ for every } i,\ 1 \leq i \leq n\}.$$

Say $f(Z_1, \ldots, Z_n)$ is holomorphic near the origin and suppose $f(0, \ldots, 0) = \frac{\partial f}{\partial Z_1}(0) = \cdots = \frac{\partial^{r-1} f}{\partial Z_1^{r-1}}(0) = 0$ and $\frac{\partial^r f}{\partial Z_1^r}(0) \neq 0$, Consider f as a function of Z_1, with $\|(Z_2, \ldots, Z_n)\| < \epsilon$, for some $\epsilon > 0$. Then, f will have r zeros, each as a function of $Z_2, \ldots, Z_n$ in the ϵ-disc. Now, we know (by one-dimensional Cauchy theory) that

$$\eta_1^q + \cdots + \eta_r^q = \frac{1}{2\pi i} \int_{|\xi| = R} \frac{\xi^q \frac{\partial f}{\partial Z_1}(\xi, Z_2, \ldots, Z_n)}{f(\xi, Z_2, \ldots, Z_n)}\, d\xi,$$

where $\eta_1, \ldots, \eta_r$ are the roots (as functions of $Z_2, \ldots, Z_n$) and q is any integer ≥ 0. Therefore, the power sums of the roots are holomorphic functions of $Z_2, \ldots, Z_n$. By Newton's identities, the elementary symmetric functions $\sigma_j(\eta_1, \ldots, \eta_r)$, for $j = 1, \ldots, r$, are polynomials in the power sums, call these elementary symmetric functions $g_1(Z_2, \ldots, Z_n), \ldots, g_r(Z_2, \ldots, Z_n)$. Then, the polynomial

$$w(Z_1, \ldots, Z_n) = Z_1^r - g_1(Z_2, \ldots, Z_n) Z_1^{r-1} + \cdots + (-1)^r g_r(Z_2, \ldots, Z_n)$$

vanishes *exactly* where f vanishes. Look at $f(Z_1, \ldots, Z_n)/w(Z_1, \ldots, Z_n)$ off the zeros. Then, the latter as a function of Z_1 has only removable singularities. Thus, by Riemann's theorem, this function extends to a *holomorphic* function in Z_1 on the whole disc. As f/w is holomorphic in Z_1, by the Cauchy integral formula, we get

$$u(Z_1, \ldots, Z_n) = \frac{f(Z_1, \ldots, Z_n)}{w(Z_1, \ldots, Z_n)} = \frac{1}{2\pi i} \int_{|\xi| = R} \frac{u(\xi, Z_2, \ldots, Z_n)}{\xi - Z_1}\, d\xi.$$

Yet, the right hand side is holomorphic in $Z_2, \ldots, Z_n$, which means that $u(Z_1, \ldots, Z_n)$ is holomorphic in a polydisc and as we let the Z_j go to 0, the function $u(Z_1, \ldots, Z_n)$ does not vanish as the zeros cancel. Consequently, we have

$$f(Z_1, \ldots, Z_n) = u(Z_1, \ldots, Z_n)(Z_1^r + g_1(Z_2, \ldots, Z_n) Z_1^{r-1} + \cdots + g_r(Z_2, \ldots, Z_n)).$$

Theorem 1.8 *(Weierstrass Preparation Theorem (Convergent Case)) Given $f \in \mathbb{C}\{Z_1, \ldots, Z_n\}$, suppose*

$$f(0, \ldots, 0) = \frac{\partial f}{\partial Z_1}(0) = \cdots = \frac{\partial^{r-1} f}{\partial Z_1^{r-1}}(0) = 0, \quad \textit{yet} \quad \frac{\partial^r f}{\partial Z_1^r}(0) \neq 0.$$

Then, there exist unique power series, $u(Z_1, \ldots, Z_n)$, $g_j(Z_2, \ldots, Z_n)$ in $\mathbb{C}\{Z_1, \ldots, Z_n\}$, with $1 \leq j \leq r$, and some $\epsilon > 0$, so that

(1) $u(Z_1, \ldots, Z_n)$ is a unit

(2) $g_j(0, \ldots, 0) = 0$, where $1 \leq j \leq r$ and

(3) $f(Z_1, \ldots, Z_n) = u(Z_1, \ldots, Z_n)(Z_1^r + g_1(Z_2, \ldots, Z_n) Z_1^{r-1} + \cdots + g_r(Z_2, \ldots, Z_n))$, in some polydisc $PD(0, \epsilon)$.

Proof. Existence has already been proved. If more than one solution exists, read in $\mathbb{C}[[Z_1, \ldots, Z_n]]$ and apply uniqueness there. $\square$

As a consequence, we obtain the implicit function theorem and the inverse function theorem in the convergent case.

Theorem 1.9 *(Implicit Function Theorem (First Form–Convergent Case)) Let $f \in \mathbb{C}\{Z_1, \ldots, Z_n\}$ and suppose that $f(0, \ldots, 0) = 0$, but*

$$\frac{\partial f}{\partial Z_1}(0, \ldots, 0) \neq 0.$$

Then, there exists a unique power series $g(Z_2, \ldots, Z_n) \in \mathbb{C}\{Z_2 \ldots, Z_n\}$ and there is some $\epsilon > 0$, so that in the polydisc $PD(0, \epsilon)$, we have $f(Z_1, \ldots, Z_n) = 0$ if and only if $Z_1 = g(Z_2, \ldots, Z_n)$. Furthermore, the map $h \mapsto \widetilde{h} = h(g(Z_2, \ldots, Z_n), Z_2, \ldots, Z_n)$ gives rise to the commutative diagram

$$\begin{tikzcd} \mathbb{C}\{Z_1, \ldots, Z_n\} \arrow[rr] & & \mathbb{C}\{Z_1, \ldots, Z_n\}/(f) \\ & \mathbb{C}\{Z_2, \ldots, Z_n\} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof. (DX). $\square$

An easy induction yields

Theorem 1.10 *(Convergent implicit function theorem) Let $f_1, \ldots, f_r \in \mathbb{C}\{Z_1, \ldots, Z_n\}$. If $f_j(0, \ldots, 0) = 0$ for $j = 1, \ldots, r$ and*

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right) = r$$

(so that $n \geq r$), then there is a permutation of the variables so that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right) = r, \quad \textit{where } 1 \leq i, j \leq r$$

and there exist r unique power series $g_j(Z_{r+1}, \ldots, Z_n) \in \mathbb{C}\{Z_{r+1} \ldots, Z_n\}$ $(1 \leq j \leq r)$ and an $\epsilon > 0$, so that in the polydisc $PD(0, \epsilon)$, we have

$$f_1(\xi) = \cdots = f_r(\xi) = 0 \quad \textit{iff} \quad \xi_j = g_j(\xi_{r+1}, \ldots, \xi_n), \quad \textit{for } j = 1, \ldots, r.$$

Moreover

$$\mathbb{C}\{Z_1, \ldots, Z_n\}/(f_1, \ldots, f_r) \cong \mathbb{C}\{Z_{r+1}, \ldots, Z_n\}$$

and the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}\{Z_1, \ldots, Z_n\} \arrow[rr] & & \mathbb{C}\{Z_1, \ldots, Z_n\}/(f_1, \ldots, f_r) \\ & \mathbb{C}\{Z_{r+1}, \ldots, Z_n\} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

When $r = n$, we have another form of the convergent implicit function theorem also called the *inverse function theorem.*

Theorem 1.11 *(Inverse function theorem) Let $f_1, \ldots, f_n \in \mathbb{C}\{Z_1, \ldots, Z_n\}$ and suppose that $f_j(0, \ldots, 0) = 0$ for $j = 1, \ldots, n$, but*

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0, \ldots, 0)\right) = n.$$

Then, there exist n unique power series $g_j(W_1, \ldots, W_n) \in \mathbb{C}\{W_1 \ldots, W_n\}$ $(1 \leq j \leq n)$ and there are some open neighborhoods of $(0, \ldots, 0)$ (in the Z's and in the W's), call them U and V, so that the holomorphic maps

$$(Z_1, \ldots, Z_n) \mapsto (W_1 = f_1(Z_1, \ldots, Z_n), \ldots, W_n = f_n(Z_1, \ldots, Z_n)) \colon U \to V$$
$$(W_1, \ldots, W_n) \mapsto (Z_1 = g_1(W_1, \ldots, W_n), \ldots, Z_n = g_n(W_1, \ldots, W_n)) \colon V \to U$$

are inverse isomorphisms.

In order to use these theorems, we need a linear analysis *via* some kind of "tangent space." Recall that a variety, V, is a union of affine opens

$$V = \bigcup_{\alpha} V_\alpha.$$

Take $\xi \in V$, then there is some α (perhaps many) so that $\xi \in V_\alpha$. Therefore, assume at first that V *is* affine and say $V \subseteq \mathbb{C}^n$ and is cut out by the radical ideal $\mathfrak{A} = \mathfrak{I}(V) = (f_1, \ldots, f_p)$. Pick any $f \in \mathfrak{A}$ and write the Taylor expansion for f at $\xi \in \mathbb{C}^n$:

$$f(Z_1, \ldots, Z_n) = f(\xi) + \sum_{j=1}^{n} \left(\frac{\partial f}{\partial Z_j}\right)_\xi (Z_j - \xi_j) + O(\text{quadratic}).$$

Since $\xi \in V$, we have $f(\xi) = 0$. This suggests looking at the linear form

$$l_{f,\xi}(\lambda_1, \ldots, \lambda_n) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial Z_j}\right)_\xi \lambda_j, \quad \text{where} \quad \lambda_j = Z_j - \xi_j.$$

Examine the linear subspace

$$\bigcap_{f \in \mathfrak{A}} \operatorname{Ker} l_{f,\xi} = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \;\middle|\; (\forall f \in \mathfrak{A}) \left(\sum_{j=1}^{n} \left(\frac{\partial f}{\partial Z_j}\right)_\xi \lambda_j = 0 \right) \right\}. \tag{$*$}$$

Note that as $f = \sum_{i=1}^{t} h_i f_i$, where $h_i \in \mathbb{C}[Z_1, \ldots, Z_n]$, we get

$$\frac{\partial f}{\partial Z_j} = \sum_{i=1}^{t} \left(h_i \frac{\partial f_i}{\partial Z_j} + f_i \frac{\partial h_i}{\partial Z_j} \right),$$

and, since $f_i(\xi) = 0$,

$$\left(\frac{\partial f}{\partial Z_j}\right)_\xi = \sum_{i=1}^{t} h_i(\xi) \left(\frac{\partial f_i}{\partial Z_j}\right)_\xi.$$

The equation in $(*)$ becomes

$$\sum_{j=1}^{n} \sum_{i=1}^{t} \left(h_i(\xi) \left(\frac{\partial f_i}{\partial Z_j}\right)_\xi \right) (Z_j - \xi_j) = 0,$$

which yields

$$\sum_{i=1}^{t} h_i(\xi) \left(\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial Z_j}\right)_\xi (Z_j - \xi_j) \right) = 0.$$

Hence, the vector space defined by $(*)$ is also defined by

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial Z_j}\right)_\xi (Z_j - \xi_j) = 0, \quad \text{for } i = 1, \ldots, t. \tag{$**$}$$

Definition 1.1 The linear space at $\xi \in V$ defined by $(**)$ is called the *Zariski tangent space at* ξ *of* V. It is denoted by $T_{V,\xi}$.

Note that Definition 1.1 is an extrinsic definition. It depends on the embedding of V in $\mathbb{C}^n$, but assume for the moment that it is independent of the embedding. We have the following proposition:

Proposition 1.12 *Let V be an irreducible complex variety. The function*

$$\xi \mapsto \dim_{\mathbb{C}} T_{V,\xi}$$

is upper-semicontinuous on V, i.e.,

$$S_l = \{\xi \in V \mid \dim_{\mathbb{C}} T_{V,\xi} \geq l\}$$

is Z-closed in V, and furthermore, $S_{l+1} \subseteq S_l$ and it is Z-closed in S_l.

Proof. Since we are assuming that $T_{V,\xi}$ is independent of the particular affine patch where ξ finds itself, we may assume that V is affine. So, $T_{V,\xi}$ is the vector space given by the set of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial Z_j}\right)_{\xi} \lambda_j = 0, \quad \text{for } i = 1, \ldots, m,$$

where $f_1, \ldots, f_m$ generate the ideal $\mathfrak{A} = \mathfrak{I}(V)$. Hence, $T_{V,\xi}$ is the kernel of the linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ given by the $m \times n$ matrix

$$\left(\left(\frac{\partial f_i}{\partial Z_j}\right)_{\xi}\right).$$

It follows that

$$\dim_{\mathbb{C}} T_{V,\xi} = n - \mathrm{rk}\left(\left(\frac{\partial f_i}{\partial Z_j}\right)_{\xi}\right).$$

Consequently, $\dim_{\mathbb{C}} T_{V,\xi} \geq l$ iff

$$\mathrm{rk}\left(\left(\frac{\partial f_i}{\partial Z_j}\right)_{\xi}\right) \leq n - l;$$

and this holds iff the $(n-l+1) \times (n-l+1)$ minors are all singular at ξ. But the latter is true when and only when the corresponding determinants vanish at ξ. These give additional equations on V at ξ in order that $\xi \in S_l$ and this implies that S_l is Z-closed in V. That $S_{l+1} \subseteq S_l$ is obvious and since S_{l+1} is given by more equations, S_{l+1} is Z-closed in S_l. $\square$

We now go back to the question: Is the definition of the tangent space intrinsic?

It is possible to give an intrinsic definition. For this, we review the notion of $\mathbb{C}$-derivation. Let M be a $\mathbb{C}$-module and recall that $A[V] = \mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{I}(V)$, the affine coordinate ring of V.

Definition 1.2 A $\mathbb{C}$-*derivation of $A[V]$ with values in M centered at ξ* consists of the following data:

(1) A $\mathbb{C}$-linear map $D\colon A[V] \to M$. (values in M)

(2) $D(fg) = f(\xi)Dg + g(\xi)Df$ (Leibnitz rule) (centered at ξ)

(3) $D(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. ($\mathbb{C}$-derivation)

The set of such derivations is denoted by $\mathrm{Der}_{\mathbb{C}}(A[V], M; \xi)$.

The composition

$$\mathbb{C}[Z_1, \ldots, Z_n] \longrightarrow A[V] \xrightarrow{D} M$$

is again a $\mathbb{C}$-derivation (on the polynomial ring) centered at ξ with values in M. Note that a $\mathbb{C}$-derivation on the polynomial ring (call it D again) factors as above iff $D \restriction \mathfrak{I}(V) = 0$. This shows that

$$\mathrm{Der}_{\mathbb{C}}(A[V], M; \xi) = \{D \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[Z_1, \ldots, Z_n], M; \xi) \mid D \restriction \mathfrak{I}(V) = 0\}.$$

However, a $\mathbb{C}$-derivation $D \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[Z_1, \ldots, Z_n], M; \xi)$ is determined by its values $D(Z_j) = \lambda_j$ at the variables Z_j. Clearly (DX),

$$D(f(Z_1, \ldots, Z_n)) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial Z_j}\right)_{\xi} D(Z_j).$$

But, observe that for any $(\lambda_1, \ldots, \lambda_n)$, the restriction of D to $\mathfrak{I}(V)$ vanishes iff

$$\sum_{j=1}^{n} \left(\frac{\partial f}{\partial Z_j}\right)_{\xi} \lambda_j = 0, \quad \text{for every } f \in \mathfrak{I}(V),$$

that is, iff

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial Z_j}\right)_{\xi} \lambda_j = 0, \quad \text{for every } i = 1, \ldots, m,$$

where $f_1, \ldots, f_m$ generate the ideal $\mathfrak{I}(V)$. Letting $\eta_j = \lambda_j + \xi_j \in M$ (with $\xi_i \in M$), we have a bijection between
$\mathrm{Der}_{\mathbb{C}}(A[V], M; \xi)$ and

$$\left\{ (\eta_1, \ldots, \eta_n) \in M^n \;\middle|\; \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial Z_j}\right)_{\xi} \lambda_j = 0,\; 1 \le i \le m \right\}.$$

It is given by the map

$$D \mapsto (\eta_1, \ldots, \eta_n),$$

with $\eta_j = D(Z_j) + \xi_j$. This gives the isomorphism

$$T_{V,\xi} \cong \mathrm{Der}_{\mathbb{C}}(A[V], \mathbb{C}; \xi).$$

We conclude that $T_{V,\xi}$ is independent of the embedding of V into $\mathbb{C}^n$, up to isomorphism.

Take V to be irreducible to avoid complications. Then, $A[V]$ is an integral domain (as $\mathfrak{I}(V)$ is a prime ideal) and so, $\mathcal{O}_{V,\xi} = A[V]_{\mathfrak{I}(\xi)}$, the localization of $A[V]$ at the prime ideal $\mathfrak{I}(\xi)$ consisting of all $g \in A[V]$ where $g(\xi) = 0$. This is because elements of the local ring $\mathcal{O}_{V,\xi}$ are equivalence classes of ratios f/g, where $f, g \in A[V]$ with $g(\xi) \neq 0$ (where g is zero is Z-closed and so, the latter is Z-open), with $f/g \sim \widetilde{f}/\widetilde{g}$ iff f/g and $\widetilde{f}/\widetilde{g}$ agree on a small neighborhood of ξ. On the Z-open where $g\widetilde{g} \neq 0$, we get $f\widetilde{g} - g\widetilde{f} = 0$ iff $f/g \sim \widetilde{f}/\widetilde{g}$. By analytic continuation, we get $f\widetilde{g} - g\widetilde{f} = 0$ in $A[V]$. Therefore, $\mathcal{O}_{V,\xi} = A[V]_{\mathfrak{I}(\xi)}$.

It follows that

$$\mathcal{O}_{V,\xi} = \left\{ \left[\frac{f}{g}\right] \;\middle|\; f, g \in A[V],\; g \notin \mathfrak{I}(\xi) \right\} = \left\{ \left[\frac{f}{g}\right] \;\middle|\; f, g \in A[V],\; g(\xi) \neq 0 \right\}.$$

Any $\mathbb{C}$-derivation $D \in \mathrm{Der}_{\mathbb{C}}(A[V], M; \xi)$ is uniquely extendable to $\mathcal{O}_{V,\xi}$ *via*

$$D\left(\frac{f}{g}\right) = \frac{g(\xi)Df - f(\xi)Dg}{g(\xi)^2}.$$

Therefore,

$$\mathrm{Der}_{\mathbb{C}}(A[V], \mathbb{C}; \xi) = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{V,\xi}, \mathbb{C}; \xi).$$

There are some difficulties when V is reducible. As an example in $\mathbb{C}^3$, consider the union of a plane and an algebraic curve piercing that plane, with ξ any point of intersection.

Remark: Since the S_l manifestly form a nonincreasing chain as l increases, there is a largest l for which $S_l = V$. The set S_{l+1} is closed in V, and its complement $\{\xi \mid \dim_{\mathbb{C}} T_{V,\xi} = l\}$ is Z-open. This gives us the *tangent space stratification* (a disjoint union) by locally closed sets (a locally closed set is the intersection of an open set with a closed set)

$$V = U_0 \uplus U_1 \uplus \cdots \uplus U_t,$$

where $U_0 = \{\xi \mid \dim_{\mathbb{C}} T_{V,\xi} = l\}$ is open, and $U_i = \{\xi \mid \dim_{\mathbb{C}} T_{V,\xi} = l + i\}$. We have U_1 open in $V - U_0 = S_{l+1}$, etc.

Now, we have the first main result.

Theorem 1.13 *Say V is a complex variety and suppose $\dim_{\mathbb{C}} V = \sup_\alpha \dim_{\mathbb{C}} V_\alpha < \infty$, where V_α is an affine open in V), e.g., V is quasi-compact (which means that V is a finite union of open affines). Then, there is a nonempty open (in fact, Z-open), U, in V so that for all $\xi \in U$,*

$$\dim_{\mathbb{C}} T_{V,\xi} = \dim_{\mathbb{C}} V.$$

Moreover, for all $\xi \in V$, if V is irreducible, then $\dim_{\mathbb{C}} T_{V,\xi} \geq \dim_{\mathbb{C}} V$.

Proof. We have $V = \bigcup_\alpha V_\alpha$, where each V_α is affine open and $\dim_{\mathbb{C}} V = \sup_\alpha \dim_{\mathbb{C}} V_\alpha \leq \infty$ so that $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V_\alpha$, for some α and if the first statement of the theorem is true for V_α, then there is some open, $U \subseteq V_\alpha$, and as V_α is open itself, U is an open in V with the desired property. So, we may assume that V is affine. Let

$$V = V_1 \cup \cdots \cup V_t$$

be an irredundant decomposition into irreducible components. At least one of the V_j's has dimension $\dim(V)$. Say it is $j = 1$. Look at $V_1 \cap V_j$, $j = 2, \ldots, t$. Each $V_1 \cap V_j$ is a closed set, and so

$$W = V - \bigcup_{j=2}^{t} V_1 \cap V_j$$

is Z-open. Also, $W \cap V_1$ is Z-open in V_1 because it is the complement of all the closed sets $V_1 \cap V_j$ with $j \geq 2$. Take any open subset, U, of $V - \bigcup_{j=2}^{t} V_j$ for which U is a good open in V_1, that is, where $\dim_{\mathbb{C}} T_{V,\xi} = \dim_{\mathbb{C}} V_1$ whenever $\xi \in U$. Then, $U \cap W$ also has the right property. Hence, we may assume that V is affine *and* irreducible.

Write $A[V]$ for the coordinate ring $\mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{I}(V)$, where $\mathfrak{I}(V)$ is a prime ideal. Then, $A[V]$ is a domain and write $K = \mathrm{Frac}(A[V])$. I claim that

$$\dim_K \mathrm{Der}_{\mathbb{C}}(K, K) = \dim_{\mathbb{C}} V. \qquad (*)$$

We know $\dim V = \mathrm{tr.d}_{\mathbb{C}}\, K = \mathrm{tr.d}_{\mathbb{C}}\, A[V]$. Pick a transcendence basis $\zeta_1, \ldots, \zeta_r$ for $A[V]$, then $A[V]$ is algebraic over $\mathbb{C}[\zeta_1, \ldots, \zeta_r]$; therefore, $A[V]$ is separable over $\mathbb{C}[\zeta_1, \ldots, \zeta_r]$ ($\mathbb{C}$ has characteristic zero). We have the isomorphism

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[\zeta_1, \ldots, \zeta_r], K) \xrightarrow{\sim} K^r,$$

and if $\alpha \in A[V]$, then α satisfies an irreducible polynomial

$$\xi_0 \alpha^m + \xi_1 \alpha^{m-1} + \cdots + \xi_m = 0,$$

where $\xi_j \in \mathbb{C}[\zeta_1, \ldots, \zeta_r]$ and α is a simple root. Let $f(T) = \sum_{i=0}^{m} \xi_i T^{m-i}$, where T is some indeterminate. We have $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. For any $D \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[\zeta_1, \ldots, \zeta_r], K)$, we have

$$0 = D(f(\alpha)) = f'(\alpha) D(\alpha) + \sum_{i=0}^{m} \alpha^{m-i} D\xi_i,$$

and as $f'(\alpha) \neq 0$, we see that $D(\alpha)$ exists and is uniquely determined. Therefore,

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[\zeta_1, \ldots, \zeta_r], K) \cong \mathrm{Der}_{\mathbb{C}}(A[V], K)$$

which proves $(*)$.

Now, we have $\mathfrak{I}(V) = (f_1, \ldots, f_p)$ and as we observed earlier

$$\mathrm{Der}_{\mathbb{C}}(A[V], K) \cong \left\{ (\lambda_1, \ldots, \lambda_n) \in K^n \;\middle|\; \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial Z_j} \right) \lambda_j = 0, \;\; 1 \leq i \leq p \right\}.$$

Therefore,

$$\dim_{\mathbb{C}} V = \dim_K \mathrm{Der}(A[V], K) = n - \mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j} \right) \right).$$

Let

$$s = \mathrm{rk}\left(\left(\frac{\partial f_i}{\partial x_j} \right) \right)$$

where the above matrix has entries in K. By linear algebra, there are matrices A, B (with entries in K) so that

$$A \left(\frac{\partial f_i}{\partial x_j} \right) B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\alpha(X_1, \ldots, X_n)$ and $\beta(X_1, \ldots, X_n)$ be the common denominators of entries in A and B, respectively. So, $A = (1/\alpha)\widetilde{A}$ and $B = (1/\beta)\widetilde{B}$, and the entries in $\widetilde{A}$ and $\widetilde{B}$ are in $A[V]$. Let U be the open set where the polynomial $\alpha\beta \det(\widetilde{A}) \det(\widetilde{B})$ is nonzero. Then, as

$$\frac{1}{\alpha\beta} \widetilde{A} \left(\frac{\partial f_i}{\partial x_j} \right) \widetilde{B} = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \quad \text{in } K,$$

for any $\xi \in U$, we have

$$\frac{1}{\alpha(\xi)\beta(\xi)} \widetilde{A}(\xi) \left(\left(\frac{\partial f_i}{\partial x_j} \right)_{\xi} \right) \widetilde{B}(\xi) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\left(\left(\frac{\partial f_i}{\partial x_j} \right)_{\xi} \right)$$

has rank s, a constant.

Now, if V is irreducible, we must have a big open subset U_0 of V where $\dim T_{V,\xi}$ is equal to the minimum it takes on V. Also, we have an open, $\widetilde{U_0}$, where $\dim T_{V,\xi} = \dim(V)$. Since these opens are dense, we find

$$U_0 \cap \widetilde{U_0} \neq \emptyset.$$

Therefore, we must have

$$U_0 = \widetilde{U_0},$$

and the minimum value taken by the dimension of the Zariski tangent space is just $\dim(V)$. In summary, the set

$$U_0 = \{\xi \in V \mid \dim T_{V,\xi} = \dim(V)\} = \min_{\xi \in V} \dim T_{V,\xi}$$

is a Z-open dense subset of V. $\square$

Remark: Say V_i and V_j are irreducible in some irredundant decomposition of V. If $\xi \in V_i \cap V_j$ $(i \neq j)$, check (DX) that $\dim T_{V,\xi} \geq \dim T_{V_i,\xi} + \dim T_{V_j,\xi}$.

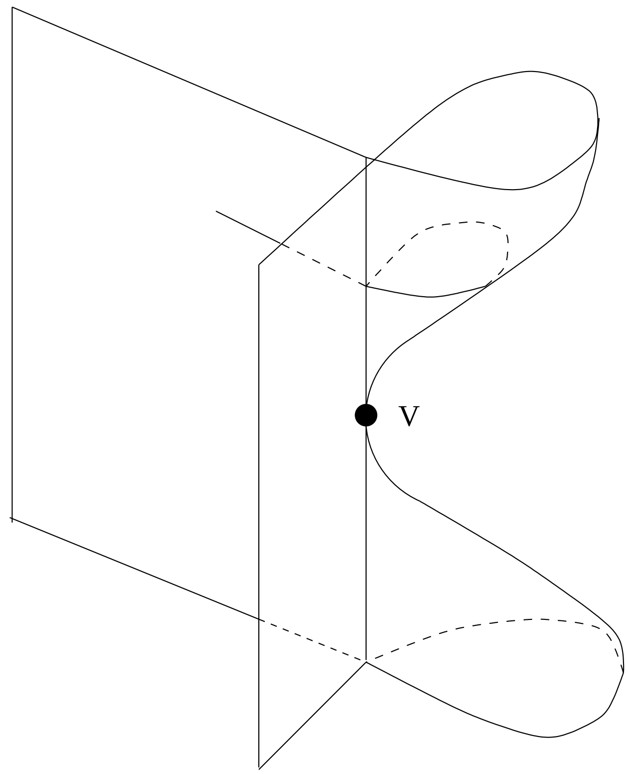

Figure 1.2: Example of A Surface with Singularites

Definition 1.3 If V is an irreducible variety, a point $\xi \in V$ is *nonsingular* if

$$\dim_{\mathbb{C}} T_{V,\xi} = \dim_{\mathbb{C}}(V).$$

Otherwise, we say that ξ is *singular*. If V is quasi-compact but not irreducible and $\xi \in V_i \cap V_j$ for two distinct irreducible (irredundant) components of V, we also say that ξ is *singular*. The singular locus of V is denoted by $\mathrm{Sing}(V)$.

Remark: In the interest of brevity, from now on, we will assume that a complex variety is a quasi-compact (in the Z-topology) complex algebraic variety. A *generalized complex variety* is a complex variety that is Hausdorff but not necessarily quasi-compact.

From previous observations, the singular locus, $\mathrm{Sing}(V)$, of V is a Z-closed set, so it is a complex variety. This leads to the Zariski stratification. Let U_0 be the set of nonsingular points in V, write $V_1 = \mathrm{Sing}(V) = V - U_0$, and let U_1 be the set of nonsingular points in V_1. We can set $V_2 = V_1 - U_1$, and so on. Then, we obtain the *Zariski-stratification of* V into disjoint locally closed strata

$$V = U_0 \,\dot{\cup}\, U_1 \,\dot{\cup} \cdots \dot{\cup}\, U_t,$$

where each U_i is a nonsingular variety and U_0 is the open subset of nonsingular points in V.

Example 1.1 In this example (see Figure 1.2), $\mathrm{Sing}(V)$ consists of a line with a bad point on it (the origin). V_1 is that line, and $V_2 = \mathrm{Sing}(V_1)$ is the bad point.

Let us take a closer look at the tangent space $T_{V,\xi}$.

Pick, ξ, a point of an irreducible variety V. We know that

$$T_{V,\xi} = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{V,\xi}, \mathbb{C}; \xi)$$

and $\mathcal{O}_{V,\xi} = A[W]_{\mathfrak{p}}$, where W is an affine open with $\xi \in W$ and $\mathfrak{p}$ is the prime ideal of $A[W]$ consisting of all g so that $g(\xi) = 0$. We have

$$\mathbb{C} \hookrightarrow \mathcal{O}_{V,\xi} \longrightarrow \mathcal{O}_{V,\xi}/\mathfrak{m}_\xi = \mathbb{C}.$$

Therefore, we can write

$$\mathcal{O}_{V,\xi} = \mathbb{C} \coprod \mathfrak{m}_\xi,$$

where the multiplication in the ring $\mathcal{O}_{V,\xi}$ is given by

$$(\lambda, m)(\lambda', m') = (\lambda\lambda', (\lambda m' + \lambda' m + mm')).$$

Given a derivation $D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{V,\xi}, \mathbb{C}; \xi)$, we have $D(\lambda) = 0$, so $D \restriction \mathfrak{m}_\xi$ determines D. I claim that the restriction $D \restriction \mathfrak{m}_\xi$ of D to $\mathfrak{m}_\xi$ has the property that $D \restriction \mathfrak{m}_\xi^2 = 0$. Indeed,

$$D\left(\sum_i a_i b_i\right) = \sum_i D(a_i b_i) = a_i(\xi) D b_i + b_i(\xi) D a_i.$$

Since $a_i, b_i \in \mathfrak{m}_\xi$, we have $a_i(\xi) = b_i(\xi) = 0$, and so, $D\left(\sum_i a_i b_i\right) = 0$, which proves that $D \restriction \mathfrak{m}_\xi^2 = 0$. As a consequence, D is a $\mathbb{C}$-linear map from $\mathfrak{m}_\xi/\mathfrak{m}_\xi^2$ to $\mathbb{C}$.

Conversely, given a $\mathbb{C}$-linear map $L\colon \mathfrak{m}_\xi/\mathfrak{m}_\xi^2 \to \mathbb{C}$ how do we make a derivation D inducing L?

Define D on $\mathcal{O}_{V,\xi}$ *via*

$$D(\lambda, m) = L(m \ (\mathrm{mod}\ \mathfrak{m}_\xi^2)).$$

We need to check that it is a derivation. Letting $f = (\lambda, m)$ and $g = (\lambda', m')$, we have

$$\begin{aligned} D(fg) &= D(\lambda\lambda', (\lambda m' + \lambda' m + mm')) \\ &= L(\lambda m' + \lambda' m + mm' \ (\mathrm{mod}\ \mathfrak{m}_\xi^2)) \\ &= L(\lambda m' + \lambda' m \ (\mathrm{mod}\ \mathfrak{m}_\xi^2)) \\ &= \lambda L(m') + \lambda' L(m) \\ &= f(\xi) D(g) + g(\xi) D(f). \end{aligned}$$

If we recall that $\widehat{\mathfrak{m}}_\xi/\widehat{\mathfrak{m}}_\xi^2 \cong \mathfrak{m}_\xi/\mathfrak{m}_\xi^2$, we get

$$\begin{aligned} \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{V,\xi}, \mathbb{C}; \xi) &\cong (\mathfrak{m}_\xi/\mathfrak{m}_\xi^2)^D \\ &\cong (\widehat{\mathfrak{m}}_\xi/\widehat{\mathfrak{m}}_\xi^2)^D \\ &\cong \mathrm{Der}_{\mathbb{C}}(\widehat{\mathcal{O}}_{V,\xi}, \mathbb{C}; \xi). \end{aligned}$$

Finally, we get our intrinsic definition of the Zariski tangent space.

Definition 1.4 If V is a (generalized) complex variety, then the *Zariski tangent space to V at ξ*, $T_{V,\xi}$, is just the vector space $(\mathfrak{m}_\xi/\mathfrak{m}_\xi^2)^D$. The vector space $\mathfrak{m}_\xi/\mathfrak{m}_\xi^2$ is the *Zariski cotangent space to V at ξ*.

1.3 Local Structure of Complex Varieties, II; Dimension and Use of the Implicit Function Theorem

Let V be an irreducible variety. For any $\xi \in V$, we have $\mathfrak{m}_\xi$, the maximal ideal of the local ring $\mathcal{O}_{V,\xi}$. Examine maximal chains of prime ideals of $\mathcal{O}_{V,\xi}$

$$\mathfrak{m}_\xi > \mathfrak{p}_1 > \cdots > \mathfrak{p}_d = (0). \tag{†}$$

Such chains exist and end with (0) since $\mathcal{O}_{V,\xi}$ is a noetherian local domain. The length, d, of this chain is the *height of* $\mathfrak{m}_\xi$. The *Krull dimension of* $\mathcal{O}_{V,\xi}$ is the height of $\mathfrak{m}_\xi$ (denoted $\dim \mathcal{O}_{V,\xi}$). Since V is locally affine, every $\xi \in V$ belongs to some affine open, so we may assume that V is affine, $V \subseteq \mathbb{C}^n$ and V is given by a prime ideal $\mathfrak{I}(V)$. Thus, $A[V] = \mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{I}(V)$ and $\mathcal{O}_{V,\xi} = A[V]_{\mathfrak{P}}$, where

$$\mathfrak{P} = \{g \in A[V] \mid g(\xi) = 0\}.$$

Our chain (†) corresponds to a maximal chain of prime ideals

$$\mathfrak{P} > \mathfrak{P}_1 > \cdots > \mathfrak{P}_d = (0)$$

of $A[V]$ and the latter chain corresponds to a chain of irreducible subvarieties

$$\{\xi\} < V_1 < \cdots < V_d = V.$$

Therefore, the length d of our chains is on one hand the Krull dimension of $\mathcal{O}_{V,\xi}$ and on the other hand it is the *combinatorial dimension of* V *at* ξ. Therefore,

$$\mathrm{ht}\, \mathfrak{m}_\xi = \dim \mathcal{O}_{X,\xi} = \text{combinatorial dim. of } V \text{ at } \xi.$$

Nomenclature. Say V and W are affine and we have a morphism $\varphi\colon V \to W$. This gives a map of $\mathbb{C}$-algebras, $A[W] \xrightarrow{\Gamma(\varphi)} A[V]$. Now, we say that

(1) φ is an *integral morphism* if $\Gamma(\varphi)$ makes $A[V]$ a ring integral over $A[W]$.

(2) φ is a *finite morphism* if $A[V]$ a finitely generated $A[W]$-module.

(3) If V and W are not necessarily affine and $\varphi\colon V \to W$ is a morphism, then φ is an *affine morphism* iff there is an open cover of W by affines, W_α, so that $\varphi^{-1}(W_\alpha)$ is again affine as a variety. Then, we can carry over (1) and (2) to the general case *via*: A morphism φ is *integral* (resp. *finite*) iff

   (a) φ is affine

   (b) For every α, the morphism $\varphi \restriction \varphi^{-1}(W_\alpha) \longrightarrow W_\alpha$ is integral (resp. finite).

An irreducible variety, V, is a *normal variety* iff it has an open covering, $V = \bigcup_\alpha V_\alpha$, so that $A[V_\alpha]$ is integrally closed in its fraction field.

Proposition 1.14 *Let V, W be irreducible complex varieties, with W normal. If* $\dim(W) = d$ *and* $\varphi\colon V \to W$ *is a finite surjective morphism, then* $\dim(V) = d$ *and* φ *establishes a surjective map from the collection of closed Z-irreducible subvarieties of V to those of W, so that*

(1) maximal irreducible subvarieties of V map to maximal irreducible subvarieties of W

(2) inclusion relations are preserved

(3) dimensions are preserved

(4) no irreducible subvariety of V, except V itself, maps onto W.

Proof. Let W_α be an affine open in W, then so is $V_\alpha = \varphi^{-1}(W_\alpha)$ in V, because φ is affine, since it is a finite morphism. If Z is an irreducible closed variety in V, then $Z_\alpha = Z \cap V_\alpha$ is irreducible in V_α since Z_α is dense in Z. Thus, we may assume that V and W are affine. Let $A = A[W]$ and $B = A[V]$. Since φ is finite and surjective, we see that $\Gamma(\varphi)$ is an injection, so B is a finite A-module. Both A, B are integral domains, both are Noetherian, A is integrally closed, and no nonzero element of A is a zero divisor in B. These are the conditions for applying the Cohen-Seidenberg theorems I, II, and III. As $A[V]$ is integral, therefore algebraic over $A[W]$, we have

$$d = \dim W = \mathrm{tr.d}_{\mathbb{C}}\, A[W] = \mathrm{tr.d}_{\mathbb{C}}\, A[V] = \dim V,$$

which shows that $\dim W = \dim V$.

By Cohen-Seidenberg I (Zariski and Samuel [14], Theorem 3, Chapter V, Section 2, or Atiyah and Macdonald [1], Chapter 5), there is a surjective correspondence

$$\mathfrak{P} \mapsto \mathfrak{P} \cap A$$

between prime ideals of B and prime ideals of A, and thus, there is a surjective correspondence between irreducible subvarieties of V and their images in W.

Consider a maximal irreducible variety Z in V. Then, its corresponding ideal is a minimal prime ideal $\mathfrak{P}$. Let $\mathfrak{p} = \mathfrak{P} \cap A$, the ideal corresponding to $\varphi(Z)$. If $\varphi(Z)$ is not a maximal irreducible variety in W, then $\mathfrak{p}$ is not a minimal prime, and thus, there is some prime ideal $\mathfrak{q}$ of A such that

$$\mathfrak{q} \subset \mathfrak{p},$$

where the inclusion is strict. By Cohen-Seidenberg III (Zariski and Samuel [14], Theorem 6, Chapter V, Section 3, or Atiyah and Macdonald [1], Chapter 5), there is some prime ideal $\mathfrak{Q}$ in B such that

$$\mathfrak{Q} \subset \mathfrak{P}$$

and $\mathfrak{q} = \mathfrak{Q} \cap A$, contradicting the fact that $\mathfrak{P}$ is minimal. Thus, φ takes maximal irreducible varieties to maximal irreducible varieties.

Finally, by Cohen-Seidenberg II (Zariski and Samuel [14], Corollary to Theorem 3, Chapter V, Section 2, or Atiyah and Macdonald [1], Chapter 5), inclusions are preserved, and since φ is finite, dimension is preserved. The rest is clear. $\square$

We can finally prove the fundamental fact on dimension.

Proposition 1.15 *Let V and W be irreducible complex varieties with W a maximal subvariety of V. Then,*

$$\dim(W) = \dim(V) - 1.$$

Proof. We may assume that V and W are affine (using open covers, as usual). By Noether's normalization theorem (Theorem 1.2), there is a finite surjective morphism $\varphi\colon V \to \mathbb{C}^r$, where $r = \dim_{\mathbb{C}}(V)$. However, $\mathbb{C}^r$ is normal, and by Proposition 1.14, we may assume that $V = \mathbb{C}^r$. Let W be a maximal irreducible variety in $\mathbb{C}^r$. It corresponds to a minimal prime ideal $\mathfrak{P}$ of $A[T_1, \ldots, T_r]$, which is a UFD. As a consequence, since $\mathfrak{P}$ is a minimal prime, it is equal to some principal ideal, i.e., $\mathfrak{P} = (g)$, where g is not a unit. Without loss of generality, we may assume that g involves T_r.

Now, the images $t_1, \ldots, t_{r-1}$ of $T_1, \ldots, T_{r-1}$ in $A[T_1, \ldots, T_r]/\mathfrak{P}$ are algebraically independent over $\mathbb{C}$. Otherwise, there would be some polynomial $f \in A[T_1, \ldots, T_{r-1}]$ such that

$$f(t_1, \ldots, t_{r-1}) = 0.$$

But then, $f(T_1,\ldots,T_{r-1}) \in \mathfrak{P} = (g)$. Thus,

$$f(T_1,\ldots,T_{r-1}) = \alpha(T_1,\ldots,T_r)g(T_1,\ldots,T_r),$$

contradicting the algebraic independence of $T_1,\ldots,T_r$. Therefore, $\dim_{\mathbb{C}}(W) \geq r-1$, but since we also know that $\dim_{\mathbb{C}}(W) \leq r-1$, we get $\dim_{\mathbb{C}}(W) = r-1$. $\square$

Corollary 1.16 *The combinatorial dimension of V over $\{\xi\}$ is just $\dim_{\mathbb{C}} V$; consequently, it is independent of $\xi \in V$ (where V is affine irreducible).*

Corollary 1.17 *If V is affine and irreducible, then $\dim_{\mathbb{C}} \mathfrak{m}_\xi/\mathfrak{m}_\xi^2 \geq \dim_{\mathbb{C}} V =$ comb. dim. $V = \dim_{\mathbb{C}} \mathcal{O}_{V,\xi}$ (= Krull dimension) and we have equality iff ξ is a nonsingular point. Therefore, $\mathcal{O}_{V,\xi}$ is a regular local ring (i.e., $\dim_{\mathbb{C}} \mathfrak{m}_\xi/\mathfrak{m}_\xi^2 = \dim_{\mathbb{C}} \mathcal{O}_{V,\xi}$) iff ξ is a nonsingular point.*

We now use these facts and the implicit function theorem for local analysis near a nonsingular point.

Definition 1.5 Say $V \subseteq \mathbb{C}^n$ is an affine variety and write $d = \dim V$. Then, V is called a *complete intersection* iff $\mathfrak{I}(V)$ has $n-d$ generators. If V is not necessarily affine and $\xi \in V$, then V is a *local complete intersection at ξ* iff there is some affine open, $V(\xi)$, of V such that

(a) $\xi \in V(\xi) \subseteq V$

(b) $V(\xi) \subseteq \mathbb{C}^n$, for some n, as a subvariety

(c) $V(\xi)$ is a complete intersection in $\mathbb{C}^n$.

Theorem 1.18 *(Local Complete Intersection Theorem) Let V be an irreducible complex variety, $\xi \in V$ be a nonsingular point and write $\dim(V) = d$. Then, V is a local complete intersection at ξ. That is, there is some affine open, $U \subseteq V$, with $\xi \in U$, such that U can be embedded into $\mathbb{C}^n$ as a Z-closed subset (we may assume that n is minimal) and U is cut out by $r = n-d$ polynomials. This means that there exists a possibly smaller Z-open $W \subseteq U \subseteq V$ with $\xi \in W$ and some polynomials $f_1,\ldots,f_r$, so that*

$$\eta \in W \quad \text{if and only if} \quad f_1(\eta) = \cdots = f_r(\eta) = 0.$$

The local complete intersection theorem will be obtained from the following affine form of the theorem.

Theorem 1.19 *(Affine local complete intersection theorem) Let $V \subseteq \mathbb{C}^n$ be an affine irreducible complex variety of dimension $\dim(V) = d$, and assume that $V = V(\mathfrak{p})$. If $\xi \in V$ is nonsingular point, then there exist $f_1,\ldots,f_r \in \mathfrak{p}$, with $r = n-d$, so that*

$$\mathfrak{p} = \left\{ g \in \mathbb{C}[Z_1,\ldots,Z_n] \;\middle|\; g = \sum_{i=1}^{r} \frac{h_i(Z_1,\ldots,Z_n)}{l(Z_1,\ldots,Z_n)} f_i(Z_1,\ldots,Z_n)\,, \quad \textit{and} \quad l(\xi) \neq 0 \right\}, \tag{$\dagger$}$$

where h_i and $l \in \mathbb{C}[Z_1,\ldots,Z_n]$. The f_i's having the above property are exactly those $f_i \in \mathfrak{p}$ whose differentials df_i cut out the tangent space $T_{V,\xi}$ (i.e., these differentials are linearly independent).

Proof of the local complete intersection theorem (Theorem 1.18). We show that the affine local complete intersection theorem (Theorem 1.19) implies the general one (Theorem 1.18). There is some open affine set, say U, with $\xi \in U$. By working with U instead of V, we may assume that V is affine. Let $V = V(\mathfrak{p})$, and let $\mathfrak{A} = (f_1,\ldots,f_r)$, in $\mathbb{C}[Z_1,\ldots,Z_n]$. Suppose that $g_1, \ldots, g_t$ are some generators for $\mathfrak{p}$. By the affine local complete intersection theorem (Theorem 1.19), there are some $l_1,\ldots,l_t$ with $l_j(\xi) \neq 0$, so that

$$g_j = \sum_{i=1}^{r} \frac{h_{ij}}{l_j} f_i, \quad \text{for } j = 1,\ldots,t.$$

Let $l = \prod_{j=1}^{t} l_j$ and let W be the Z-open of $\mathbb{C}^n$ where l does not vanish. We have $\xi \in W$, and we also have

$$A[V \cap W] = A[V]_l = \left\{ \frac{\alpha}{l^k} \;\middle|\; \alpha \in A[V],\ k \geq 0 \right\}.$$

But,

$$l_j g_j = \sum_{i=1}^{r} h_{ij} f_i,$$

and on $V \cap W$, the l_j's are units. Therefore,

$$\mathfrak{p} A[V \cap W] = \mathfrak{A} A[V \cap W].$$

Thus, on $V \cap W$, we have $\mathfrak{p} = \mathfrak{A}$ in the above sense, and so, $V \cap W$ is the variety given by the f_j's. The affine version of the theorem implies that $r = n - d$. $\square$

We now turn to the proof of the affine theorem.

Proof of the affine local complete intersection theorem (Theorem 1.19). Let the righthand side of (†) be $\mathfrak{P}$. Given any $g \in \mathfrak{P}$, there is some l so that

$$lg = \sum_{i=1}^{r} h_i f_i.$$

Since $f_i \in \mathfrak{p}$, we have $lg \in \mathfrak{p}$. But $l(\xi) \neq 0$, so $l \notin \mathfrak{p}$; and since $\mathfrak{p}$ is prime, we must have $g \in \mathfrak{p}$. Thus, we have

$$\mathfrak{P} \subseteq \mathfrak{p}.$$

By translation, we can move p to the origin, and we may assume that $\xi = 0$. Now, the proof of our theorem rests on the following proposition:

Proposition 1.20 *(Zariski) Let $f_1, \ldots, f_r \in \mathbb{C}[Z_1, \ldots, Z_n]$ be polynomials with $f_1(0, \ldots, 0) = \cdots = f_r(0, \ldots, 0) = 0$, and linearly independent linear terms at $(0, \ldots, 0)$. Then, the ideal*

$$\mathfrak{P} = \left\{ g \in \mathbb{C}[Z_1, \ldots, Z_n] \;\middle|\; g = \sum_{i=1}^{r} \frac{h_i(Z_1, \ldots, Z_n)}{l(Z_1, \ldots, Z_n)} f_i(Z_1, \ldots, Z_n)\,, \quad \textit{and} \quad l(0, \ldots, 0) \neq 0 \right\}$$

is a prime ideal and $V(\mathfrak{P})$ has dimension $n - r$. Moreover, $(0, \ldots, 0) \in V(\mathfrak{P})$ is a nonsingular point and $V(f_1, \ldots, f_r) = V(\mathfrak{P}) \cup Y$, where Y is Z-closed and $(0, \ldots, 0) \notin Y$.

If we assume Zariski's Proposition 1.20, we can finish the proof of the affine local complete intersection theorem (Theorem 1.19): Since $\xi = (0, \ldots, 0)$ is nonsingular, we find $\dim T_{V,0} = d$, the differentials of $f_1, \ldots, f_r$ are linearly independent if and only if they cut out $T_{V,0}$. Then, $V(\mathfrak{P})$ has dimension $n - r = d$. By Proposition 1.20, $\mathfrak{P}$ is prime, and we have already proved $\mathfrak{P} \subseteq \mathfrak{p}$. However,

$$\dim V(\mathfrak{P}) = \dim V(\mathfrak{p});$$

so, we get $V(\mathfrak{P}) = V(\mathfrak{p})$, and thus, $\mathfrak{P} = \mathfrak{p}$. This proves the affine local complete intersection theorem. $\square$

It remains to prove Zariski's proposition.

Proof of Proposition 1.20. We have the three rings

$$\begin{aligned} R &= \mathbb{C}[Z_1, \ldots, Z_n], \\ R' &= \mathbb{C}[Z_1, \ldots, Z_n]_{(Z_1, \ldots, Z_n)} = \mathcal{O}_{\mathbb{C}^n, 0}, \quad \text{and} \\ R'' &= \mathbb{C}[[Z_1, \ldots, Z_n]]. \end{aligned}$$

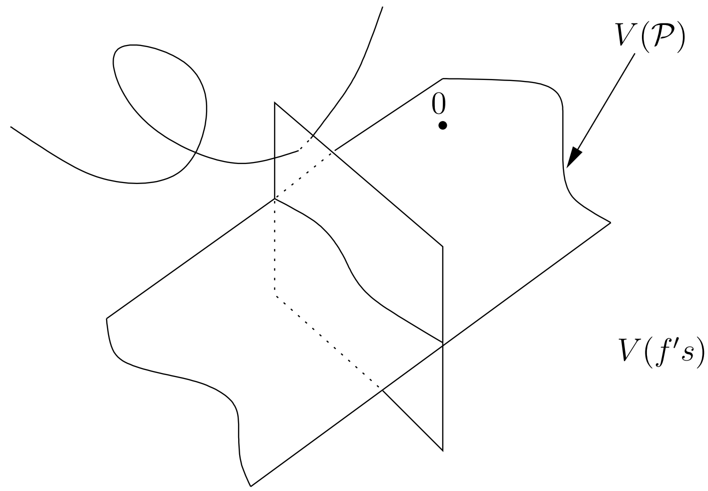

Figure 1.3: Illustration of Proposition 1.20

If $l \in \mathcal{O}_{\mathbb{C}^n,0} \cap \mathbb{C}[Z_1, \ldots, Z_n]$ and $l(0) \neq 0$, then

$$l(Z_1, \ldots, Z_n) = l(0) \left(1 + \sum_{j=1}^{n} a_j(Z_1, \ldots, Z_n) Z_j \right),$$

where $a_j(Z_1, \ldots, Z_n) \in \mathbb{C}[Z_1, \ldots, Z_n]$. But then,

$$\frac{1}{1 + \sum_{j=1}^{n} a_j(Z_1, \ldots, Z_n) Z_j} = \sum_{r=0}^{\infty} (-1)^r \left(\sum_{j=1}^{n} a_j(Z_1, \ldots, Z_n) Z_j \right)^r,$$

which belongs to $\mathbb{C}[[Z_1, \ldots, Z_n]]$. Hence, we have inclusions

$$R \hookrightarrow R' \hookrightarrow R''.$$

Let $\mathfrak{P}' = (f_1, \ldots, f_r)R'$ and write $\mathfrak{P}'' = (f_1, \ldots, f_r)R''$. By definition, $\mathfrak{P} = \mathfrak{P}' \cap R$. If we can show that $\mathfrak{P}'$ is a prime ideal, then $\mathfrak{P}$ will be prime, too.

Claim: $\mathfrak{P}' = \mathfrak{P}'' \cap R'$.

Let $g \in \mathfrak{P}'' \cap R'$. Then,

$$g = \sum_{i=1}^{r} h_i f_i,$$

with $g \in R'$, by assumption, and with $h_i \in R''$. We can define the notion of "vanishing to order t of a power series," and with "obvious notation," we can write

$$h_i = \widetilde{h_i} + O(X^t),$$

where $\deg \widetilde{h_i} < t$. Because $f_i(0, \ldots, 0) = 0$ for each i, we find that

$$g = \sum_{i=1}^{r} \widetilde{h_i} f_i + O(X^{t+1}),$$

and thus,

$$g \in \mathfrak{P}' + (Z_1, \ldots, Z_n)^{t+1} R', \quad \text{for all } t.$$

As a consequence,

$$g \in \bigcap_{t=1}^{\infty} \left(\mathfrak{P}' + (Z_1, \ldots, Z_n)^{t+1} R'\right);$$

so,

$$\mathfrak{P}'' \cap R' \subseteq \bigcap_{t=1}^{\infty} \left(\mathfrak{P}' + (Z_1, \ldots, Z_n)^{t+1} R'\right).$$

But R' is a Noetherian local ring, and by Krull's intersection theorem (Zariski and Samuel [14], Theorem 12′, Chapter IV, Section 7), $\mathfrak{P}'$ is closed in the $\mathfrak{M}$-adic topology of R' (where, $\mathfrak{M} = (Z_1, \ldots, Z_n)R'$). Consequently,

$$\mathfrak{P}' = \bigcap_{t=1}^{\infty} \left(\mathfrak{P}' + \mathfrak{M}^{t+1}\right),$$

and we have proved

$$\mathfrak{P}'' \cap R' \subseteq \mathfrak{P}'.$$

Since we already know that $\mathfrak{P}' \subseteq \mathfrak{P}'' \cap R'$, we get our claim. Thus, if we knew $\mathfrak{P}''$ were prime, then so would be $\mathfrak{P}'$. Now, the linear terms of $f_1, \ldots, f_r$ at $(0, \ldots, 0)$ are linearly independent, thus,

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(0)\right) = r,$$

and we can apply the formal implicit function theorem (Theorem 1.7). As a result, we get the isomorphism

$$R''/\mathfrak{P}'' \cong \mathbb{C}[[Z_{r+1}, \ldots, Z_n]].$$

However, since $\mathbb{C}[[Z_{r+1}, \ldots, Z_n]]$ is an integral domain, $\mathfrak{P}''$ must be a prime ideal. Hence, our chain of arguments proved that $\mathfrak{P}$ is a prime ideal. To calculate the dimension of $V(\mathfrak{P})$, observe that

$$\mathfrak{P}'' \cap R = \mathfrak{P}'' \cap R' \cap R = \mathfrak{P}' \cap R = \mathfrak{P},$$

and we also have

$$\mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{P} \hookrightarrow \mathbb{C}[[Z_1, \ldots, Z_n]]/\mathfrak{P}'' \cong \mathbb{C}[[Z_{r+1}, \ldots, Z_n]].$$

Therefore, $Z_{r+1}, \ldots, Z_n \pmod{\mathfrak{P}}$ are algebraically independent over k, which implies that $\dim V(\mathfrak{P}) \geq n-r$. Now, the linear terms of $f_1, \ldots, f_r$ cut out the linear space $T_{V,0}$, and by linear independence, this space has dimension $n-r$. Then,

$$n - r = \dim T_{V,0} \geq \dim V(\mathfrak{P}) \geq n - r,$$

so that $\dim V(\mathfrak{P}) = n - r$, and 0 is nonsingular.

If $g \in \mathfrak{P}$, there exists some l with $l(0) \neq 0$ such that

$$g = \sum_{i=1}^{r} \frac{h_i}{l} f_i,$$

which implies that $lg \in (f_1, \ldots, f_r)$. Applying this fact to each of the generators of $\mathfrak{P}$, say, $g_1, \ldots, g_t$, and letting $l = \prod_{i=1}^{t} l_i$, we have

$$l\mathfrak{P} \subseteq (f_1, \ldots, f_r) \subseteq \mathfrak{P}.$$

As a consequence,

$$V(\mathfrak{P}) \subseteq V(f_1, \ldots, f_r) \subseteq V(l\mathfrak{P}) = V(l) \cup V(\mathfrak{P}).$$

If we let $Y = V(l) \cap V(f_1, \ldots, f_r)$, we have

$$V(f_1, \ldots, f_r) = V(\mathfrak{P}) \cup Y.$$

Since $l(0) \neq 0$, we have $0 \notin Y$. $\square$

Remarks:

(1) Given an irreducible complex variety, V, we have the intuition that $\mathcal{O}_{V,\xi}$ should control the structure of V near ξ. Since the question is local, we may assume that V is affine, say $V = V(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{C}[Z_1, \ldots, Z_n]$. We have the diagram

$$\begin{tikzcd} \mathbb{C}[Z_1, \ldots, Z_n] \arrow[r, hook] \arrow[d] & \mathbb{C}[Z_1, \ldots, Z_n]_{\mathfrak{I}(\xi)} = \mathcal{O}_{\mathbb{C}^n,\xi} \arrow[d] \\ \mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{p} = A[V] \arrow[r, hook] & A[V]_{\overline{\mathfrak{I}(\xi)}} = \mathcal{O}_{V,\xi}. \end{tikzcd}$$

The kernel on the left hand side is $\mathfrak{p}$ and the kernel on the right hand side is $\mathfrak{p}^e$. By the LCIT, we have $f_1, \ldots, f_r$ (where $n = r + d$ and $\dim V = d$) cutting out V. What is the right hand side of ALCIT? It is $\mathfrak{A}^{ec}$, where $\mathfrak{A} = (f_1, \ldots, f_r) \subseteq \mathbb{C}[Z_1, \ldots, Z_n]$ and $\mathfrak{A}^e = \mathfrak{A}\mathcal{O}_{\mathbb{C}^n,\xi}$. Therefore, the ALCIT says

$$\mathfrak{p} = \mathfrak{A}^{ec}.$$

Thus, $\mathfrak{p}^e = \mathfrak{A}^e$. It follows that if $\mathfrak{p}^e$ as ideal of $\mathcal{O}_{\mathbb{C}^n,\xi}$ has generators $f_1, \ldots, f_r$, then these generators cut out V (from $\mathbb{C}^n$) in a Z-neighborhood of ξ (ξ is nonsingular).

(2) Suppose ξ is nonsingular and V is affine, irreducible and look at the diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^n,\xi} = \mathbb{C}[Z_1 - \xi_1, \ldots, Z_n - \xi_n]_{\mathfrak{I}(\xi)} \arrow[r, hook] \arrow[d] & \mathbb{C}[[Z_1 - \xi_1, \ldots, Z_n - \xi_n]] = \widehat{\mathcal{O}}_{\mathbb{C}^n,\xi} \arrow[d] \\ \mathcal{O}_{V,\xi} = A[V]_{\overline{\mathfrak{I}(\xi)}} \arrow[r, hook] & \widehat{\mathcal{O}}_{V,\xi}. \end{tikzcd}$$

By the LCIT, V is cut out by $f_1, \ldots, f_r$, where $df_1, \ldots, df_r$ cut out $T_{V,\xi}$ from $\mathbb{C}^n$, i.e., $df_1, \ldots, df_r$ are linearly independent forms on $T_{\mathbb{C}^n,\xi} \cong \mathbb{C}^n$. Thus, $\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}\right) = r$ is maximal. By the FIFT, we have

$$\widehat{\mathcal{O}}_{V,\xi} = \mathbb{C}[[Z_1 - \xi_1, \ldots, Z_n - \xi_n]]/(f_1, \ldots, f_r) \cong \mathbb{C}[[Z_{r+1} - \xi_{r+1}, \ldots, Z_n - \xi_n]].$$

By the same theorem, if we pick $y_1, \ldots, y_d$ ($d = n - r$) in $\mathfrak{m}_\xi$, with images $\overline{y_i}$ linearly independent in $\mathfrak{m}_\xi/\mathfrak{m}_\xi^2$ ($\overline{y_j} = dy_j$), then we know that $\widehat{\mathcal{O}}_{V,\xi} \cong \mathbb{C}[[y_1, \ldots, y_d]]$. Therefore, at ξ, as point of $\mathbb{C}^n$, the differentials $df_1, \ldots, df_r, dy_1, \ldots, dy_d$ are linearly independent on $T_{\mathbb{C}^n,\xi}$, i.e., they are a basis. So, by the FIFT, again,

$$\mathbb{C}[[Z_1 - \xi_1, \ldots, Z_n - \xi_n]] \cong \mathbb{C}[[f_1, , \ldots, f_r, y_1, \ldots, y_d]].$$

We conclude

(a) For all $f \in \widehat{\mathcal{O}}_{V,\xi}$, there is a unique power series in $y_1, \ldots, y_d$ equal to f (Taylor series).

(b) Near ξ (in the Z-topology), by Remark 1, $\mathbb{C}^n$ has a "splitting" into coordinates locally on V and coordinates locally transverse to V. Therefore, $T_{V,\eta}$ as η ranges over a Z-open neighborhood of the nonsingular point ξ is locally constant, i.e., just given by $dy_1, \ldots, dy_d$ in this neighborhood.

For the complex analytic case, we have:

Theorem 1.21 *Let V be an irreducible complex algebraic variety of dimension d and let $\xi \in V$ be a nonsingular point. If locally in the Zariski topology near ξ, the variety V may be embedded in $\mathbb{C}^n$, then there exist d of the coordinates (of $\mathbb{C}^n$), say $Z_{r+1}, \ldots, Z_n$ ($r = n - d$) so that*

(a) $dZ_{r+1}, \ldots, dZ_n$ are linearly independent forms on $T_{V,\xi}$,

(b) there exists some $\epsilon > 0$ and we have r converging power series $g_1(Z_{r+1} - \xi_{r+1}, \ldots, Z_n - \xi_n)$, $\ldots$, $g_r(Z_{r+1} - \xi_{r+1}, \ldots, Z_n - \xi_n)$, so that

$$(Z_1, \ldots, Z_n) \in PD(\xi, \epsilon) \cap V \quad \textit{iff} \quad Z_i - \xi_i = g_i(Z_{r+1} - \xi_{r+1}, \ldots, Z_n - \xi_n),\ i = 1, \ldots, r.$$

(c) Any choice of d of the coordinates $Z_1, \ldots, Z_n$ so that the corresponding dZ_i's are linearly independent on $T_{V,\xi}$ will serve, and the map

$$V \cap PD(\xi, \epsilon) \longrightarrow PD(0, \epsilon)$$

given by

$$(Z_1, \ldots, Z_n) \mapsto (Z_{r+1} - \xi_{r+1}, \ldots, Z_n - \xi_n)$$

is an analytic isomorphism. Hence, if we take $(V - \mathrm{Sing}\, V)^{\mathrm{an}}$, it has the natural structure of a complex analytic manifold. Furthermore, V^{an} is a complex analytic manifold if and only if V is a nonsingular variety.

Proof. Since ξ is nonsingular, by the local complete intersection theorem (Theorem 1.18), we can cut out V locally (in the Zariski topology) by $f_1, \ldots, f_r$ and then we know that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(\xi)\right) = r$$

is maximal. By the convergent implicit function theorem (Theorem 1.10), there is some $\epsilon > 0$ and there are some power series $g_1, \ldots, g_r$ so that on $PD(\xi, \epsilon)$, we have

$$f_i(Z_1, \ldots, Z_n) = 0 \quad \text{iff} \quad Z_i - \xi_i = g_i(Z_{r+1} - \xi_{r+1}, \ldots, Z_n - \xi_n) \quad \text{for } i = 1, \ldots, r. \qquad (*)$$

The lefthand side says exactly that

$$(Z_1, \ldots, Z_n) \in V \cap PD(\xi, \epsilon).$$

We get a map by projection on the last d coordinates

$$V \cap PD(\xi, \epsilon) \longrightarrow PD(0, \epsilon),$$

whose inverse is given by the righthand side of equation $(*)$; and thus, the map is an analytic isomorphism. By the formal implicit function theorem (Theorem 1.7),

$$\mathbb{C}[[Z_1, \ldots, Z_n]]/(f_1, \ldots, f_r) \cong \mathbb{C}[[Z_{r+1}, \ldots, Z_n]].$$

Hence, $dZ_{r+1}, \ldots, dZ_n$ are linearly independent on $T_{V,\xi}$. If conversely, the last d coordinates have linearly independent differentials $dZ_{r+1}, \ldots, dZ_n$, then

$$\dim T_{V,\xi} \leq d.$$

But ξ is nonsingular, and thus, $dZ_{r+1}, \ldots, dZ_n$ form a basis of $T_{V,\xi}$. Now, $T_{\mathbb{C}^n,\xi}$ is cut out by $df_1, \ldots, df_r$, $dZ_{r+1}, \ldots, dZ_n$, where $f_1, \ldots, f_r$ cut out V locally (in the Zariski topology) at ξ, by the local complete intersection theorem. It follows that

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial Z_j}(\xi)\right)$$

is maximal (that is, $r = n - d$) and we can repeat our previous arguments. The last statement of the theorem is just a recap of what has already been proved. $\square$

Remark: In the nonsingular case, the differential geometric definition of $T_{V,\xi}$ and ours then agree and the notions of dimension also agree. Consider a morphism, $\varphi\colon V \to W$, of varieties and let ξ be a point of V (perhaps singular). We know that we have the local morphism $(\varphi^*)_\xi\colon \mathcal{O}_{W,\varphi(\xi)} \longrightarrow \mathcal{O}_{V,\xi}$ and $\mathfrak{m}_{\varphi(\xi)} \longrightarrow \mathfrak{m}_\xi$. Thus, we have a linear map $\mathfrak{m}_{\varphi(\xi)}/\mathfrak{m}^2_{\varphi(\xi)} \longrightarrow \mathfrak{m}_\xi/\mathfrak{m}^2_\xi$, but $T^D_{W,\varphi(\xi)} \cong \mathfrak{m}_{\varphi(\xi)}/\mathfrak{m}^2_{\varphi(\xi)}$ and $T^D_{V,\xi} = \mathfrak{m}_\xi/\mathfrak{m}^2_\xi$. If we dualize, we get a linear map $T_{V,\xi} \longrightarrow T_{W,\varphi(\xi)}$, as expected.

To go further and understand the local structure of an irreducible variety near a nonsingular point on it, we need the following famous theorem first proved by Zariski (1947) in the case at hand [13]. However, the theorem is more general and holds for an arbitrary regular local ring as was proved by M. Auslander and D. Buchsbaum, and independently Jean-Pierre Serre (all in 1959).

Theorem 1.22 *Let V be an irreducible complex algebraic variety and let ξ be a nonsingular point of V, then $\mathcal{O}_{V,\xi}$ is a UFD.*

In order to prove Theorem 1.22, we need and will prove the following algebraic theorem:

Theorem 1.23 *If A is a local noetherian ring and if its completion $\widehat{A}$ is a UFD, then, A itself is a UFD.*

Proof of Theorem 1.22. Assume Theorem 1.23, then, $\mathcal{O}_{V,\xi}$ is a noetherian local ring and as ξ is nonsingular,
$$\widehat{\mathcal{O}}_{V,\xi} \cong \mathbb{C}[[Z_1, \ldots, Z_d]],$$
for some d, by the LCIT and implicit function theorem. However, the latter ring is a UFD, by elementary algebra. Therefore, Theorem 1.23 implies Theorem 1.22. $\square$

Observe that we also obtain the fact that $\mathbb{C}\{Z_1, \ldots, Z_d\}$ is a UFD.

Proof of Theorem 1.23. The proof proceeds in three steps.

Step 1. I claim that for every ideal $\mathfrak{A} \subseteq A$ we have
$$\mathfrak{A} = A \cap \mathfrak{A}\widehat{A}.$$
Clearly, $\mathfrak{A} \subseteq A \cap \mathfrak{A}\widehat{A}$. We need to prove that
$$A \cap \mathfrak{A}\widehat{A} \subseteq \mathfrak{A}.$$
Pick $f \in A \cap \mathfrak{A}\widehat{A}$, then, $f \in A$ and
$$f = \sum_{i=1}^t \alpha_i a_i,$$
and $\alpha_i \in \widehat{A}$ and $a_i \in \mathfrak{A}$. Write
$$\alpha_i = \alpha_i^{(n)} + O(\widehat{\mathfrak{m}}^{n+1}),$$
where $\alpha_i^{(n)} \in A$, and $\mathfrak{m}$ is the maximal ideal of A. Then,
$$f = \sum_i \alpha_i^{(n)} a_i + \sum_i O(\widehat{\mathfrak{m}}^{n+1}) a_i,$$
and $\sum_i \alpha_i^{(n)} a_i \in \mathfrak{A}$. So,
$$f \in \mathfrak{A} + \mathfrak{A}\widehat{\mathfrak{m}}^{n+1} = \mathfrak{A} + \mathfrak{A}\mathfrak{m}^{n+1}\widehat{A},$$

and this is true for all n. The piece of f in $\mathfrak{A}\mathfrak{m}^{n+1}\widehat{A}$ lies in A, and thus, in $\mathfrak{m}^{n+1}$. We find that $f \in \mathfrak{A} + \mathfrak{m}^{n+1}$ for all n, and we have

$$f \in \bigcap_{n\geq 0}(\mathfrak{A} + \mathfrak{m}^{n+1}) = \mathfrak{A},$$

by Krull's intersection theorem.

Step 2. I claim that

$$\mathrm{Frac}(A) \cap \widehat{A} = A.$$

This means that given $f/g \in \mathrm{Frac}(A)$ and $f/g \in \widehat{A}$, then $f/g \in A$. Equivalently, this means that if g divides f in $\widehat{A}$, then g divides f in A. Look at

$$\mathfrak{A} = gA.$$

If $f/g \in \widehat{A}$, then $f \in g\widehat{A}$, and since $f \in A$, we have

$$f \in A \cap g\widehat{A}.$$

But $g\widehat{A} = \mathfrak{A}\widehat{A}$, and by Step 1, we find that

$$gA = \mathfrak{A} = A \cap \mathfrak{A}\widehat{A},$$

so, $f \in gA$, as claimed.

We now come to the heart of the proof.

Step 3. Let $f, g \in A$ with f irreducible. I claim that either f divides g in A or $(f, g) = 1$ in $\widehat{A}$ (where (f, g) denotes the gcd of f and g).

Assuming this has been established, here is how we prove Theorem 1.23: Firstly, since A is noetherian, factorization into irreducible factors exists (but not necessarily uniquely). By elementary algebra, one knows that to prove uniqueness, it suffices to prove that if f is irreducible then f is prime. That is, if f is irreducible and f divides gh, then we must prove either f divides g or f divides h.

If f divides g, then we are done. Otherwise, $(f, g) = 1$ in $\widehat{A}$, by Step 3. Now, f divides gh in $\widehat{A}$ and $\widehat{A}$ is a UFD, so that as $(f, g) = 1$ in $\widehat{A}$ we find that f divides h in $\widehat{A}$. By Step 1 2, we get that f divides h in A, as desired.

Proof of Step 3. Let $f, g \in \widehat{A}$ and let d be the gcd of f and g in $\widehat{A}$. Thus,

$$f = dF, \quad \text{and} \quad g = dG,$$

where $d, F, G \in \widehat{A}$, and

$$(F, G) = 1 \quad \text{in} \quad \widehat{A}.$$

Let $\mathrm{ord}_{\widehat{\mathfrak{m}}} F = n_0$ (that is, n_0 is characterized by the fact that $F \in \widehat{\mathfrak{m}}^{n_0}$ but $F \notin \widehat{\mathfrak{m}}^{n_0+1}$). Either F is a unit or a nonunit in $\widehat{A}$. If F is a unit in $\widehat{A}$, then $n_0 = 0$, and $f = dF$ implies that $F^{-1}f = d$; then,

$$F^{-1}fG = g,$$

which implies that f divides g in $\widehat{A}$. By Step 2, we get that f divides g in A.

We now have to deal with the case where $\mathrm{ord}(F) = n_0 > 0$. We have

$$F = \lim_{n\mapsto\infty} F_n \quad \text{and} \quad G = \lim_{n\mapsto\infty} G_n,$$

in the $\mathfrak{m}$-adic topology, with F_n and $G_n \in A$, and $F - F_n$ and $G - G_n \in \widehat{\mathfrak{m}}^{n+1}$. Look at

$$\frac{g}{f} - \frac{G_n}{F_n} = \frac{gF_n - fG_n}{fF_n}.$$

Now,

$$\begin{aligned} gF_n - fG_n &= g(F_n - F) + gF - fG_n \\ &= g(F_n - F) + dGF - fG_n \\ &= g(F_n - F) + fG - fG_n \\ &= g(F_n - F) + f(G - G_n). \end{aligned}$$

The righthand side belongs to $(f, g)\widehat{\mathfrak{m}}^{n+1}$, which means that it belongs to $(f, g)\mathfrak{m}^{n+1}\widehat{A}$. However, the lefthand side is in A, and thus, the righthand side belongs to

$$A \cap (f, g)\mathfrak{m}^{n+1}\widehat{A}.$$

Letting $\mathfrak{A} = (f, g)\mathfrak{m}^{n+1}$, we can apply Step 1, and thus, the lefthand side belongs to $(f, g)\mathfrak{m}^{n+1}$. This means that there are some $\sigma_n, \tau_n \in \mathfrak{m}^{n+1} \subseteq A$ so that

$$gF_n - fG_n = f\sigma_n + g\tau_n;$$

It follows that

$$g(F_n - \tau_n) = f(G_n + \sigma_n);$$

so, if we let

$$\alpha_n = G_n + \sigma_n \quad \text{and} \quad \beta_n = F_n - \tau_n,$$

we have the following properties:

(1) $g\beta_n = f\alpha_n$, with $\alpha_n, \beta_n \in A$,

(2) $\alpha_n \equiv G_n \pmod{\mathfrak{m}^{n+1}}$ and $\beta_n \equiv F_n \pmod{\mathfrak{m}^{n+1}}$,

(3) $G_n \equiv G \pmod{\mathfrak{m}^{n+1}\widehat{A}}$ and $F_n \equiv F \pmod{\mathfrak{m}^{n+1}\widehat{A}}$.

Choose $n = n_0$. Since $\mathrm{ord}(F) = n_0 > 0$, we have $\mathrm{ord}(F_{n_0}) = n_0$, and thus, $\mathrm{ord}(\beta_{n_0}) = n_0$. Look at (1):

$$g\beta_{n_0} = f\alpha_{n_0},$$

so

$$dG\beta_{n_0} = dF\alpha_{n_0},$$

and, because $\widehat{A}$ is an integral domain,

$$G\beta_{n_0} = F\alpha_{n_0}.$$

However, $(F, G) = 1$ in $\widehat{A}$ and F divides $G\beta_{n_0}$. Hence, F divides β_{n_0}, so that there is some $H \in \widehat{A}$ with $\beta_{n_0} = FH$ and

$$\mathrm{ord}(\beta_{n_0}) = \mathrm{ord}(F) + \mathrm{ord}(H).$$

But $\mathrm{ord}(F) = n_0$, and consequently, $\mathrm{ord}(H) = 0$, and H is a unit. Since $\beta_{n_0} = FH$, we see that β_{n_0} divides F, and thus,

$$F = \beta_{n_0}\delta$$

for some $\delta \in \widehat{A}$. Again, $\mathrm{ord}(\delta) = 0$, and we conclude that δ is a unit. Then,

$$\beta_{n_0}\delta d = dF = f,$$

so that β_{n_0} divides f in $\widehat{A}$. By step 2, β_{n_0} divides f in A. But f is irreducible and β_{n_0} is not a unit, and so $\beta_{n_0}u = f$ where u *is* a unit. Thus, $\delta d = u$ is a unit, and since δ is a unit, so is d, as desired. $\square$

The unique factorization theorem just proved has important consequences for the local structure of a variety near a nonsingular point:

Theorem 1.24 *Say V is an irreducible complex variety and let ξ be a point in V. Let $f \in \mathcal{O}_{V,\xi}$ be a locally defined holomorphic function at ξ and assume that $f \not\equiv 0$ on V and $f(\xi) = 0$. Then, the locally defined subvariety, $W = \{x \in V \mid f(x) = 0\}$, given by f is a subvariety of codimension 1 in V. If ξ is nonsingular and f is irreducible, then W is irreducible. More generally, if ξ is nonsingular then the irreducible components of W through ξ correspond bijectively to the irreducible factors of f in $\mathcal{O}_{V,\xi}$. Conversely, suppose ξ is nonsingular and W is a locally defined pure codimension 1 subvariety of V through ξ, then, there is some irreducible $f \in \mathcal{O}_{V,\xi}$ so that near enough ξ, we have*

$$W = \{x \in V \mid f(x) = 0\} \quad \textit{and} \quad \mathfrak{I}(W)\mathcal{O}_{V,\xi} = f\mathcal{O}_{V,\xi}.$$

Proof. Let ξ be a point in V and let f be in $\mathcal{O}_{V,\xi}$, with f irreducible. As the question is local on V we may assume that V is affine. Also,

$$\mathcal{O}_{V,\xi} = \varinjlim_{g \notin \mathfrak{I}(\xi)} A_g,$$

with $A = A[V]$. Thus, we may assume that $f = F/G$, with $G(\xi) \neq 0$ and with $F, G \in A$. Upon replacing V by V_G (where V_G is an open such that $\xi \in V_G$), we may assume that f is the image of some $F \in A = A[V]$. The variety V is irreducible and $V = V(\mathfrak{p})$, where $\mathfrak{p}$ is some prime ideal. Near ξ (i.e., on some open affine subset U_0 with $\xi \in U_0$), let

$$\mathfrak{A} = \{g \in \mathbb{C}[Z_1, \ldots, Z_n] \mid lg \in \mathfrak{p} + (f), \quad \text{where} \quad l(\xi) \neq 0\}, \tag{$*$}$$

and let $\mathfrak{m}$ be the ideal of ξ on V. This means that $\mathfrak{m} = \{g \in A[V] \mid g(\xi) = 0\}$. We have

$$\mathfrak{p} \subseteq \mathfrak{A} \subseteq \mathfrak{m}.$$

Reading the above in A, we get $\overline{\mathfrak{A}} \subseteq \overline{\mathfrak{m}}$, and in $\mathcal{O}_{V,\xi}$, we find from ($*$) that $\overline{\mathfrak{A}}^e = f\mathcal{O}_{V,\xi}$. Thus, $\overline{\mathfrak{A}}^e$ is a prime ideal, because f is irreducible and $\mathcal{O}_{V,\xi}$ is a UFD. Then, $\overline{\mathfrak{A}}$ is prime and $W = V(\overline{\mathfrak{A}})$ is a variety locally defined by $f = 0$, and is irreducible. We have $W \not\subseteq V$, since $f = 0$ on W but not on V, and we find that

$$\dim(W) \leq \dim(V) - 1.$$

We will prove equality by a tangent space argument.

Claim. There is some affine open $U \subseteq W_\alpha$ with $\xi \in U$ so that for all $u \in U$: $T_{W_\alpha,u}$ is cut out from $T_{V,u}$ by the equation $df = 0$, where W_α is some irreducible component of W through ξ.

Let $g_1, \ldots, g_t$ be generators for $\mathfrak{A}$. Thus, $dg_1 = \cdots = dg_t = 0$ cut out $T_{W,u}$ near ξ, i.e., in some suitable open set U_0 with $\xi \in U_0$. By ($*$), on U_0, there exist $l_1, \ldots, l_t$ so that

$$l_i g_i = p_i + \lambda_i f,$$

where $p_i \in \mathfrak{p}$, and the λ_i's are polynomials. Let $l = \prod l_i$, and take

$$U = U_0 \cap \{\eta \mid l(\eta) \neq 0\}.$$

The set U is open and affine. By differentiating, we get

$$l_i dg_i + (dl_i)g_i = dp_i + (d\lambda_i)f + \lambda_i df. \tag{$\dagger$}$$

On $U \subseteq W \subseteq V$, we have

(1) $f = 0$ (in W).

(2) $p_i = 0$ (in V).

(3) $g_i = 0$ (in W).

(4) $l_i \neq 0$.

(5) $dp_i = 0$, as we are in $T_{V,u}$, with $u \in U$.

In view of (†), we get

$$l_i(u)dg_i(u) = \lambda_i(u)df(u).$$

Assume that $df(u) = 0$. Since $l_i(u) \neq 0$, we get $dg_i(u) = 0$, which implies that the equation $df(u) = 0$ cuts out a subspace of $T_{W,u}$. Then, $T_{W,u}$ contains the hyperplane $df = 0$ of $T_{V,u}$, which implies that

$$\dim(T_{W,u}) \geq \dim(T_{V,u}) - 1.$$

Now, Z-open are dense in irreducible; consequently on some irreducible component W_α of W, $U \cap W_\alpha$ is open, dense and some u in $U \cap W_\alpha$ is nonsingular, so

$$\dim W \geq \dim W_\alpha = \dim(T_{W,u}) \geq \dim(T_{V,u}) - 1 \geq \dim V - 1.$$

Thus, (by previous work),

$$\dim(W) = \dim(V) - 1.$$

Conversely, assume that W is locally defined near ξ, and is of codimension 1. Replacing V by this affine neighborhood, we may assume that $W \subseteq V$, is globally defined, and of codimension 1. Also recall that ξ is assumed to be nonsingular. We have the ideal $\mathfrak{I}(W)\mathcal{O}_{V,\xi}$ in $\mathcal{O}_{V,\xi}$, and we can write

$$\mathfrak{I}(W)\mathcal{O}_{V,\xi} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t,$$

where the $\mathfrak{p}_j$'s are minimal primes of $\mathcal{O}_{V,\xi}$, each of height 1. Since $\mathcal{O}_{V,\xi}$ is a UFD, every $\mathfrak{p}_i$ is principal, i.e., $\mathfrak{p}_i = f_i\mathcal{O}_{V,\xi}$, where f_i is irreducible. As

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = \mathfrak{p}_1 \cdots \mathfrak{p}_t,$$

we get

$$\mathfrak{I}(W)\mathcal{O}_{V,\xi} = f\mathcal{O}_{V,\xi},$$

where $f = f_1 \cdots f_t$. The above argument implies that $\mathfrak{I}(W) = (F)$ in some A_G, where $A = A[V]$; $G(\xi) \neq 0$; $G \in A$. Thus, $\mathfrak{I}(W)$ is locally principal. Observe also that if W is irreducible, then $\mathfrak{I}(W)$ is prime; so, $f = f_j$ for some j, i.e., f is irreducible.

Now, consider $f \in \mathcal{O}_{V,\xi}$, where ξ is not necessarily nonsingular, and look at the local variety through ξ defined by $f = 0$ (remember, $f(\xi) = 0$). The radical ideal $\mathfrak{A} = \mathfrak{I}(W)$ (in $A = \mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{p}$) defining W has a decomposition

$$\mathfrak{A} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t,$$

and since $\mathfrak{A} = \sqrt{\mathfrak{A}}$, the $\mathfrak{p}_j$'s are the minimal primes containing $\mathfrak{A}$ (the isolated primes of $\mathfrak{A}$). Let $g_1, \ldots, g_t$ be generators of $\mathfrak{A}$. The image of g_j in $\mathcal{O}_{V,\xi}$ has the form $\lambda_j f$ (remember, W is locally principal by hypothesis). Since

$$\mathcal{O}_{V,\xi} = \varinjlim_{G \notin \mathfrak{I}(\xi)} A_G,$$

take G enough for $g_1, \ldots, g_t$, and then the open V_G so that $g_j = \widetilde{\lambda_j} F$, where $\mathfrak{I}(W)$ in A_G is just (F), and $F/G = f$ in $\mathcal{O}_{V,\xi}$. Thus,

$$\mathfrak{A} = \mathfrak{I}(W) = \bigcap_{j=1}^{s} \mathfrak{p}_j,$$

where in the above intersection, we find only the primes surviving in $\mathcal{O}_{V,\xi}$, i.e., those with $\mathfrak{p}_j \subseteq \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{I}(\xi)$. By Krull's principal ideal theorem (Zariski and Samuel [14], Theorem 29, Chapter IV, Section 14), these $\mathfrak{p}_j$'s are minimal ideals, and thus, the components of W have codimension 1.

If ξ is actually nonsingular, then these surviving $\mathfrak{p}_j$'s are minimal in the UFD $\mathcal{O}_{V,\xi}$. Hence, locally enough, each $\mathfrak{p}_j$ is principal; say $\mathfrak{p}_j = (f_j)$. Then,

$$(f) = \mathfrak{A} = (f_1) \cap \cdots \cap (f_s) = (f_1 \cdots f_s);$$

so that

$$f = uf_1 \cdots f_s$$

where u is a unit. The irreducible branches of W through ξ are the irreducible factors of the local equation $f = 0$ defining W locally. $\square$

Nomenclature: If V is a complex variety, then a *Cartier divisor on* V is a subvariety, $W \subseteq V$, so that for all $\xi \in W$, there is U_ξ open in V, with $\xi \in U_\xi$ and there is some function f_ξ locally defined on U_ξ so that

$$W \cap U_\xi = \{\eta \mid f_\xi(\eta) = 0\}.$$

Then, the translation of Theorem 1.24 is:

Corollary 1.25 *If V is a complex variety, then every Cartier divisor is a pure codimension one subvariety of V. If ξ is nonsingular, then a pure codimension one subvariety of V through ξ is a Cartier divisor near ξ.*

Corollary 1.26 *(Hypersurface Intersection Theorem) Say V is a complex variety and f is a global function ($f \not\equiv 0$) on V. Let $W = \{v \in V \mid f(v) = 0\}$ (e.g., $V \subseteq \mathbb{C}^N$ is affine, $H =$ a hyperplane in $\mathbb{C}^N$ given by $H = \{x \in \mathbb{C}^N \mid F(x) = 0\}$ and $W = V \cap H$, with $V \not\subseteq H$ and $V \cap H \neq \emptyset$.) If $f \not\equiv 0$ and f is not a unit, then each irreducible component of W has codimension 1.*

Corollary 1.27 *(Intersection Dimension Theorem) Say V, W, Z are irreducible complex varieties and $V, W \subseteq Z$. Pick $\xi \in V \cap W$ and assume ξ is a nonsingular point of Z. Then, $V \cap W = (\bigcup_\alpha T_\alpha) \cup Q$, where*

(1) The T_α are irreducible components of $V \cap W$ through ξ.

(2) $Q =$ the rest of $V \cap W$.

(3) $\dim T_\alpha \geq \dim V + \dim W - \dim Z$, for all α.

Proof. (1) and (2) simply set up the notation and we just have to prove (3). We may assume that V, W, Z are affine (by density of a Z-open in an irreducible), say $Z \subseteq \mathbb{C}^N$. As ξ is nonsingular, if $\dim Z = n$, there exist functions $g_1, \ldots, g_n$ on Z with $dg_1, \ldots, dg_n$ a basis for $T^D_{Z,\xi}$ (by the Local Complete Intersection Theorem). Observe that

$$V \cap W = (V \prod W) \cap \Delta_Z,$$

where $\Delta_Z \subseteq Z \prod Z$ is the diagonal. Consider the functions $f_1, \ldots, f_n$ on $Z \prod Z$ given (near ξ) by

$$f_i(z_1, \ldots, z_n; w_1, \ldots, w_n) = g_i(z_1, \ldots, z_n) - g_i(w_1, \ldots, w_n), \quad i = 1, \ldots, n.$$

Clearly, $f_i \restriction \Delta_Z \equiv 0$ (near (ξ, ξ)) and $df_1, \ldots, df_n$ are l.i. at (ξ, ξ) on the tangent space. By the LCIT, we must have

$$C(f_1, \ldots, f_n) = \Delta_Z \cup R,$$

where $R =$ the union of the irreducible components of $V(f_1, \ldots, f_n)$ through (ξ, ξ). Therefore, near ξ, we have

$$V \cap W = (V \cap W) \cap V(f_1) \cap \cdots \cap V(f_n).$$

By Corollary 1.26, $\dim T_\alpha \geq \dim(V \prod W) - n$. However, $\dim(V \prod W) = \dim V + \dim W$, so

$$\dim T_\alpha \geq \dim V + \dim W - \dim Z. \qquad \square$$

If V and W are contained in some affine variety Z not $\mathbb{C}^q$, the intersection dimension theorem may be **false** at a singular point. Indeed, consider the following example.

Example 1.2 Let Z be the quadric cone in $\mathbb{C}^4$ given by

$$x_1x_2 - x_3x_4 = 0.$$

The cone Z has dimension 3 (it is a hypersurface). Let V be the plane

$$x_1 = x_3 = 0,$$

and W the plane

$$x_2 = x_4 = 0.$$

Observe that $V, W \subseteq Z$. Since V and W have dimension 2 and $V \cap W \neq \emptyset$, the intersection dimension theorem would yield $\dim(V \cap W) \geq 2 + 2 - 3 = 1$. However $V \cap W = \{(0,0,0,0)\}$, the origin, whose dimension is zero!

What is the problem? The answer is that near 0, $\Delta \cap Z$ is not the locus of three equations, but rather of four equations.

Corollary 1.28 *(Intersection Dimension Theorem in $\mathbb{C}^N$ and $\mathbb{P}^N$)*

(1) Say V, W are complex affine varieties $V, W \subseteq \mathbb{C}^N$. Then, every irreducible component of $V \cap W$ has dimension at least $\dim V + \dim W - N$.

(2) Say V, W are complex varieties $V, W \subseteq \mathbb{P}^N$, with $\dim V + \dim W \geq N$. *Then, $V \cap W \neq \emptyset$ and every irreducible component of $V \cap W$ has dimension at least* $\dim V + \dim W - N$.

Proof. (1) As each point of $\mathbb{C}^N$ is nonsingular, we can apply Corollary 1.28.

(2) Write $C(V), C(W)$ for the cones over V and W in $\mathbb{C}^{N+1}$, i.e., the affine varieties given by the same homogeneous equations regarded in affine space $\mathbb{C}^{N+1}$. We have $\dim C(V) = \dim V + 1$ and
$\dim C(W) = \dim W + 1$. As $0 \in C(V) \cap C(W)$, by part (1), every irreducible component of $C(V) \cap C(W)$ through 0 has dimension at least $\dim V + 1 + \dim W + 1 - (N+1)$, i.e., $(\dim V + \dim W - N) + 1$. But, each such irreducible component is the cone over the corresponding irreducible component of $V \cap W$ and, as $\dim V + \dim W \geq N$, we deduce that these irreducible affine cones have dimension at least 1, so that the corresponding irreducible components in $\mathbb{P}^N$ are nonempty. The rest is clear. $\square$

1.4 Elementary Global Theory of Varieties

We begin by observing that the category of complex varieties has fibred products. The set-up is the following: V and W are two given varieties and we have two morphisms $p\colon V \to Z$ and $q\colon W \to Z$, where W is another complex variety. We seek a variety, P, together with two morphisms, $pr_1\colon P \to V$ and $pr_2\colon P \to W$, so that the diagram below commutes

$$\begin{tikzcd} P \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & W \arrow[d, "q"] \\ V \arrow[r, "p"'] & Z \end{tikzcd}$$

and so that for every test variety, T, and morphisms, $\varphi\colon T \to V$ and $\psi\colon T \to W$, so that the diagram

$$\begin{tikzcd} T \arrow[r, "\psi"] \arrow[d, "\varphi"'] & W \arrow[d, "q"] \\ V \arrow[r, "p"'] & Z \end{tikzcd}$$

commutes, there is a *unique* morphism, $(\varphi, \psi)\colon T \to P$, with

$$\varphi = pr_1 \circ (\varphi, \psi) \quad \text{and} \quad \psi = pr_2 \circ (\varphi, \psi).$$

Such a variety, P, always exists and is unique up to (unique) isomorphism (use glueing, DX). This variety is called the *fibred product of* V *and* W *over* Z and is denoted $V \prod_Z W$.

Say we have a morphism $p\colon V \to Z$, and let $z \in Z$ be any point in Z. Then $\{z\} \subseteq Z$ is Z-closed, i.e., the inclusion map, $q\colon \{z\} \to Z$ is a morphism. Consequently, we can make the fibred product $V \prod_Z \{z\}$, a variety. Set theoretically, it is the fibre $p^{-1}(z)$.

Theorem 1.29 *(Fibre Dimension Theorem) If $\varphi\colon V \to Z$ is a surjective morphism of complex, irreducible varieties, then*

(1) For every $z \in Z$,

$$\dim \varphi^{-1}(z) \geq \dim V - \dim Z.$$

(2) There is a Z-open, $U \subseteq Z$, so that, for every $u \in U$,

$$\dim \varphi^{-1}(u) = \dim V - \dim Z.$$

Proof. You check (DX), we may assume Z is affine, say $Z \subseteq \mathbb{C}^N$.

(1) Pick $\xi \in Z$, $\xi \neq z$. There is a hyperplane $H \subseteq \mathbb{C}^N$ such that $z \in H$ and $\xi \notin H$. Thus, Z is not contained in H. [In fact, if $L = 0$ is a linear form defining H, L^d $(d \geq 1)$ is a form of degree d defining a hypersurface of degree d, call it H'; Z is not contained in H', but $z \in H'$.] By the hypersurface intersection theorem (Corollary 1.26), the dimension of any irreducible component of $Z \cap H$ is $\dim(Z) - 1$. Pick, $\xi_1, \ldots \xi_s$ in each of the components of $Z \cap H$. Then, there is a hyperplane $\widetilde{H}$ so that $\xi_j \notin \widetilde{H}$ for all j, $1 \leq j \leq s$, but $z \in \widetilde{H}$. Then, by Corollary 1.26 again, the dimension of any component of $Z \cap H \cap \widetilde{H}$ is $\dim(Z) - 2$. Using this process, we get some hyperplanes $H = H_1, H_2, \ldots, H_m$ such that

$$z \in \bigcap_{j=1}^{m} H_j,$$

and if we write

$$Z_j = Z_{j-1} \cap H_j,$$

with $Z_1 = Z \cap H_1$, we get a chain

$$Z \supset Z_1 \supset Z_2 \supset \cdots \supset Z_m \quad \text{with} \quad m = \dim Z.$$

Here, $z \in Z_m$, and

$$\dim(Z_j) = \dim(Z) - j.$$

Thus, the linear forms $L_1, \ldots, L_m$ associated with the H_j's define Z_m in Z and

$$\dim(Z_m) = 0.$$

Consequently, Z_m is a finite set of points:

$$Z_m = \{z = z_1, z_2, \ldots, z_t\}.$$

Let

$$U_0 = Z - \{z_2, \ldots, z_t\},$$

it is a Z-open dense subset of Z. We can replace Z by U_0, and thus, we may assume that $Z_m = \{z\}$. We have the morphism $\varphi\colon V \to Z$, and so, each $\varphi^*(L_j)$ is a global function on V (where $\varphi^*\colon \varphi^*\mathcal{O}_Z \to \mathcal{O}_V$). But then, $\varphi^{-1}(z)$ is the locus in V cut out by $\varphi^*(L_1), \ldots, \varphi^*(L_m)$; so, by the intersection dimension theorem, we get

$$\dim \varphi^{-1}(z) \geq \dim(V) - m = \dim(V) - \dim Z.$$

(2) We need to prove that there is a Z-open, $U \subseteq Z$, so that $\dim \varphi^{-1}(u) = \dim V - \dim Z$, for all $u \in U$. As this statement is local on the base, we may assume that Z is affine. Assume at first that V is affine as well. By hypothesis, $\varphi\colon V \to Z$ is onto ($\varphi(V)$ Z-dense in Z is enough), so the corresponding map $A[Z] \longrightarrow A[V]$ is an injection (DX). If we let $m = \dim Z$ and $n = \dim V$, as $\mathrm{tr.d}_{\mathbb{C}}\, A[Z] = m$ and $\mathrm{tr.d}_{\mathbb{C}}\, A[V] = n$, we have

$$\mathrm{tr.d}_{A[Z]}\, A[V] = n - m.$$

Now, we have $V \hookrightarrow \mathbb{C}^N$, with $A[V] = \mathbb{C}[v_1, \ldots, v_N]$ and $Z \hookrightarrow \mathbb{C}^M$, with $A[Z] = \mathbb{C}[z_1, \ldots, z_M]$, for some M, N. We can choose and reorder the v_j's so that $v_1, \ldots, v_{n-m}$ form a transcendence basis of $A[V]$ over $A[Z]$. Then, each v_j ($j = n-m+1, \ldots, N$) is algebraic over $A[Z][v_1, \ldots, v_{n-m}]$, and there are polynomials $G_j(T_1, \ldots, T_{n-m}, T)$ (coefficients in $A[Z]$) so that

$$G_j(v_1, \ldots, v_{n-m}, v_j) = 0.$$

Pick $g_j(T_1, \ldots, T_{n-m})$ as the coefficient of highest degree of G_j in T. The set

$$\{z \in Z \mid g_j(z) = 0\} = Z_j$$

is a Z-closed subset of Z. Let

$$U = Z - \bigcup_{j=n-m+1}^{N} Z_j.$$

The Z-open U is nonempty, since Z is irreducible. On U, the polynomial G_j is not identically zero as a polynomial in $T_1, \ldots, T_{n-m}, T$, yet

$$G_j(v_1, \ldots, v_{n-m}, v_j) = 0.$$

Thus, v_j is algebraically dependent on $v_1, \ldots, v_{n-m}$ over $A[U]$. Letting $\widetilde{v_j}$ denote the restriction of v_j to $\varphi^{-1}(z)$ (i.e., the image of v_j in $A[V] \otimes_{A[Z]} \mathbb{C}$), where $z \in U$, we see that $\widetilde{v_j}$ is also algebraically dependent on $\widetilde{v_1}, \ldots, \widetilde{v}_{n-m}$. Now,

$$A[\varphi^{-1}(z)] = \mathbb{C}[\widetilde{v_1}, \ldots, \widetilde{v}_{n-m}] = A[V] \otimes_{A[Z]} \mathbb{C},$$

which implies that

$$\dim(\varphi^{-1}(z)) = \mathrm{tr.d}_{\mathbb{C}}\, \mathbb{C}[\widetilde{v_1}, \ldots, \widetilde{v}_{n-m}] \leq n - m.$$

However, by (1),

$$\dim(\varphi^{-1}(z)) \geq n - m,$$

and so, $\dim(\varphi^{-1}(z)) = n - m$ on U.

If V is not affine, cover V by affine opens, V_α; each V_α is Z-dense in V (since V is irred.) Then, for every $z \in Z$, we have $\varphi^{-1}(z) = \bigcup_\alpha (V_\alpha \cap \varphi^{-1}(z)) = \bigcup_\alpha \varphi_\alpha^{-1}(z)$. Here, $\varphi_\alpha\colon V_\alpha \hookrightarrow V \xrightarrow{\varphi} Z$. Now, $\varphi_\alpha(V_\alpha)$ is Z-dense in Z, so the above implies that there exist some opens U_α in Z so that $\dim \varphi_\alpha^{-1}(z) = n - m$ if $z \in U_\alpha$. We can take $U = \bigcap_\alpha U_\alpha$ (a finite intersection since Z is a variety, and thus, is quasi-compact). $\square$

Corollary 1.30 *Assume that we are in the same situation as in the fibre dimension theorem. Let*

$$Z_l = \{w \in Z \mid \dim(\varphi^{-1}(w)) \geq l\}.$$

Then, Z_l is Z-closed in Z, i.e., the function

$$w \mapsto \dim(\varphi^{-1}(w))$$

is upper semi-continuous on Z. Hence, Z possesses a stratification

$$Z = U_0 \cup U_1 \cup \cdots \cup U_n,$$

where $U_j = Z_j - Z_{j+1} = \{z \in Z \mid \dim \varphi^{-1}(z) = l\}$ is locally closed and $\dim(\varphi^{-1}(w)) = j$ for all $w \in U_j$.

Proof. The proof is by induction on $\dim(Z)$. The case where $\dim(Z) = 0$ is easy. Given Z, Theorem 1.29 part (2) implies that there is some open set $U \subseteq Z$ and some Z_l ($l \geq 1$ and l minimum) so that

$$Z_l \subseteq \widetilde{Z} = Z - U.$$

Also, $\widetilde{Z}$ is closed and we have some irredundant decomposition

$$\widetilde{Z} = \bigcup_{j=1}^{t} Z_j,$$

where Z_j is irreducible and strictly contained in Z. Then, $\dim(Z_j) < \dim(Z)$, and we can apply the induction hypothesis to the maps $\varphi_j\colon \varphi^{-1}(Z_j) \to Z_j$, the details are left as an exercise (DX). $\square$

Observe that given a morphism $\varphi\colon V \to Z$, we can write

$$V = \bigcup_{z \in Z} \varphi^{-1}(z).$$

Each $\varphi^{-1}(z)$ is a complex algebraic variety and these are indexed by an algebraic variety. Therefore, V is an algebraic family of varieties.

Note that the dimension of the fibres may jump, as shown by the following example (which is nothing but the "blowing-up" at a point in $\mathbb{C}^2$).

Example 1.3 Let $Z = \mathbb{C}^2$, and consider $\mathbb{C}^2 \prod \mathbb{P}^1$. We use z_1, z_2 as cooordinates on Z, and ξ_1, ξ_2 as homogeneous coordinates on $\mathbb{P}^1$. Write $B_0(\mathbb{C}^2)$ for the subvariety of $\mathbb{C}^2 \prod \mathbb{P}^1$ given by the equation

$$z_1\xi_2 = z_2\xi_1.$$

This equation is homogeneous in ξ_1, ξ_2, and it defines a closed subvariety of $\mathbb{C}^2 \prod \mathbb{P}^1$. We get a morphism $\varphi\colon B_0(\mathbb{C}^2) \to Z$ *via*

$$\varphi\colon B_0(\mathbb{C}^2) \hookrightarrow \mathbb{C}^2 \prod \mathbb{P}^1 \xrightarrow{pr_1} Z = \mathbb{C}^2.$$

If $z = (z_1, z_2) \neq (0,0)$, then the fibre over z is $\{(\xi_1 \colon \xi_2) \mid z_1\xi_2 = z_2\xi_1\}$.

1. If $z_1 \neq 0$, then $\xi_2 = (z_2/z_1)\xi_1$.

2. If $z_2 \neq 0$, then $\xi_1 = (z_1/z_2)\xi_2$.

In both cases, we get a single point $(z_1, z_2; (1\colon z_2/z_1))$ in case (1) and $(z_1, z_2; (z_1/z_2\colon 1))$ in case (2)) and the dimension of the fibre $\varphi^{-1}(z)$ is zero for $z \in \mathbb{C} - \{(0,0)\}$. In fact, observe that

$$B_0(\mathbb{C}^2) - pr_1^{-1}((0,0)) \xrightarrow{pr1} \mathbb{C}^2 - \{(0,0)\}.$$

Therefore, pr_1 is a birational morphism between $B_0(\mathbb{C}^2)$ and $\mathbb{C}^2$. Observe that $pr_2\colon B_0(\mathbb{C}^2) \to \mathbb{P}^1$ is also surjective and the fibre over every point of $\mathbb{P}^1$ is a line. It turns out that $B_0(\mathbb{C}^2)$ is a line bundle over $\mathbb{P}^1$.

When $z = (0,0)$, the fibre is the whole $\mathbb{P}^1$. Thus, the dimension of the fibre at the origin jumps from 0 to 1.

In algebraic geometry, we have the analog of the notions of being Hausdorff or compact, but here working for the Zariski topology.

Definition 1.6 A morphism $\varphi\colon V \to W$ is *separated* iff $\Delta_W V$ = the diagonal in $V \prod_W V$ is Z-closed in $V \prod_W V$. A variety, V, is *separated* when the morphism $V \longrightarrow *$ is separated, where $*$ is any one-point variety.

Examples of separated varieties.

(1) Any affine variety and any projective variety is separated.

(2) Any closed or open subvariety of a separated variety is separated (DX).

(3) We say that V is *quasi-affine* (resp. *quasi-projective*) iff it is open in an affine (resp. open in a projective) variety. These are separated.

(4) The product of separated varieties is separated.

(5) The variety we get from two copies of $\mathbb{P}^1$ by glueing the opens $\mathbb{C}^1 - \{0\}$ by the identity map is **not** separated.

Definition 1.7 A morphism, $\varphi\colon V \to W$, is *proper* iff

(a) It is separated and

(b) For every variety, T, the second projection map, $pr_2\colon V \prod_W T \to T$, is a Z-closed morphism.

We say that a variety, V, is *proper* iff it is separated and if property (b) holds ($W = \{*\}$).

As we said, the notion of properness of a variety is the algebraic substitute for compactness. An older terminology is the term *complete variety*. As an illustration of the similarity of properness and compactness, we have the following property (well known for continuous maps on compact spaces):

Proposition 1.31 *If V is proper and W is separated, then any morphism $\varphi\colon V \to W$ is Z-closed.*

Proof. Consider the graph morphism
$$\Gamma_\varphi\colon V \to V \prod W,$$
given by
$$\Gamma_\varphi(v) = (v, \varphi(v)).$$
Note that the image of Γ_φ is closed in $V \prod W$ because W is separated. Indeed, consider the morphism
$$(\varphi, \mathrm{id})\colon V \prod W \to W \prod W$$
given by
$$(\varphi, \mathrm{id})(v, w) = (\varphi(v), w).$$
It is obvious that $\mathrm{Im}\,\Gamma_\varphi = (\varphi, \mathrm{id})^{-1}(\Delta_W)$, where Δ_W is the diagonal in $W \prod W$. But, W is separated, so Δ_W is Z-closed in $W \prod W$ and consequently, $(\varphi, \mathrm{id})^{-1}(\Delta_W)$ is Z-closed in $V \prod W$. Since V is proper, by setting $T = W$ in condition (b), we see that $pr_2\colon V \prod W \to W$ is Z-closed and so, $pr_2(\mathrm{Im}\,\Gamma_\varphi)$ is Z-closed in W. However, $pr_2(\mathrm{Im}\,\Gamma_\varphi) = \mathrm{Im}\,\varphi$, therefore, $\mathrm{Im}\,\varphi$ is closed. If C is closed in V, then the composition $C \hookrightarrow V \longrightarrow *$ is proper (DX). Replace V by C and apply the above argument to get $\mathrm{Im}\,C$ is closed. $\square$

Theorem 1.32 *(Properness of projective varieties) A projective variety, V, is a proper variety. This means that for every variety W,*

$$pr_2 \colon V \prod W \longrightarrow W$$

is a Z-closed map.

Proof. (1) We reduce the proof to the case where W is affine. Assume that the theorem holds when W is affine. Cover W with affine opens W_α, so that $W = \bigcup_\alpha W_\alpha$. Check that

$$V \prod W \cong \bigcup_\alpha \left(V \prod W_\alpha \right).$$

Let $C \subseteq V \prod W$ be a closed subvariety. If C_α denotes $C \cap (V \prod W_\alpha)$, then,

$$pr_2(C) \cap W_\alpha = pr_2(C_\alpha).$$

But, $pr_2(C_\alpha)$ is closed in W_α, which implies that $pr_2(C)$ is closed in W.

(2) We reduce the proof to the case where $V = \mathbb{P}^n$. Assume that the theorem holds for $\mathbb{P}^n$. We have a closed immersion, $V \hookrightarrow \mathbb{P}^n$, so

$$V \prod W \hookrightarrow \mathbb{P}^n \prod W$$

is also a closed immersion (= embedding) (DX). Hence, we have the commutative diagram

$$\begin{tikzcd}
C \arrow[r, hook] & V \prod W \arrow[rr, hook] \arrow[dr, "pr_2"'] & & \mathbb{P}^n \prod_n W \arrow[dl, "pr_2"] \\
& & W &
\end{tikzcd}$$

and this shows that we may assume that $V = \mathbb{P}^n$.

(3) Lastly, we reduce the proof to the case: $W = \mathbb{C}^m$. Assume that the theorem holds for $W = \mathbb{C}^m$. By (1), we may assume that W is closed in $\mathbb{C}^m$, then we have the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, hook] & \mathbb{P}^n \prod W \arrow[r, hook] \arrow[d, "(pr_2)_W"'] & \mathbb{P}^n \prod \mathbb{C}^m \arrow[d, "(pr_2)_{\mathbb{C}^m}"] \\
& W \arrow[r, hook] & \mathbb{C}^m
\end{tikzcd}$$

where the arrows in the top line are closed immersions (for the second arrow, this is because $W \hookrightarrow \mathbb{C}^m$, as in (2)). So,

$$(pr_2)_W(C) = (pr_2)_{\mathbb{C}^m}(C) \cap W,$$

and, since by hypothesis, $(pr_2)_{\mathbb{C}^m}(C)$ is closed, and W is closed, we find $(pr_2)_W(C)$ is also closed.

We are now reduced to the essential case: Which is to prove that $pr_2 \colon \mathbb{P}^n \prod \mathbb{C}^m \to \mathbb{C}^m$ is a closed map. Let C be a closed subvariety of $\mathbb{P}^n \prod \mathbb{C}^m$. Then, C is the common solution set of a system of equations of form

$$f_j(X_0, \ldots, X_n; Y_1, \ldots, Y_m) = 0, \quad \text{for } j = 1, \ldots, p, \tag{†}$$

where f_j is homogeneous in the X_j's and we restrict to solutions for which $X_j \neq 0$ for some j, with $0 \leq j \leq n$. Pick $y \in \mathbb{C}^m$, and write $y = (y_1, \ldots, y_m)$; also write $(\dagger)(y)$ for the system $(\dagger)$ in which we have set $Y_j = y_j$ for $j = 1, \ldots, m$.

Plan of the proof: We will prove that $(pr_2(C))^c$ (the complement of $pr_2(C)$) is open. Observe that

$$\begin{aligned} y \in pr_2(C) \quad &\text{iff} \quad (\exists x)((x,y) \in C) \\ &\text{iff} \quad (\exists x)(x_j \neq 0 \text{ for some } j, \text{ and } (\dagger)(y) \text{ holds}). \end{aligned}$$

Thus,

$$y \in (pr_2(C))^c \quad \text{iff} \quad (\forall x)(\text{if } (\dagger)(y) \text{ holds, then } x_j = 0, \quad \text{for } 0 \leq j \leq n).$$

Let $\mathfrak{A}(y)$ be the ideal generated by the polynomials, $f_j(X_0, \ldots, X_n, y_1, \ldots, y_m)$, occurring in $(\dagger)(y)$. Hence,

$$y \in (pr_2(C))^c \quad \text{iff} \quad (\exists d \geq 0)((X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y)).$$

Let

$$\mathcal{N}_d = \{y \in \mathbb{C}^m \mid (X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y)\}.$$

Then,

$$(pr_2(C))^c = \bigcup_{d=1}^{\infty} \mathcal{N}_d.$$

Now,

$$\mathcal{N}_d \subseteq \mathcal{N}_{d+1},$$

and so,

$$(pr_2(C))^c = \bigcup_{d>>0}^{\infty} \mathcal{N}_d,$$

where $d >> 0$ means that d is sufficiently large.

Claim. If $d > \max\{d_1, \ldots, d_p\}$, where d_j is the homogeneous degree of $f_j(X_0, \ldots, X_n, Y_1, \ldots, Y_m)$ in the X_i's, then $\mathcal{N}_d$ is open in $\mathbb{C}^m$. This will finish the proof.

Write $S_d(y)$ for the vector space (over $\mathbb{C}$) of polynomials in $k[y_1, \ldots, y_m][X_0, \ldots, X_n]$ of *exact degree d*. We have a map of vector spaces

$$\psi_d(y) \colon S_{d-d_1}(y) \oplus \cdots \oplus S_{d-d_p}(y) \longrightarrow S_d(y)$$

given by

$$\psi_d(y)(g_1, \ldots, g_p) = \sum_{j=1}^{p} f_j(X_0, \ldots, X_n, y_1, \ldots, y_m) g_j.$$

If we assume that $\psi_d(y)$ is surjective, then all monomials of degree d are in the range of $\psi_d(y)$. Thus, $\mathfrak{A}(y)$ will contain all the generators of $(X_0, \ldots, X_n)^d$, i.e.,

$$(X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y),$$

and this means

$$y \in \mathcal{N}_d.$$

Conversely, if $y \in \mathcal{N}_d$, then $(X_0, \ldots, X_n)^d \subseteq \mathfrak{A}(y)$, and thus, $\mathfrak{A}(y)$ contains every monomial of degree d. But then, each monomial of degree d is in the range of $\psi_d(y)$, and since these monomials form a basis of $S_d(y)$, the map $\psi_d(y)$ is surjective.

Therefore, $y \in \mathcal{N}_d$ iff $\psi_d(y)$ is surjective.

Pick bases for all of the S_{d-d_j}'s and for S_d. Then, $\psi_d(Y)$ is given by a matrix whose entries are polynomials in the Y_j's. We know that $\psi_d(y)$ is surjective iff $\mathrm{rk}\,\psi_d(y) = n_d$, where $n_d = \dim(S_d)(y)$. Therefore, $\psi_d(y)$ will be surjective iff some $n_d \times n_d$ minor of our matrix is nonsingular. This holds if and only if the determinant of this minor is nonzero. However, these determinants for $\psi_d(y)$ are polynomials $q(Y_1, \ldots, Y_m)$. Therefore, $\psi_d(y)$ will be surjective iff y belongs to the Z-open such that some $q(y) \neq 0$. This proves that $\mathcal{N}_d$ is open, and finishes the proof. $\square$

Remark: Homogeneity in the X_i's allowed us to control degrees.

Corollary 1.33 *Let V be a proper variety (e.g., by Theorem 1.32, any projective variety). If W is any quasi-affine variety (i.e., an open in an affine) or any affine variety, then for any morphism $\varphi\colon V \to W$, the image,* Im φ, *of φ is a finite set of points. If V is Z-connected (e.g, norm connected or Z-connected), then φ is constant. In particular, every holomorphic function on V has finitely many values and if V is Z-connected, φ is constant.*

Proof. Since $\mathbb{C}^n$ is separated, W is separated. We have

$$V \longrightarrow W \hookrightarrow \mathbb{C}^n,$$

and thus, we may assume that $W = \mathbb{C}^n$. Pick j, with $1 \leq j \leq n$, and look at

$$V \longrightarrow \mathbb{C}^n \xrightarrow{pr_j} \mathbb{C}^1.$$

If we knew the result for $\mathbb{C}^1$, by a simple combinatorial argument, we would have the result for $\mathbb{C}^n$. Thus, we are reduced to the case $W = \mathbb{C}^1$. In this case, either Im $\varphi = \mathbb{C}^1$, or a finite set of points, since $\mathbb{C}^1$ is irreducible. Furthermore, in the latter case, if V is Z-connected, then Im φ consists of a single point. We need to prove that $\varphi\colon V \to \mathbb{C}^1$ is never surjective. Assume it is. Consider the diagram

$$\begin{tikzcd} V \prod \mathbb{C}^1 \arrow[r, "\varphi \prod \mathrm{id}"] \arrow[dr, "pr_2"'] & \mathbb{C}^1 \prod \mathbb{C}^1 \arrow[d, "\pi_2"] \\ & \mathbb{C}^1 \end{tikzcd}$$

and let

$$H = \{(x, y) \in \mathbb{C}^1 \times \mathbb{C}^1 \mid xy = 1\}.$$

The map $\varphi \prod \mathrm{id}$ is onto. Therefore, $(\varphi \prod \mathrm{id})^{-1}(H)$ is closed and

$$\varphi \prod \mathrm{id}\colon (\varphi \prod \mathrm{id})^{-1}(H) \to H$$

is surjective. Let $C = (\varphi \prod \mathrm{id})^{-1}(H)$. By the definition of proper, $pr_2(C)$ is closed. However, by commutativity of the diagram,

$$pr_2(C) = \pi_2(H),$$

and yet, $\pi_2(H) = \mathbb{C}^1 - \{0\}$ is Z-open, a contradiction on the properness of $\mathbb{C}^1$. $\square$

Corollary 1.34 *(Kronecker's main theorem of elimination) Consider p polynomials $f_1(X_0, \ldots, X_n; Y_1, \ldots, Y_m)$, $\ldots$, $f_p(X_0, \ldots, X_n; Y_1, \ldots, Y_m)$, with coefficients in $\mathbb{C}$ and homogeneous in the X_i's (of varying degrees). Consider further the simultaneous system*

$$f_j(X_0, \ldots, X_n; Y_1, \ldots, Y_m) = 0, \quad \textit{for } j = 1, \ldots, p. \tag{†}$$

Then, there exist polynomials $g_1(Y_1, \ldots, Y_m), \ldots, g_t(Y_1, \ldots, Y_m)$ with coefficients in $\mathbb{C}$ involving only the Y_j's so that (†) has a solution in which not all X_i's are 0 iff the system

$$g_j(Y_1, \ldots, Y_n) = 0, \quad \textit{for } j = 1, \ldots, t, \tag{††}$$

has a solution. (The X_i's have been eliminated).

Proof. The system (†) defines a closed subvariety C of $\mathbb{P}^n \prod \mathbb{C}^m$.

Claim. The set $pr_2(C)$, which, by Theorem 1.32, is closed in $\mathbb{C}^m$, gives us the system (††) by taking the g_j's as a set of polynomials defining $pr_2(C)$. To see this, note that $C = \emptyset$ iff $pr_2(C) = \emptyset$; note further that $(x, y) \in C$ iff (†) has a solution with not all X_i's all zero. Consequently, (†) has a solution with not all X_i zero iff (††) has a solution in the Y_j's. $\square$

Theorem 1.35 *(Irreducibility criterion) Say $\varphi\colon V \to W$ a surjective morphism with W separated and assume that*

(1) V is proper

(2) W is irreducible.

(3) $\varphi^{-1}(w)$ is irreducible for every $w \in W$.

(4) $\dim(\varphi^{-1}(w)) = d$, a constant for all $w \in W$.

Then, V is irreducible.

Proof. Let $V = \bigcup_{j=1}^q V_j$ be an irredundant decomposition of V into irreducible components. Consider V_j. It is closed in V, and thus, $\varphi(V_j)$ is closed in W, because V is proper and W is separated. Since $\varphi\colon V \to W$ is surjective,

$$W = \bigcup_{j=1}^q \varphi(V_j).$$

But W is irreducible; so, it follows (after renumbering, if needed) that $\varphi(V_j) = W$ for $j = 1, \ldots, s$, and $\varphi(V_j)$ is strictly contained in W for $j = s+1, \ldots, q$. Thus,

$$\bigcup_{j=s+1}^q \varphi(V_j)$$

is a Z-closed subset of W strictly contained in W, and

$$\widetilde{W} = W - \bigcup_{j=s+1}^q \varphi(V_j)$$

is a Z-open dense subset of W, as W is irreducible. Let $\widetilde{V} = \varphi^{-1}(\widetilde{W})$, write $\widetilde{V_j} = \widetilde{V} \cap V_j$, and let φ_j be the restriction of φ to $\widetilde{V_j}$. Note,

$$\varphi_j(\widetilde{V_j}) = \varphi(\widetilde{V_j}) = \widetilde{W} \qquad (1 \leq j \leq s),$$

because, given any $w \in \widetilde{W}$, there exists $v \in V_j$ with $\varphi(v) = w$. Since $\varphi(v) \in \widetilde{W}$, the element v is in $\widetilde{V}$. Therefore, $v \in \widetilde{V} \cap V_j$; hence, $v \in \widetilde{V_j}$, as required. Write

$$\mu_j = \min\{\dim(\varphi_j^{-1}(w)) \mid w \in \widetilde{W}\}.$$

By the fibre dimension theorem (Theorem 1.29), there is some nonempty open subset $U_j \subseteq \widetilde{W}$ so that if $w \in U_j$, then $\dim(\varphi_j^{-1}(w)) = \mu_j$. Thus, as $U = \bigcap_{j=1}^s U_j$ is a dense Z-open subset of $\widetilde{W}$, we have a nontrivial Z-dense open, U, so that if $w \in U$, then $\dim(\varphi_j^{-1}(w)) = \mu_j$, for $j = 1, \ldots, s$. Pick $w_0 \in U$. Then

$$\varphi^{-1}(w_0) = \bigcup_{j=1}^s \varphi_j^{-1}(w_0).$$

However, $\varphi^{-1}(w_0)$ is irreducible, and thus, there is some j such that

$$\varphi^{-1}(w_0) = \varphi_j^{-1}(w_0).$$

We may assume that $j = 1$. Since the dimension of the fibres is constant, we get

$$\mu_1 = d.$$

By the fibre dimension theorem, $\dim \varphi_1^{-1}(w) \geq \dim \varphi_1^{-1}(w_0) = d$, for all $w \in W$. Now,

$$\varphi^{-1}(w) = \bigcup_{j=1}^{s} \varphi_j^{-1}(w), \tag{$*$}$$

and since $\dim \varphi_1^{-1}(w) \leq \dim \varphi^{-1}(w) = d$, we must have $\dim \varphi_1^{-1}(w) = d$ for all $w \in W$ and $(*)$ together with the irreducibility of $\varphi^{-1}(w)$ imply that $\varphi^{-1}(w) = \varphi_1^{-1}(w)$, for all $w \in W$. It follows that

$$V = \bigcup_{w\in W} \varphi^{-1}(w) = \bigcup_{w\in W} \varphi_1^{-1}(w) = V_1$$

and since V_1 is irreducible, so is V. $\square$

The conditions of the irreducibilty criterion though sufficient are not necessary. For example, take $\pi\colon B_0(\mathbb{C}^n) \to \mathbb{C}^n$. Outside 0 and $\pi^{-1}(0)$ we have an isomorphism and so, $B_0(\mathbb{C}^n)$ is irreducible (it contains a dense open which is irreducible), yet the fibres don't have constant dimension.

Generally, given a morphism, $\varphi\colon V \to W$, of algebraic varieties, the image, $\varphi(Z)$, of a closed subset of U is *not* closed. Nevertheless, it is natural to ask what kind of set $\varphi(Z)$ is. The answer is given by a theorem of Chevalley and involves sets called constructible.

Definition 1.8 Given a topological space (for example, an affine variety) V, we say that a set Z is *locally closed in V* if

$$Z = U \cap W$$

where U is open and W is closed.

Observe that open and closed sets in a variety are locally closed. Let $Z_i = U_i \cap W_i$, $i = 1, 2$. Then,

$$Z_1 \cap Z_2 = U_1 \cap U_2 \cap W_1 \cap W_2,$$

so that $Z_1 \cap Z_2$ is locally closed. Thus, any finite intersection of locally closed sets is locally closed.

If $Z = U \cap W$, then $Z^c = U^c \cup W^c$, where U^c is closed and W^c is open. It follows that the Boolean algebra generated by the open and closed sets is just the set of finite unions of locally closed sets, denoted $\mathrm{Constr}(V)$. Finite unions of locally closed sets are called *constructible sets*. We have the following important theorem.

Theorem 1.36 *(Chevalley) If $\varphi\colon V \to W$ is a morphism of complex varieties and Z is a constructible set in V, then $\varphi(Z)$ is constructible in W. If φ is dominant (i.e, $\overline{\varphi(V)} = W$), then there is a nonempty open, $U \subseteq W$, so that $U \subseteq \varphi(V) \subseteq W$.*

Proof. Consider the statements

(1) The image of a constructible set in V is a constructible set in W.

(2) If $\varphi(V)$ is dense in W, there is some nonempty Z-open set U in W so that

$$U \subseteq \varphi(V) \subseteq W.$$

We first prove (2). I claim that (2) follows from the case where both V and W are irreducible. Let V' be any irreducible component of V. Then, $\varphi(V')$ is irreducible in W, and thus, $\widetilde{W} = \overline{\varphi(V')}$ is again irreducible and closed in W. Let

$$W = \bigcup_{j=1}^{t} W_j,$$

where the W_j are the irredundant components of W. Then

$$\widetilde{W} = \bigcup_{j=1}^{t} \widetilde{W} \cap W_j,$$

and since $\widetilde{W}$ is irreducible, there is some j, $1 \leq j \leq t$, such that $\widetilde{W} = \widetilde{W} \cap W_j$, i.e.,

$$\widetilde{W} \subseteq W_j.$$

But, if

$$V = \bigcup_{i=1}^{s} V_i$$

is an irredundant decomposition of V, we showed that for every i, $1 \leq i \leq s$, there is some $j = j(i)$ so that

$$\varphi(V_i) \subseteq W_{j(i)}.$$

However,

$$\varphi(V) = \varphi\left(\bigcup_{i=1}^{s} V_i\right) = \bigcup_{i=1}^{s} \varphi(V_i).$$

Therefore,

$$W = \overline{\varphi(V)} = \overline{\bigcup_{i=1}^{s} \varphi(V_i)} = \bigcup_{i=1}^{s} \overline{\varphi(V_i)} \subseteq \bigcup_{i=1}^{s} \overline{W_{j(i)}} = \bigcup_{i=1}^{s} W_{j(i)} \subseteq \bigcup_{j=1}^{t} W_j = W,$$

and the inclusions are all equalities. Since the decompositions are irredundant, the $W_{j(i)}$ run over all the W_j's and, by denseness, $\varphi(V_i)$ is dense in $W_{j(i)}$.

Assume that the theorem (2) holds when V is irreducible (so is W, since $W = \overline{\varphi(V)}$). Then, for every i, there is some Z-open subset $U_i \subseteq W_{j(i)}$ so that

$$U_i \subseteq \varphi(V_i) \subseteq W_{j(i)}.$$

If $C_i = W_{j(i)} - U_i$, then C_i is closed in $W_{j(i)}$, which implies that C_i is closed in W. The image $\varphi(V)$ misses at most

$$C = \bigcup_{i=1}^{s} C_i,$$

which is closed. Therefore, $U = C^c$ is a nonempty Z-open contained in $\varphi(V)$. Therefore, we may assume that V and W are irreducible.

I now claim that we may also assume that V and W are affine. Since W is affine, we have $W = \bigcup_\alpha W_\alpha$ for some affine Z-open sets W_α. Let $V_\alpha = \varphi^{-1}(V_\alpha)$. Note that W_α is Z-dense in W and V_α is Z-dense in V. Thus, if V_α^0 is open and Z-dense in V_α, then V_α^0 is also Z-dense in V. Let V_α^0 be an affine open in V_α, then

(i) V_α^0, W_α are affine.

(ii) V_α^0, W_α are irreducible.

(iii) $\varphi(V_\alpha^0) = W_\alpha$.

Then, there is some open subset U_α of W_α so that

$$U_\alpha \subseteq \varphi(V_\alpha^0) \subseteq \varphi(V_\alpha) \subseteq \varphi(V) \subseteq W,$$

and U_α is open in W (as an open in an open).

We are finally reduced to the *basic case*: V and W are irreducible, affine and φ is dominant.

As $\varphi(V)$ is Z-dense in W, we know that $A[W] \hookrightarrow A[V]$ is an inclusion. Letting $r = \mathrm{tr.d}_{A[W]} A[V]$, we pick some transcendence base $\xi_1, \ldots, \xi_r$ ($\xi_j \in A[V]$) over $A[W]$, so that $A[V]$ is algebraic over $A[W][\xi_1, \ldots, \xi_r]$. Since

$$A[W][\xi_1, \ldots, \xi_r] \cong A[W] \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \ldots, \xi_r],$$

the map

$$A[W] \hookrightarrow A[W][\xi_1, \ldots, \xi_r] \hookrightarrow A[V] \tag{$*$}$$

is just the map

$$\widetilde{\varphi}\colon A[W] \hookrightarrow A[W] \otimes_{\mathbb{C}} A[\xi_1, \ldots, \xi_r] \hookrightarrow A[V].$$

Reading the above geometrically, we get the map

$$\varphi\colon V \xrightarrow{\varphi_1} W \prod \mathbb{C}^r \xrightarrow{pr_1} W.$$

Since each $\eta \in A[V]$ is algebraic over $A[W \prod \mathbb{C}^r]$, we have equations

$$a_0(\xi_1, \ldots, \xi_r)\eta^s + a_1(\xi_1, \ldots, \xi_r)\eta^{s-1} + \cdots + a_s(\xi_1, \ldots, \xi_r) = 0,$$

where the coefficients $a_j(\xi_1, \ldots, \xi_r)$ are functions over W, and thus, depend on $w \in W$, but we omit w for simplicity of notation. If we multiply by $a_0(\xi_1, \ldots, \xi_r)^{s-1}$ and let $\zeta = a_0(\xi_1, \ldots, \xi_r)\eta$, we get

$$\zeta^s + b_1(\xi_1, \ldots, \xi_r)\zeta^{s-1} + \cdots + b_s(\xi_1, \ldots, \xi_r) = 0.$$

Therefore, for every $\eta \in A[V]$, there is some $\alpha \in A[W \prod \mathbb{C}^r]$ so that

$$\zeta = \alpha\eta$$

is integral over $A[W \prod \mathbb{C}^r]$. Since $A[V]$ is finitely generated, there exist $\eta_1, \ldots, \eta_t$ so that

$$A[V] = A[W \prod \mathbb{C}^r][\eta_1, \ldots, \eta_t],$$

and each η_j comes with its corresponding α_j and $\alpha_j\eta_j$ is integral over $A[W \prod \mathbb{C}^r]$. Let

$$b = \prod_{j=1}^{t} \alpha_j(\xi_1, \ldots, \xi_s) \in A[W \prod \mathbb{C}^r].$$

Let U_1 be the Z-open of $W \prod \mathbb{C}^r$ where b is invertible; it is that affine variety in $W \prod \mathbb{C}^r$ whose coordinate ring is $A[W \prod \mathbb{C}^r]_b$. We have $U_1 \subseteq W \prod \mathbb{C}^r$, and on U_1, b and all the α_j's are invertible. Let us look at $\widehat{b} = \widehat{\varphi_1}(b) \in A[V]$, where $\widehat{\varphi_1}\colon A[W \prod \mathbb{C}^r] \to A[V]$ is the algebra homomorphism associated with the morphism $\varphi_1\colon V \to W \prod \mathbb{C}^r$. Then, we get

$$V_{\widehat{b}} \xrightarrow{\varphi_1} U_1.$$

Since each α_j is invertible, on $V_{\widehat{b}}$, each η_j is integral over $A[U_1]$. But $V_{\widehat{b}}$ is generated by the η_j's; so $V_{\widehat{b}}$ is integral over U_1. And therefore, the image of the morphism

$$V_{\widehat{b}} \xrightarrow{\varphi_1} U_1 \tag{†}$$

is closed (DX). Consequently, as $V_{\widehat{b}} \longrightarrow U_1$ has dense image, (†) is a surjection of varieties, and we find

$$U_1 = \varphi_1(V_{\widehat{b}}) \subseteq \varphi_1(V) \subseteq W \prod \mathbb{C}^r.$$

Even though U_1 is a nonempty open, we still need to show that there is some nonempty open $U \subseteq W$ such that $U \subseteq pr_1(U_1)$. For then, we will have $U \subseteq \varphi(V)$. Now, $b \in A[W \prod \mathbb{C}^r]$ means that b can be expressed by a formula of the form

$$b = \sum_{(\beta)} b_{(\beta)}(w)\xi^{(\beta)},$$

where (β) denotes the multi-index $(\beta) = (\beta_1, \ldots, \beta_r)$, $x^{(\beta)} = \xi_1^{\beta_1} \cdots \xi_r^{\beta_r}$, and $b_{(\beta)} \in A[W]$. Let

$$U = \{w \in W \mid \exists(\beta),\, b_{(\beta)}(w) \neq 0\}.$$

The set U is a Z-open set in W. If $w \in U$, since b is a polynomial in the ξ_j's which is not identically null, there is some (β) such that $b_{(\beta)}(w) \neq 0$. Now, $\mathbb{C}$ is infinite, so there are some elements $t_1, \ldots, t_r \in \mathbb{C}$ such that $b(w, t_1, \ldots, t_r) \neq 0$. However, $(w, t_1, \ldots, t_r) \in W \prod \mathbb{C}^r$ and $b(w, t_1, \ldots, t_r) \neq 0$, so that $(w, t_1, \ldots, t_r) \in U_1$ and $pr_1(w, t_1, \ldots, t_r) = w$. Therefore, $U \subseteq pr_1(U_1)$, which concludes the proof of (2).

(1) Say Z is constructible in V, then

$$Z = (U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n),$$

where each $U_i \subseteq V$ is open and each $V_i \subseteq V$ is closed. Since $\varphi(Z) = \bigcup_{j=1}^n \varphi(U_i \cap U_j)$, we may assume that Z is locally closed.

Say we know that $\varphi(Z)$ is constructible if Z is a variety and further, U open (in a variety) implies $\varphi(U)$ is constructible. Take Z locally closed in V, then

$$Z = U_1 \cap V_1 \hookrightarrow V_1 \hookrightarrow V \xrightarrow{\varphi} W,$$

and if we let φ_1 be the composition $V_1 \hookrightarrow V \xrightarrow{\varphi} W$, then φ_1 is a morphism. As Z is open in V_1, by assumption, $\varphi_1(Z) = \varphi(Z)$ is constructible in W.

Now, assume that $\varphi(V)$ is constructible if V is an affine variety. Take U, any open in V. Since V is a variety, we can write $V = \bigcup_\alpha V_\alpha$, where each V_α is affine open. Then, $U = \bigcup_\alpha (U \cap V_\alpha)$ and $\varphi(U) = \bigcup_\alpha \varphi(U \cap V_\alpha)$, where each $U \cap V_\alpha$ is open in an affine. Consequently, we may assume that U is open in an affine. In this case,

$$U = V_{f_1} \cup \cdots \cup V_{f_l},$$

a union of affine varieties and

$$\varphi(U) = \bigcup_{j=1}^{l} \varphi(V_{f_j}),$$

a union of constructible as each V_{f_j} is an affine variety. Therefore, we are reduced to the case where Z is an affine variety and we have to prove that $\varphi(Z)$ is constructible.

I claim that we can assume that $\varphi(V)$ is Z-dense in W. For, Let $\widetilde{W} = \overline{\mathrm{Im}\ \varphi}$. If $\mathrm{Im}\ \varphi$ is constructible in $\widetilde{W}$, then

$$\mathrm{Im}\ \varphi = U_1 \cap \widetilde{W_1} \cup \ldots \cup U_n \cap \widetilde{W_n},$$

where U_j is open in $\widetilde{W}$, and $\widetilde{W_j}$ is closed in $\widetilde{W}$, which implies that $\widetilde{W_j}$ is closed in W. By definition of the relative topology, there are some open sets U_j' in W so that

$$\widetilde{W} \cap U_j' = U_j.$$

Then, we have

$$\begin{aligned}\mathrm{Im}\,\varphi &= (\widetilde{W} \cap U_1') \cap \widetilde{W_1} \cup \cdots \cup (\widetilde{W} \cap U_n') \cap \widetilde{W_n} \\ &= \widetilde{W} \cap (U_1' \cap \widetilde{W_1} \cup \cdots \cup U_n' \cap \widetilde{W_n}) \\ &= U_1' \cap \widetilde{W_1} \cup \cdots \cup U_n' \cap \widetilde{W_n},\end{aligned}$$

a constructible set in W. As a consequence, we may assume that $W = \widetilde{W}$, i.e., that $\mathrm{Im}\,\varphi$ is dense in W. Now, as $\varphi(V)$ is dense in W and V is irreducible, it follows that W is also irreducible and we can finish the proof by induction on $\dim V$.

If $\dim V = 0$, then both V and $\varphi(V)$ consist of a single point and (1) holds trivially.

Assume the induction hypothesis holds if $\dim V = r - 1$ and let $\dim V = r$. By (2), there is some nonempty open subset U of W such that $U \subseteq \varphi(V)$. Let $T = \varphi^{-1}(U)$. This is a Z-open subset of V and moreover, $\varphi(T) = U$. Let $Z = V - T$. The set Z is Z-closed in V, and thus

$$\dim Z < \dim V,$$

and by induction, Chevalley's result holds for Z. But then,

$$\varphi(V) = \varphi(Z) \cup \varphi(T) = \varphi(Z) \cup U,$$

and since $\varphi(Z)$ is constructible and U is open, $\varphi(Z) \cup U$ is also constructible. $\square$

Corollary 1.37 *(of the proof) Say $\varphi\colon V \to W$ is a surjective quasi-finite morphism of complex varieties (i.e., all the fibres are finite). Then, there exist a nonempty Z-open, $U \subseteq W$, so that $\varphi \restriction \varphi^{-1}(U)\colon \varphi^{-1}(U) \to W$ is an integral morphism.*

In order to prove the topological comparison theorem, we need some material on projections.

Let $p \in \mathbb{P}^n$, and let H be a hyperplane such that $p \notin H$. Consider the collection of lines through p, and take any $q \in \mathbb{P}^n$ such that $q \neq p$. Then, p and q define a unique line l_{pq} not contained in H, since otherwise, we would have $p \in H$. The line l_{pq} intersects H in a single point, $\pi_p(q)$. This defines a map

$$\pi_p\colon \mathbb{P}^n - \{p\} \longrightarrow H,$$

called the *projection onto H from p*. We claim that this map is a morphism. For this, let

$$\sum_{j=0}^{n} a_j X_j = 0$$

be an equation defining the hyperplane H; let $p = (p_0 \colon \cdots : p_n)$ and $q = (q_0 \colon \cdots : q_n)$. The line l_{pq} has the parametric equation

$$(s\colon t) \mapsto (sp_0 + tq_0 \colon \cdots : sp_n + tq_n),$$

where $(s\colon t) \in \mathbb{P}^1$. The line l_{pq} intersects H in the point whose coordinates satisfy the equation

$$\sum_{j=0}^{n} a_j(sp_j + tq_j) = 0,$$

and we get

$$s\sum_{j=0}^{n} a_j p_j + t\sum_{j=0}^{n} a_j q_j = 0.$$

However, $\sum_{j=0}^{n} a_j p_j \neq 0$, since $p \notin H$, and thus, we can solve for s in terms of t. We find that $l_{pq} \cap H$ is the point with homogeneous coordinates

$$t\left(-\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_0 + q_0\colon \cdots\colon -\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_n + q_n\right),$$

and this is,

$$\left(-\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_0 + q_0\colon \cdots\colon -\left(\frac{\sum_{j=0}^{n} a_j q_j}{\sum_{j=0}^{n} a_j p_j}\right)p_n + q_n\right),$$

since $t \neq 0$, because $p \notin H$. These coordinates are linear in the q_j's, and thus, the projection map is a morphism.

We may perform a linear change of coordinates so that the equation of the hyperplane H becomes

$$X_n = 0.$$

We get

$$\pi_p(q_0\colon \cdots\colon q_n) = (l_1(q_0, \ldots, q_n)\colon \cdots\colon l_n(q_0, \ldots, q_n)\colon 0),$$

where $l_i(q_0, \ldots, q_n) = -(p_{i-1}/p_n)q_n + q_{i-1}$ is a linear form, for $i = 1, \ldots, n$. Furthermore, these n linear forms do not vanish simultaneously for any $q = (q_0\colon \cdots\colon q_n)$, unless $q = p$, which implies that they are linearly independent.

Conversely, let us take any n linearly independent linear forms $l_1(X_0, \ldots, X_n)$, …, $l_n(X_0, \ldots, X_n)$. These linear forms define some hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}^n$ whose intersection is a point $p \in \mathbb{P}^n$. Then, we have the map $\pi_p\colon (\mathbb{P}^n - \{p\}) \to \mathbb{P}^{n-1}$, defined by

$$\pi_p(X_0\colon \cdots\colon X_n) = (l_1(X_0, \ldots, X_n)\colon \cdots\colon l_n(X_0, \ldots, X_n)).$$

Geometrically, π_p is the projection from p onto the hyperplane $X_n = 0$. We have the following corollary of Theorem 1.32:

Corollary 1.38 *Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension $r < n$ and let $p \in \mathbb{P}^n - X$. Then, projection from p, when restricted to X, is a morphism from X to $\mathbb{P}^{n-1}$. Further, we have the following properties:*

(a) If $X' = \pi_p(X)$, then $\pi_p \restriction X\colon X \to X'$ is a morphism.

(b) X' is closed in $\mathbb{P}^{n-1}$ and r-dimensional.

(c) The fibres of $\pi_p \restriction X$ are finite and there is an open $U \subseteq \pi_p(X)$ so that $\pi_p \restriction \pi_p^{-1}(U)\colon \pi_p^{-1}(U) \to U$ is an integral morphism.

Proof. The map π_p is a morphism outside p, and since $p \notin X$, it is a morphism on X. Since X is closed in $\mathbb{P}^n$, by Theorem 1.32, X' is closed in $\mathbb{P}^{n-1}$. For (c), pick $q \in X'$. Note that $\pi_p^{-1}(q)$ corresponds to the line l_{pq} intersected with X. However, $l_{pq} \not\subseteq X$, since $p \notin X$, and thus, $l_{pq} \cap X \neq l_{pq}$. Then, $l_{pq} \cap X$ is closed in l_{pq}, and since l_{pq} has dimension 1, it follows that $l_{pq} \cap X$ is finite. $\square$

Let L be a linear subspace of $\mathbb{P}^n$, which means that $C(L)$, the cone over L, is a linear subspace of $\mathbb{A}^{n+1}$. Assume that $\dim(L) = \delta$, and let $r = n - \delta - 1$. Then, we can define a morphism $\pi_L\colon (\mathbb{P}^n - L) \to \mathbb{P}^r$. Indeed, if L is cut out by $n - \delta = r + 1$ hyperplanes defined by linear forms $l_0, \ldots l_r$, we let

$$\pi_L(q_0\colon \cdots\colon q_n) = (l_0(q_0, \ldots, q_n)\colon \cdots\colon l_r(q_0, \ldots, q_n)).$$

Geometrically, π_L can described as follows: Let H be a linear subspace of $\mathbb{P}^n$ of dimension $n-\delta-1=r$, disjoint from L. Consider any linear subspace F of dimension $\delta+1=n-r$ through L. Then,

$$\dim(F)+\dim(H)-n=\delta+1+r-n=0.$$

By the projective version of the intersection dimension theorem, $F\cap H$ is nonempty, and $F\cap H$ consists of a single point, $\pi_L(F)$. Thus, we get a map as follows: For every $q\notin L$, if F_q is the span of q and L, then $\dim(F_q)=\delta+1$, and we let

$$\pi_L(q)=F_q\cap H.$$

If we take points $p_1,\ldots,p_{\delta+1}$ spanning L, then we can successively project from the p_j's and we get

$$\pi_L\colon \mathbb{P}^n-L\xrightarrow{\pi_{p_1}}\mathbb{P}^{n-1}-\langle p_2,\ldots,p_{\delta+1}\rangle\xrightarrow{\pi_{p_2}}\cdots\xrightarrow{\pi_{p_\delta}}\mathbb{P}^{n-\delta}-p_{\delta+1}\xrightarrow{\pi_{p_{\delta+1}}}\mathbb{P}^r.$$

Therefore, π_L is the composition of π_{p_j}'s.

We can iterate Proposition 1.38 to prove *Noether's normalization lemma in the projective case.*

Theorem 1.39 *(Noether's normalization lemma–projective case) Let $X\subseteq\mathbb{P}^n$ be an irreducible projective variety, and assume that* $\dim(X)=r<n$. *Then, there is a linear subspace, $L\subseteq\mathbb{P}^n$, so that $\pi_L\colon X\to\mathbb{P}^r$ is surjective and has finite fibres. Moreover, we can choose L and the embedding $\mathbb{P}^r\longrightarrow\mathbb{P}^n$ by a linear change of coordinates so that $\pi_L\restriction X\colon X\to\mathbb{P}^r$ is a finite morphism, i.e., for the projective coordinate rings* $\mathbb{C}[T_0,\ldots,T_r]\hookrightarrow\mathbb{C}[Z_0,\ldots,Z_n]/\mathfrak{I}(X)$*, the right hand side is a finitely generated module over the former.*

Proof. If L has dimension $n-r+1$, then we can choose L so that $L\cap X=\emptyset$. From the previous discussion, π_L is the composition of π_{p_j}'s. Project from p_1. Corollary 1.38 says that $\pi_{p_1}(X)=X_1\subseteq\mathbb{P}^{n-1}$, and that the fibres are finite. Then, $\dim(X_1)=\dim(X)=r$, by the fibre dimension theorem. If $r\neq n-1$, repeat the process. We get a sequence of projections

$$X\xrightarrow{\pi_{p_1}}X_1\longrightarrow\cdots\longrightarrow X'\subseteq\mathbb{P}^r.$$

Since X is irreducible, X' is also irreducible, and $\dim(X')=r=\dim(\mathbb{P}^r)$. Hence, $X'=\mathbb{P}^r$, since $\mathbb{P}^r$ is irreducible. The fibres of π are finite.

In order to prove the second statement of the theorem, we only need to consider a single step, since being a finite module is a transitive property, and we can finish by induction. Pick $p\in\mathbb{P}^n-X$. Using a preliminary linear transformation, we may assume that $p=(0\colon\cdots\colon 0\colon 1)$ and that the linear forms l_j defining p are $l_j(Z_0,\ldots,Z_n)=Z_j$, for $j=0,\ldots,n-1$. Then,

$$\pi_p(q)=(q_0\colon\cdots\colon q_{n-1}).$$

Our result is a question about the affine cones $C(X)$ and $C(X')$, whose rings are
$A[C(X)]=\mathbb{C}[Z_0,\ldots,Z_n]/\mathfrak{I}(X)$ and $A[C(X')]=\mathbb{C}[Z_0,\ldots,Z_{n-1}]/\mathfrak{I}(X')$, where the map of affine rings

$$A[C(X')]\longrightarrow A[C(X)]$$

is given by $Z_j\mapsto Z_j$, $j=0,\ldots,n-1$. There is some $f\in\mathfrak{I}(X)$ such that $f(p)\neq 0$, since $p\notin X$. Let $\deg(f)=\delta$.

Claim. The monomial Z_n^δ appears in f.

If not, all of the monomials appearing in f are of the form

$$Z_n^\epsilon Z_0^{\alpha_0}\cdots Z_{n-1}^{\alpha_{n-1}}$$

where $\epsilon + \alpha_1 + \cdots + \alpha_{n-1} = \delta$ and $\epsilon < \delta$. But then, some $\alpha_i > 0$, and these monomials all vanish at p, a contradiction. Thus,

$$f(Z_0, \ldots, Z_n) = Z_n^\delta + f_1(Z_0, \ldots, Z_{n-1})Z_n^{\delta-1} + \cdots + f_\delta(Z_0, \ldots, Z_n).$$

We know that the map

$$\mathbb{C}[Z_0, \ldots, Z_{n-1}] \longrightarrow \mathbb{C}[Z_0, \ldots, Z_n]/\mathfrak{I}(X)$$

factors through $A[C(X')]$. We only need to prove that $\mathbb{C}[Z_0, \ldots, Z_n]/\mathfrak{I}(X)$ is a finite $\mathbb{C}[Z_0, \ldots, Z_{n-1}]$-module. This will be the case if $\mathbb{C}[Z_0, \ldots, Z_n]/(f)$ is a finite $\mathbb{C}[Z_0, \ldots, Z_{n-1}]$-module. But $\mathbb{C}[Z_0, \ldots, Z_n]/(f)$ is a free $\mathbb{C}[Z_0, \ldots, Z_{n-1}]$-module on the basis $1, Z_n, \ldots, Z_n^{\delta-1}$, and this proves the second statement of the theorem. $\square$

Remark: We can use Proposition 1.38 to show that the degree of a curve is well-defined. Let $C \subseteq \mathbb{P}^n$ be a complex projective curve. We wish to prove that there is an integer, $d \geq 1$, so that for every hyperplane, H, of $\mathbb{P}^n$ the number of intersection points $\#(C \cap H)$ is at most d. The idea is to pick a "good" point, p, outside C, and to project C from p onto $\mathbb{P}^{n-1}$ in such a way that the hyperplanes through p that cut C in at most d points are in one-to-one correspondence with the hyperplanes in $\mathbb{P}^{n-1}$ that cut the projection, C', of C in at most d points. Repeating this procedure, we will ultimately be reduced to the case $n = 3$. A "good" point is a point not on the secant variety of C, i.e., a point so that no line through it meets the curve in at least two distinct points or is tangent to the curve. Since the secant variety has dimension 3, a good point can always be found provided $n \geq 4$. When $n = 3$, a good point may not exist. However, C only has a finite number of singular points and the projection C' of C in $\mathbb{P}^2$ only has finitely more singular points than C. Then, in the case $n = 2$, as C is irreducible, C' is also irreducible and it is given by a single homogeneous equation of degree d which is the desired number.

In order to prove the projective comparison theorem, we will need a refined version of Noether's normalization.

Theorem 1.40 *Let $X \subseteq \mathbb{P}^n$ be an irreducible projective complex variety of dimension r, let L be a linear subspace of dimension $n - r - 1$ so that $L \cap X = \emptyset$, and let p_L be the projection with center L. For any $\xi \in X$, there is some linear subspace M of L of dimension $n - r - 2$, so that the following properties hold:*

(1) If $\pi = p_M \upharpoonright X$, then

$$(\pi)^{-1}(\pi(\xi)) = \{\xi\}.$$

(2) p_L factors as

$$p_L = p_x \circ \pi$$

according to the following commutative diagram, for any $x \notin p_M(X)$:

$$\begin{tikzcd}
 & \mathbb{P}^{r+1} - \{x\} \arrow[r, "p_x"] & \mathbb{P}^r \\
X \arrow[r, "\pi"'] & p_M(X) \arrow[u] \arrow[ur, "p_x"'] &
\end{tikzcd}$$

Proof. We have

$$p_L(\xi) = L(\xi) \cap \mathbb{P}^r,$$

where $L(\xi)$ is the join of L and ξ. Given y, we have

$$p_L(y) = p_L(\xi) \quad \text{iff} \quad y \in L(\xi).$$

Thus, $y \in p_L^{-1}(p_L(\xi))$ iff $y \in L(\xi)$. By the standard version of Noether's normalization, $L(\xi) \cap X$ is a finite set containing ξ, i.e.,

$$L(\xi) \cap X = \{\xi, \eta_1, \dots, \eta_t\}.$$

Let $L^0(\xi)$ be a hyperplane in $L(\xi)$ so that $\xi \in L^0(\xi)$ but $\eta_j \notin L^0(\xi)$ for $j = 1, \dots, t$. Write $M = L^0(\xi) \cap L$. Then, M is a hyperplane in L, since $\xi \notin L$ (recall that $L \cap X = \emptyset$ and $\xi \in X$). Observe that

$$M(\xi) = L^0(\xi).$$

For any $y \in X$, we have

$$\pi(y) = \pi(\xi) \quad \text{iff} \quad y \in M(\xi) \cap X \quad \text{iff} \quad y \in L^0(\xi) \cap X.$$

But $L^0(\xi) \cap X = \{\xi\}$, by construction of $L^0(\xi)$. Thus, $y \in (\pi)^{-1}(\pi(\xi))$ iff $y = \xi$, proving (1).

To prove (2) is now very easy. Take x so that $x \notin p_M(X)$ and $M(x) = L$. The rest is clear. $\square$

Theorem 1.41 *(Comparison theorem) Suppose X is a complex variety and Y is a Z-constructible subset of X. If Y is Z-dense in X then Y is norm-dense in X.*

Proof. (Mumford and Stolzenberg) Note if $\overline{\overline{U}}$ = the Z-closure of U and $\overline{U}$ = the norm-closure of U (clearly, $\overline{U} \subseteq \overline{\overline{U}}$), then the assertion of the theorem is that $\overline{\overline{U}} = \overline{U}$ if U is constructible.

First, assume Y is Z-open in X. Write $X = \bigcup_\alpha X_\alpha$, where X_α is affine, open (and there are only finitely many α, since X is a variety). Assume the theorem holds for affines. Then, $Y = X \cap Y = \bigcup_\alpha Y_\alpha$, where $Y_\alpha = Y \cap X_\alpha \subseteq X_\alpha$. We get

$$\overline{\overline{Y}} = \overline{\overline{\bigcup_\alpha Y_\alpha}} = \bigcup_\alpha \overline{\overline{Y_\alpha}},$$

and as Y_α is open in the affine X_α, by hypothesis, we have $\overline{\overline{Y_\alpha}} = \overline{Y_\alpha}$. Thus,

$$\overline{\overline{Y}} = \bigcup_\alpha \overline{\overline{Y_\alpha}} = \bigcup_\alpha \overline{Y_\alpha} = \overline{Y}.$$

Therefore, we may assume that X is affine. Then, $X \hookrightarrow \mathbb{C}^n$ is Z-closed and $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$ is Z-dense. Let $\widetilde{X}$ be the Z-closure of X in $\mathbb{P}^n$. Then, X is Z-dense in $\widetilde{X}$ and Y is Z-dense in X, so Y is Z-open in $\widetilde{X}$. If we assume that the theorem holds for projective varieties, we get $\overline{\overline{Y}} = \widetilde{X} = \overline{Y}$ and $\overline{Y} \cap \mathbb{C}^n = \widetilde{X} \cap \mathbb{C}^n = X$. Then, Y is norm-dense in X (as X is norm-closed in $\mathbb{C}^n$). So, we may assume that X is actually projective. Finally, assume that the theorem holds for irreducible projective varieties. Write $X = \bigcup_{j=1}^n X_j$, where each X_j is irreducible. Then, $Y = Y \cap X = \bigcup_{j=1}^n Y \cap X_j$, so

$$\overline{\overline{Y}} = \overline{\overline{\bigcup_{j=1}^n Y \cap X_j}} = \bigcup_{j=1}^n \overline{\overline{Y \cap X_j}} = \bigcup_{j=1}^n \overline{Y \cap X_j} = \overline{\bigcup_{j=1}^n Y \cap X_j} = \overline{Y}.$$

Therefore, we are reduced to the

Minimal Case: $X \subseteq \mathbb{P}^n$ is projective irreducible and Y is Z-open in X. If so, Y is automatically Z-dense. Pick $\xi \in X - Y$. We'll show that ξ is the limit in the norm topology of a sequence of points in Y. Now, $\dim(X) = r$, and we can pick M and L as in the refined version of Noether's normalization theorem with respect to ξ (Theorem 1.40). We also choose $x \notin p_M(X)$. We may choose coordinates so that

(1) M is cut out by $Y = \cdots = X_{r+1} = 0$.

(2) $\xi = (1\colon 0\colon \cdots : 0)$.

(3) L is cut out by $Y = \cdots = X_r$, and
$$x = (\underbrace{0\colon \cdots\colon 0}_{r+1}\colon 1).$$

Look at $p_L(X - Y) \subseteq \mathbb{P}^r$. The image is closed, and thus, contained in some hypersurface $f = 0$, for some homogeneous polynomial, $f(Y, \ldots, X_r)$. Therefore,

$$\{x \in X \mid f(p_L(x)) \neq 0\} \subseteq Y,$$

and we may replace Y by the above open set. By (2) of Theorem 1.40, $p_M(X)$ has dimension r, and $p_M(X) \subseteq \mathbb{P}^{r+1}$, which implies that $p_M(X)$ is a hypersurface. Thus,

$$p_M(X) = \{y = (y_0\colon \cdots\colon y_{r+1}) \mid F(y) = 0\},$$

for some homogeneous form, $F(Y_0, \ldots, Y_{r+1})$ (of degree d). The rest of the argument has three stages:

Stage 1: Approximating in $\mathbb{P}^r$. Since $f \neq 0$, there is some nontrivial $(\alpha_0, \ldots, \alpha_r) \in \mathbb{C}^{r+1}$ such that $f(\alpha_0, \ldots, \alpha_r) = 0$ (because $\mathbb{C}$ is algebraically closed). Let

$$\xi_0 = p_L(\xi) \in \mathbb{P}^r.$$

By choice, $\xi_0 = (1\colon 0\colon \cdots\colon 0) \in \mathbb{P}^r$. Look at points

$$\xi_0 + t\alpha = (1 + t\alpha_0, t\alpha_1, \ldots, t\alpha_r).$$

Then, $f(\xi_0 + t\alpha) = f(1 + t\alpha_0, t\alpha_1, \ldots, t\alpha_r)$ is a polynomial in t. However, a polynomial in one variable has finitely many zeros. Thus, there exists a sequence $(t_i)_{i=1}^{\infty}$ so that

(1) $f(\xi_0 + t_i\alpha) \neq 0$.

(2) $t_i \to 0$ as $i \to \infty$.

(3) $\xi_0 + t_i\alpha \to \xi_0$ as $i \to \infty$.

Stage 2: Approximating in $\mathbb{P}^{r+1}$. We know that $p_M(X)$ is the hypersurface given by $F(Y, \ldots, X_{r+1}) = 0$, and $x = (0\colon \cdots\colon 0\colon 1)$. Write F as

$$F(Y, \ldots, X_{r+1}) = \gamma X_{r+1}^d + a_1(Y, \ldots, X_r)X_{r+1}^{d-1} + \cdots + a_d(Y, \ldots, X_r). \qquad (*)$$

Claim. There exists a sequence (b_i) so that

(1) $b_i \in p_M(X)$.

(2) $b_i \to \xi_0 + t_i\alpha$ (under p_x).

(3) $\lim_{i\to\infty} b_i = (1\colon 0\colon \cdots\colon 0) = p_M(\xi)$.

In order to satisfy (2), the b_i must be of the form

$$b_i = (1 + t_i\alpha_0\colon t_i\alpha_1\colon \cdots\colon t_i\alpha_r\colon \beta^{(i)}),$$

for some $\beta^{(i)}$ yet to be determined. We also need to satisfy (1); that is, we must have

$$F(1 + t_i\alpha_0\colon t_i\alpha_1\colon \cdots\colon t_i\alpha_r\colon \beta^{(i)}) = 0.$$

We know that $x \notin p_M(X)$, which implies that $F(x) \neq 0$, and since $x = (0\colon \cdots : 0\colon 1)$, by $(*)$, we must have $\gamma \neq 0$. The fact that $p_M(\xi) \in p_M(X)$ implies that $F(p_M(\xi)) = 0$. Since $p_M(\xi) = (1\colon 0\colon \cdots : 0)$, from $(*)$, we get $a_d(\xi_0) = F(p_M(\xi)) = 0$. Also, by $(*)$, $\beta^{(i)}$ must be a root of

$$\gamma Y^d + a_1(\xi_0 + t_i\alpha)Y^{d-1} + \cdots + a_d(\xi_0 + t_i\alpha) = 0. \tag{$**$}$$

Thus, we get (2). To get (3), we need $\beta^{(i)} \to 0$ when $i \to \infty$. Now, as $i \to \infty$, $t_i \to 0$; but the product of the roots in $(**)$ is

$$\pm\frac{a_d(\xi_0 + t_i\alpha)}{\gamma},$$

and this term tends to 0 as i tends to infinity. Then, some root must tend to 0, and we can pick $\beta^{(i)}$ in such a manner, so that $\lim_{i\to\infty} \beta^{(i)} = 0$. Thus, we get our claim.

Stage 3: Lifting back to $\mathbb{P}^n$. Lift each b_i in any arbitary manner to some $\eta_i \in X \subseteq \mathbb{P}^n$. We know that $\mathbb{P}^n$ is compact, since $\mathbb{C}$ is locally compact. Thus, the sequence (η_i) has a convergent subsequence. By restriction to this subsequence, we may assume that (η_i) converges, and we let η be the limit. Now, $\eta_i \in X$ and X is closed, so that $\eta \in X$. We have

$$p_M(\eta) = \lim_{i\to\infty} p_M(\eta_i) = \lim_{i\to\infty} b_i = p_M(\xi),$$

since p_M is continuous. Therefore,

$$\eta \in p_M^{-1}(p_M(\xi)) = \{\xi\},$$

and thus, $\eta = \xi$. Now,

$$f(p_L(\eta_i)) = f(p_x(p_M(\eta_i))) = f(p_x(b_i)) = f(\xi_0 + t_i\alpha) \neq 0,$$

and thus, $\eta_i \in Y$. This proves that Y is norm-dense.

Having shown that the theorem holds when Y is Z-open, let Y be Z-constructible. Then, we can write $Y = \bigcup_{j=1}^n Y_j \cap X_j$, where Y_j is open in X and X_j is closed in X. We have

$$\overline{\overline{Y}} = \overline{\overline{\bigcup_{j=1}^n Y_j \cap X_j}} = \bigcup_{j=1}^n \overline{\overline{Y_j \cap X_j}}.$$

But, $Y_j \cap X_j$ is Z-open in X_j, where X_j is some variety. By the open case, $\overline{\overline{Y_j \cap X_j}} = \overline{Y_j \cap X_j}$ in X_j; since X_j is closed in X, we see that $\overline{Y_j \cap X_j}$ is the closure of $Y_j \cap X_j$ in X. Therefore, $\overline{\overline{Y_j \cap X_j}} = \overline{Y_j \cap X_j}$ in X and $\overline{\overline{Y}} = \overline{Y}$, as required. $\square$

Now, we can get results comparing the Z-topology and the norm topology.

Theorem 1.42 *(Theorem A) Say X is a complex variety (not necessarily separated). Then, X is separated (in the Z-topology) iff X is Hausdorff (in the norm topology).*

Proof. ($\Leftarrow$). By hypothesis, X is Hausdorff in the norm topology. We know Δ is closed in $X \prod X$ in the norm topology ($U \cap V = \Delta^{-1}(\Delta \cap (U \prod V))$). But, $\Delta = \Delta(X)$ is the image of a morphism; it follows that Δ is constructible, by Chevalley's Theorem. Then, by the comparison theorem (Theorem 1.41), $\overline{\overline{\Delta}} = \overline{\Delta}$ and since Δ is norm closed, $\overline{\Delta} = \Delta$, which yields $\overline{\overline{\Delta}} = \Delta$, i.e., Δ is Z-closed. By definition, this means that X is separated.

($\Rightarrow$). Suppose X separated, so Δ is Z-closed in $X \prod X$. It follows that Δ is norm closed in $X \prod X$, which implies X is Hausdorff. $\square$

To deal with compacteness and properness, we need a comparison theorem between projective varieties and proper varieties. This is Chow's lemma:

Theorem 1.43 *(Chow's Lemma) Say $\pi\colon X \to Y$ is a proper morphism. Then, there exists a Y-projective variety, P, i.e., $P \hookrightarrow \mathbb{P}^M \prod Y$ as a closed subvariety, and a surjective birational morphism, $f\colon P \to X$, so that the diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & \mathbb{P}^M \prod Y \arrow[d, "pr_2"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

commutes. In particular, when Y is a point, then for every X, proper over $\mathbb{C}$, there exists a projective, complex, variety, P, and a surjective birational morphism, $f\colon P \to X$. If X is irreducible, we may choose P irreducible.

Theorem 1.44 *(Theorem B) Assume $\pi\colon X \to Y$ is a morphism of complex (separated) varieties. Then, $\pi\colon X \to Y$ is a norm-proper morphism (i.e., for every compact, $C \subseteq Y$, the set $\pi^{-1}(C)$ is compact in X) iff $\pi\colon X \to Y$ is a Z-proper morphism. In particular, when Y is a point, X is compact (in the norm topology) iff X is proper (in the Z-topology).*

Proof. ($\Rightarrow$). By hypothesis, $\pi\colon X \to Y$ is proper (in the norm topology).

Claim. For every T, a complex variety over Y, the morphism $X \prod_Y T \xrightarrow{pr_2} T$ is a norm-closed map.

Pick T, pick Q norm-closed in $X \prod_Y T$ and let $C = pr_2(Q)$. Further, pick a sequence, $\{\xi_j\}_{j\geq 1} \subseteq C$, and assume that $\{\xi_j\}$ converges to $\xi_0 \in T$. If $D = \{\xi_j\}_{j\geq 1} \cup \{\xi_0\}$, we know that D is compact. If we know that $pr_2^{-1}(D)$ is compact, then lift ξ_j ($j \neq 0$) to some $\eta_j \in Q$. Since $\{\eta_j\}$ is contained in $pr_2^{-1}(D)$, which is compact, the sequence $\{\eta_j\}$ has a converging subsequence; so, we may assume that the sequence $\{\eta_j\}$ converges and let its limit be $\eta_0 \in X \prod_Y T$. Now, $\eta_0 \in Q$, as Q is norm-closed. It follows that $\xi_0 = pr_2(\eta_0) \in pr_2(Q) = C$ and C is norm-closed. Therefore, we just have to prove that if D is compact, then $X \prod_Y D$ is compact.

Pick a sequence, $\{\eta_j\} \in X \prod_Y D$, and write $\xi_j = pr_2(\eta_j)$. We have $\eta_j = (x_j, \xi_j) \in X \prod_Y D$, so $\pi(x_j) = \nu(\xi_j)$, where $\nu\colon D \to Y$. As D is compact, we may assume that that $\{\xi_j\}$ converges to some ξ_0. Then, the $\nu(\xi_j)$'s converge to $\nu(\xi_0)$. As $\eta_j = (x_j, \xi_j)$, we deduce that $x_j \in \pi^{-1}(\bigcup_{j\geq 1} \nu(\xi_j) \cup \nu(\xi_0))$. Now, as $\bigcup_{j\geq 1} \nu(\xi_j) \cup \nu(\xi_0)$ is compact in Y, by hypothesis, $\pi^{-1}(\bigcup_{j\geq 1} \nu(\xi_j) \cup \nu(\xi_0))$ is compact, so the x_j's have a converging subsequence. We deduce that $\{\eta_j\}$ also has a converging subsequence and $X \prod_Y D$ is compact, as required.

Now, we can show that π is Z-proper. Pick T, pick Q Z-closed in $X \prod_Y T$ and let $C = pr_2(Q)$. We know pr_2 is a morphism, so C is constructible (by Chevalley). But, C is norm-closed, by the above agument. Consequenly, by the comparison theorem, $\overline{\overline{C}} = \overline{C} = C$, which shows that C is Z-closed.

($\Leftarrow$). By hypothesis, $\pi\colon X \to Y$ is a proper morphism (in the Z-topology). We need to show that $\pi^{-1}(C)$ is norm-compact whenever C is norm-compact. By Chow's Lemma, there is P projective over Y and a surjective morphism $f\colon P \to X$ so that the diagram

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "f"'] & \mathbb{P}^M \prod Y \arrow[d, "pr_2"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

commutes. Pick a compact, C, in Y. We have $pr_2^{-1}(C) = \mathbb{P}^M \prod C$, which is compact (both $\mathbb{P}^M$ and C are compact); we also have $i^{-1}(pr_2^{-1}(C)) = pr_2^{-1}(C) \cap P$. But, P is closed, so $i^{-1}(pr_2^{-1}(C))$ is compact. Yet, $f^{-1}(\pi^{-1}(C)) = i^{-1}(pr_2^{-1}(C))$ and as f is surjective and continuous, $f(f^{-1}(\pi(C))) = \pi^{-1}(C)$ and $\pi^{-1}(C)$ is compact. $\square$

Proof of Chow's Lemma (Theorem 1.43). Say the theorem holds when X is irreducible. Then, for any complex variety, X, we have $X = \bigcup_{i=1}^t X_j$, where the X_j's are irreducible. By Chow's lemma in the irreducible case, for each X_j, there is a variety, P_j, Y-projective, irreducible and a birational surjective morphism, $P_j \longrightarrow X_j$. Let $P = \coprod_{j=1}^t P_j$; I claim P is projective over Y. Since P_j is Y-projective if we cover Y by affines, Y_α (coordinate rings A_α), then P_j is covered by varieties whose homogeneous coordinate rings are

$$A_\alpha[Z_0^{(j)}, \ldots, Z_{N_j}^{(j)}]/\mathfrak{I}_\alpha$$

and the glueing is by glueing the A_α's. Fix α and look at

$$A_\alpha[T_0^{(1)}, \ldots, T_{N_1}^{(1)}, \ldots, T_0^{(t)}, \ldots, T_{N_t}^{(t)}] = B_\alpha,$$

let $M = N_1 + \cdots + N_t - 1$ and send $T_l^{(j)}$ to $Z_l^{(j)}$ (of P_j) for each j, monomials in a fixed number of variables to similar monomials and all "mized" products $Z_l^{(i)} Z_m^{(j)}$ to 0. This gives a homogeneous ideal, $\mathfrak{I}_\alpha$ of B_α and we can glue $\mathrm{Proj}(B_\alpha/\mathfrak{I}_\alpha)$ and $\mathrm{Proj}(B_\beta/\mathfrak{I}_\beta)$ *via* glueing on A_α and A_β. We get a projective variety in $\mathbb{P}^M \prod Y$. Check (DX), this is P; so, P is projective over Y. As $P_j \longrightarrow X_j$ is surjective, $\coprod_j P_j \longrightarrow \dot{\bigcup}_j X_j$ is surjective.

Let U_j be a Z-open of X_j isomorphic to a Z-open of P_j *via* our birational morphism $P_j \longrightarrow X_j$. Write $\widetilde{U_j} = U_j \cap \left(\bigcap_{i \neq j} X_i^c\right)$; by irreducibility, $\widetilde{U_j}$ is Z-open and Z-dense in X_j and isomorphic to a Z-open of P_j. The $\widetilde{U_j}$ are disjoint and so, $\bigcup_{j=1}^t \widetilde{U_j}$ is a Z-open of X isomorphic to the corresponding Z-open of P. Therefore, we may assume that X is irreducible.

Cover Y by open affines, Y_α, then the $X_\alpha = \pi^{-1}(Y_\alpha)$ cover X. Each X_α has an open affine covering (finite), say X_α^β and for all α, β, there is a closed immersion $X_\alpha^\beta \hookrightarrow \mathbb{C}^{N_\alpha^\beta} \prod Y$. Consequently, there is a locally closed immersion $X_\alpha^\beta \hookrightarrow \mathbb{P}^{N_\alpha^\beta} \prod Y$. All the X_α^β are Z-open, Z-dense in X as X is irreducible, so,

$$U = \bigcap_{\alpha,\beta} X_\alpha^\beta$$

is Z-open and Z-dense in X and we get a locally closed immersion $U \hookrightarrow \mathbb{P}^{N_\alpha^\beta} \prod Y$, for all α, β. By the Segre morphism, there is some $M > 0$ so that

$$U \hookrightarrow \mathbb{P}^M \prod Y,$$

a locally closed embedding. We also have locally closed embeddings $X_\alpha^\beta \hookrightarrow \mathbb{P}^M \prod Y$. If $\widetilde{P}_\alpha^\beta$ is the Z-closure of X_α^β in $\mathbb{P}^{N_\alpha^\beta} \prod Y$, then $\widetilde{P} = \prod_{\alpha,\beta} \widetilde{P}_\alpha^\beta$ is Z-closed in $\mathbb{P}^M \prod Y$, so $\widetilde{P}$ is Y-projective, as are the $\widetilde{P}_\alpha^\beta$. We have the diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d] & \prod_{\alpha,\beta} \widetilde{P}_\alpha^\beta = \widetilde{P} \arrow[d, "pr_\alpha^\beta"] \\ X_\alpha^\beta \arrow[r] & \widetilde{P}_\alpha^\beta \end{tikzcd} \tag{$\dagger$}$$

and we get the morphisms

(a) $U \hookrightarrow X_\alpha^\beta \hookrightarrow X$

(b) $j \colon U \hookrightarrow \widetilde{P}$ (locally closed)

(c) $j_\alpha^\beta\colon X_\alpha^\beta \hookrightarrow \widetilde{P}_\alpha^\beta$ (Y-projective)

(d) $\widetilde{P} = \prod_{\alpha,\beta} \widetilde{P}_\alpha^\beta$ is Y-projective.

The maps (a) and (b) give us a morphism

$$\psi\colon U \to X \prod_Y \widetilde{P},$$

a locally closed immersion (*via* $\psi(\xi) = (\xi, (j_\alpha^\beta(\xi)))$). Let $P = \overline{\overline{\psi(U)}}$ be the Z-closure of $\psi(U)$ in $X \prod_Y \widetilde{P}$. Note, P is closed in $X \prod_Y \widetilde{P}$ and X is proper over Y, so $pr_2(P)$ is closed in $\widetilde{P}$, thus it is Y-projective. Now, we have two maps on P:

$$f\colon P \overset{g}{\hookrightarrow} X \prod_Y \widetilde{P} \xrightarrow{pr_1} X \quad \text{and} \quad \theta\colon P \overset{g}{\hookrightarrow} X \prod_Y \widetilde{P} \xrightarrow{pr_2} \widetilde{P},$$

and we conclude that $\theta(P)$ is Y-projective.

(A) The map f is the morphism we seek.

(A1) I claim f is surjective. As $\widetilde{P}$ is projective, it is proper over Y, so $f(P)$ is closed in X. But, the diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d] \arrow[dr, "f"] & P \arrow[d, "pr_2"] \\ X_\alpha^\beta \arrow[r, hook] & X \end{tikzcd}$$

commutes, so $U \subseteq f(P)$; but, U is Z-dense and $f(P)$ is closed in X, which implies $X = f(P)$. Therefore, f is surjective.

(A2) As the fibred product $U \prod_Y \widetilde{P}$ is Z-open in $X \prod_Y \widetilde{P}$, consider the Z-open $g^{-1}(U \prod_Y \widetilde{P}) = (U \prod_Y \widetilde{P}) \cap P$. We have $f^{-1}(U) = g^{-1}(pr_1^{-1}(U)) = g^{-1}(U \prod_Y \widetilde{P})$.

Claim: $f\colon f^{-1}(U) \xrightarrow{\sim} U$.

We have the factorization

$$\psi\colon U \xrightarrow{\Gamma} U \prod_Y \widetilde{P} \longrightarrow X \prod_Y \widetilde{P}$$

and $(U \prod_Y \widetilde{P}) \cap P$ is the Z-closure of $\Gamma(U)$ in $U \prod_Y \widetilde{P}$. Yet, $\widetilde{P}$ is separated (Hausdorff), so $\Gamma(U)$ *is* Z-closed in $U \prod_Y \widetilde{P}$; this shows that $f^{-1}(U)$ is the Z-closure of $\Gamma(U)$, i.e., $f^{-1}(U) = \Gamma(U)$. But, $pr_1\colon \Gamma(U) \longrightarrow U$ is an isomorphism and it follows that $f\colon f^{-1}(U) \to U$ is an isomorphism and f is thereby birational.

(B) The map θ is an immersion. The question is local on P.

(B1) As the X_α^β cover X, if we set $X_\alpha^{\beta'} = f^{-1}(X_\alpha^\beta)$, we see that the $X_\alpha^{\beta'}$ cover P.

(B2) As $j_\alpha^\beta(X_\alpha^\beta)$ is open in $\widetilde{P}_\alpha^\beta$, the set $W_\alpha^\beta = pr_{\alpha,\beta}^{-1}(j_\alpha^\beta(X_\alpha^\beta))$ is Z-open in $\widetilde{P}$. Let $X_\alpha^{\beta''} = \theta^{-1}(W_\alpha^\beta)$, Z-open in P.

Set-theoretically, we have

(a) As $P = X \prod_Y \widetilde{P}$, any $\eta \in P$ is of the form $\eta = (y, (\eta_\gamma^\delta))$, where $\eta_\gamma^\delta \in \widetilde{P}_\alpha^\beta$ and $y \in X$. As, $f(y, (\eta_\gamma^\delta)) = y$, we see that

$$X_\alpha^{\beta'} = \{(y, (\eta_\gamma^\delta)) \in P \mid y \in X_\alpha^\beta\}.$$

(b) Since $\theta(y, (\eta_\gamma^\delta)) = (\eta_\gamma^\delta)$, we have

$$\begin{aligned} X_\alpha^{\beta''} &= \{(y, (\eta_\gamma^\delta)) \in P \mid (\eta_\gamma^\delta) \in W_\alpha^\beta\} \\ &= \{(y, (\eta_\gamma^\delta)) \in P \mid \eta_\alpha^\beta = j_\alpha^\beta(q),\ q \in X_\alpha^\beta\}. \end{aligned}$$

Consider $f^{-1}(U)$ and the incomplete diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[dr, "\varphi"'] & W_\alpha^\beta \arrow[d, "\Gamma_\alpha^\beta"] \\ & X \prod_Y W_\alpha^\beta. \end{tikzcd} \tag{†}$$

We have $f^{-1}(U) = \Gamma(U) = \{(\xi, (j_\gamma^\delta(\xi)) \mid \xi \in U\}$ and

$$\varphi(\xi, (j_\gamma^\delta(\xi)) = (\xi, (j_\gamma^\delta(\xi)) \in X \prod_Y W_\alpha^\beta,$$

since $W_\alpha^\beta = \{(\eta_\gamma^\delta) \mid \eta_\alpha^\beta = j_\alpha^\beta(x),\ x \in X_\alpha^\beta\}$. We have the map $w_\alpha^\beta \colon W_\alpha^\beta \to X$ defined as follows: $(\eta_\gamma^\delta) \mapsto x$, where $\eta_\alpha^\beta = j_\alpha^\beta(x)$. Also, let Γ_α^β be the graph of the morphism w_α^β given by $(\eta_\gamma^\delta) \mapsto (x, (\eta_\gamma^\delta))$, where $x \in X_\alpha^\beta$ and $\eta_\alpha^\beta = j_\alpha^\beta(x)$. Therefore, there is a morphism, $z_\alpha^\beta \colon f^{-1}(U) \to W_\alpha^\beta$ rendering (†) commutative. As $f^{-1}(U) \subseteq \operatorname{Im}\Gamma_\alpha^\beta$ and X is Hausdorff, we deduce that $\operatorname{Im}\Gamma_\alpha^\beta$ is Z-closed, so the Z-closure of $f^{-1}(U)$ in $X \prod_Y W_\alpha^\beta$ is contained in $\operatorname{Im}\Gamma_\alpha^\beta$. Now, $\operatorname{Im}\Gamma_\alpha^\beta \cong W_\alpha^\beta$, and so, the Z-closure of $f^{-1}(U)$ is contained in a Z-closed set isomorphic to W_α^β, which means that the Z-closure of $f^{-1}(U)$ in $X \prod_Y W_\alpha^\beta$ is $P \cap X_\alpha^{\beta''}$. Therefore, $\theta(P \cap X_\alpha^{\beta''})$ is isomorphic to a Z-closed subset of W_α^β. Now, if the $X_\alpha^{\beta''}$ were to cover P, we would find that θ is an immersion $P \hookrightarrow \widetilde{P}$. However, I claim:

(B3) For all α, β, we have $X_\alpha^{\beta'} \subseteq X_\alpha^{\beta''}$.

As f is surjective, the $X_\alpha^{\beta'}$ cover P, so the claim will show that the $X_\alpha^{\beta''}$ also cover P; this will imply that $P \hookrightarrow \widetilde{P}$ is a closed immersion and we will be done.

Look at $f^{-1}(U) = \{(\xi, j_\gamma^\delta(\xi)) \mid \xi \in U\}$. The diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "pr_2"] \arrow[d, "f"'] & \widetilde{P} \arrow[d, "pr_{\alpha,\beta}"] \\ U \arrow[r, "j_\alpha^\beta"'] & \widetilde{P}_\alpha^\beta \end{tikzcd}$$

commutes, for

$$(\xi, j_\gamma^\delta(\xi)) \xrightarrow{pr_2} (j_\gamma^\delta(\xi)) \xrightarrow{pr_{\alpha,\beta}} j_\alpha^\beta(\xi)$$

and

$$(\xi, j_\gamma^\delta(\xi)) \xrightarrow{f} \xi \longrightarrow j_\alpha^\beta(\xi).$$

Take the closure of $f^{-1}(U)$ in $X_\alpha^{\beta'}$. Now, $f^{-1}(U)$ is Z-dense in $X_\alpha^{\beta'}$ as U is Z-dense in X_α^β, so by taking closures, we get the commutative diagram

$$\begin{tikzcd} X_\alpha^{\beta'} \arrow[r, "pr_2"] \arrow[d, "f"'] & \widetilde{P} \arrow[d, "pr_{\alpha,\beta}"] \\ X_\alpha^\beta \arrow[r, "j_\alpha^\beta"'] & P_\alpha^\beta. \end{tikzcd}$$

Take $(x, (\eta_\gamma^\delta)) \in X_\alpha^{\beta'}$ where $x \in X_\alpha^\beta$, we have

$$(x, (\eta_\gamma^\delta)) \xrightarrow{pr_2} (\eta_\gamma^\delta) \longrightarrow \eta_\alpha^\beta$$

and

$$(x, (\eta_\gamma^\delta)) \xrightarrow{f} x \longrightarrow j_\alpha^\beta(x),$$

and the commutativity of the above diagram implies $\eta_\alpha^\beta = j_\alpha^\beta(x)$. Therefore, the tuple $(x, (\eta_\gamma^\delta))$ we started from belongs to $X_\alpha^{\beta''}$, which proves that $X_\alpha^{\beta'} \subseteq X_\alpha^{\beta''}$, as contended. $\square$

Now, we need to prove the connectedness theorem. We need two remarks.

(1) Say $U \neq \emptyset$ is Z-open in X, where X is a complex irreducible variety and suppose U is norm-connected, then X is norm-connected.

Proof. The set U is Z-open, Z-dense in X and by the comparison theorem, U is norm-dense in X. Yet the norm-closure of a norm-connected is norm-connected, so we conclude that X is norm-connected. $\square$

(2) Say $U \neq \emptyset$ is Z-open in $\mathbb{C}^n$. Then, U is norm-connected.

Proof. Take $\xi, \eta \in U$ and let l be the line $(\xi, \eta) \subseteq \mathbb{C}^n$. Since $l \not\subseteq U^c$, we deduce that $U^c \cap l$ is a Z-closed subset of l distinct from l. As $\dim l = 1$, we deduce that $U^c \cap l$ consists of a finite set of points. As l is a complex line, $l \cong \mathbb{C}$, and so, $U \cap l = \mathbb{C} - F$, where F is a finite set of points. But, $\mathbb{C} - F$ is arc-connected, which implies that ξ and η are arc-connected. Since ξ and η are arbitrary, we conclude that U is norm-connected. $\square$

Theorem 1.45 *(Theorem C) A complex algebraic variety, X, is Z-connected iff it is norm-connected.*

Proof. ($\Leftarrow$). This is trivial as the Z-topology is coarser than the norm-topology.

($\Rightarrow$). We may assume that X is Z-irreducible. For, suppose we know the theorem when X is irreducible. Then, for any variety, we can write $X = \bigcup_{i=1}^t X_i$, an irredundant decomposition into irreducible components and let U, V be a disconnection of X, which means that $X = U \cup V$, where U and V are norm-open and norm-dense nonempty disjoint subsets of X. Let $U_i = X_i \cap U$ and $V_i = X_i \cap V$, then U_i, V_i form a norm disconnection of X_i. As X_i is norm-connected either $U_i = X_i$ and $V_i = \emptyset$ or $U_i = \emptyset$ and $V_i = X_i$. Write $i \in V$ iff $V_i = X_i$ (iff $X_i \subseteq V_i$) and $i \in U$ iff $U_i = X_i$ iff ($X_i \subseteq U_i$), then

$$X = \left(\bigcup_{i \in U} X_i\right) \cup \left(\bigcup_{i \in V} X_i\right)$$

and the first part of the union is a subset of U whereas the second is a subset of V. Therefore, the above is a disjoint union, contradicting the hypothesis that X is Z-connected. Therefore, we may assume that X is irreducible.

Now, assume that the theorem holds if X is affine irreducible. Take a Z-open affine, U, in X (X irreducible). As the Z-open U is Z-dense and X is irreducible, U is Z-irreducible. Then, Theorem C applies to U and U is norm-connected. But, U is norm-dense by the comparison theorem; we conclude that $X = \overline{U}$ and X is norm connected.

We are reduced to the case where X is affine and irreducible. If $\dim X = n$, then by Noether's normalization theorem there exists a quasi-finite morphism, $X \longrightarrow \mathbb{C}^n$. We proved that there is an affine Z-open, $X' \subseteq X$, so that the morphism $\pi\colon X' \to \mathbb{C}^n$ is a finite morphism. Therefore, $A[X']$ is a finitely generated $\mathbb{C}[Z_1, \ldots, Z_n]$-module. Let $\xi_1, \ldots, \xi_t$ be module generators. We know $\mathcal{M}\mathrm{er}(X')$ is a finite-degree field over $\mathbb{C}(Z_1, \ldots, Z_n)$, so by Kronecker's theorem of the primitive element, there is some θ (a primitive element) so that

$$\mathcal{M}\mathrm{er}(X') = \mathbb{C}(Z_1, \ldots, Z_n)[\theta],$$

and we may assume $\theta \in A[X']$ (clear denominators). Thus,

$$\mathbb{C}[Z_1, \ldots, Z_n][\theta] \subseteq A[X'] \subseteq \mathbb{C}(Z_1, \ldots, Z_n)[\theta]$$

and, as usual, for every $\xi \in A[X']$, there is some $a_\xi \in \mathbb{C}[Z_1, \ldots, Z_n]$ so that $a_\xi \xi \in \mathbb{C}[Z_1, \ldots, Z_n][\theta]$. If we apply the latter to $\xi_1, \ldots, \xi_t$, we get $a_{\xi_1}, \ldots, a_{\xi_t}$; if $\alpha = a_{\xi_1} \cdots a_{\xi_t}$, then

$$\alpha A[X'] \subseteq \mathbb{C}[Z_1, \ldots, Z_n][\theta].$$

We know θ is the root of an irreducible polynomial, $F(T) \in \mathbb{C}[Z_1, \ldots, Z_n][T]$, of degree $m = \deg(F)$. Let δ be the discrimiant of F; we have $\delta \in \mathbb{C}[Z_1, \ldots, Z_n]$ and the locus where $\delta = 0$ (in $\mathbb{C}^n$) is called the *branch locus* of the morphism $\pi\colon X' \to \mathbb{C}^n$. Off the branch locus, all roots of F are disjoint. Let $\gamma = \alpha\delta$ and set $V = \mathbb{C}^n - V(\gamma)$. We know V *is norm-connected*, by Remark (2). Consider $U = X - V(\pi^*(\gamma))$, a Z-open in X'. We have a map $\pi\colon U \to V$, it is onto (by choices) because the coordinate ring of U is

$$A[U] = A[X']_{\pi^*(\gamma)} = A[\mathbb{C}^n]_\gamma[\theta],$$

and so $U \subseteq X'$ is affine, integral over V, which implies that π is onto and every fibre of $\pi \restriction U$ is a set of exactly m points. The Jacobian of π has full rank everywhere on U as $\delta \neq 0$ on V. So, by the convergent implicit function theorem, for every $\xi \in U$, there is an open, $U_\xi \ni \xi$, so that $\pi \restriction U_\xi$ is a complex analytic isomorphism to a small open containing $\pi(\xi)$. Therefore, by choosing $\eta_1, \ldots, \eta_m$ in the fibre over $v = \pi(\eta_i)$ and making the neighborhood small enough, the U_ξ don't intersect else we have a contradiction on the number of elements, m, in *every* fibre. It follows that U is an m-fold cover of V (in the sense of C^∞-topology). Now $U \subseteq X' \subseteq X$ and U is affine open in X. Consequently, if we prove U is norm-connected, by Remark (1), the variety X will be norm-connected.

Say U is norm-disconnected and $U = X_1 \mathbin{\dot\cup} X_2$ is a norm disconnection. The morphism, π, is both an open and a closed morphism, so $\pi(X_i)$ is nonempty, open and closed in V. Thus, each X_i is a complex analytic covering of V and if the degree is m_i, then we have $m_1 + m_2 = m$, with $m_j < m$, for $j = 1, 2$. Recall that U is integral over V. Take $\varphi \in A[U]$ and pick a point, $v \in V$, then in a small neighborhood of V, say $\widetilde{V}$, we have $\pi_j^{-1}(\widetilde{V}) = U_1^{(j)} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} U_{m_j}^{(j)}$ (where $\pi_j = \pi \restriction X_j$), disjoint open sets about each point in $\pi_j^{-1}(v)$. The function $\varphi \restriction U_i^{(j)}$ is a holomorphic function on $U_i^{(j)}$, call it $\varphi_i^{(j)}$. We know that $U_i^{(j)} \xrightarrow{\sim} \widetilde{V}$ by a complex *analytic* isomorphism. (The map $\widetilde{V} \longrightarrow U_i^{(j)}$ is holomorphic *not* algebraic.) Thus, each $\varphi_i^{(j)}$ is a holomorphic function on $\widetilde{V}$, not necessarily algebraic; namely, $\widetilde{V} \longrightarrow U_i^{(j)} \xrightarrow{\varphi_i^{(j)}} \mathbb{C}$. Let $\sigma_l^{(j)}$ be the l-th symmetric function of $\varphi_1^{(j)}, \ldots, \varphi_{m_j}^{(j)}$; this is a function on $\widetilde{V}$. Thus, $\sigma_l^{(j)}$ is a root of

$$T^{m_j} - \pi_j^* \sigma_1^{(j)} T^{m_j - 1} + \cdots + (-1)^{m_j} \pi_j^* \sigma_{m_j}^{(j)} = 0, \quad j = 1, 2.$$

I claim there exist polynomials, $P_r^{(j)} \in \mathbb{C}[Z_1, \ldots, Z_n]$, for $j = 1, 2$; $1 \leq r \leq m_j$; so that $P_r^{(j)} \restriction \widetilde{V} = \sigma_r^{(j)}$, for all r's and j's. Then, we get

$$\varphi^{m_j} - \pi_j^* P_1^{(j)} \varphi^{m_j - 1} + \cdots + (-1)^{m_j} \pi_j^* P_{m_j}^{(j)} = 0, \tag{$*$}$$

Now, $P_j[T] = T^{m_j} - \pi_j^* P_1^{(j)} T^{m_j - 1} + \cdots + (-1)^{m_j} \pi_j^* P_{m_j}^{(j)}$ belongs to $A[U][T]$ and $P_j(\varphi) = 0$. If we apply this to our primitive element, we get $P_1(\theta) = P_2(\theta) = 0$ (the first on X_1 and the second on X_2). Consequently, $(P_1P_2)(\theta) \equiv 0$ on U. Yet, $P_1P_2 \in A[U][T]$ and U is Z-connected, so $A[U][T]$ is domain and it follows that either $P_1 = 0$ or $P_2 = 0$. In either case, θ satisfies a polynomial of degree m_1 or m_2 and both $m_1, m_2 < m$, a contradiction.

So, it remains to prove our claim and, in fact, it is enough to prove it for θ. If we cover V by the $\widetilde{V}$'s we find that the $\sigma_r^{(j)}$ patch, which implies that they are global holomorphic functions on V. If $\xi \in \mathbb{C}^n$ (not on V), take a "compact open" neighborhood (i.e, an open whose closure is compact) of ξ. Now, θ is integral over $\mathbb{C}[Z_1, \ldots, Z_n]$, so we have an equation

$$\theta^m + \pi^* a_1 \varphi^{m-1} + \cdots + \pi^* a_m = 0, \quad \text{where } a_l \in \mathbb{C}[Z_1, \ldots, Z_n]. \tag{$**$}$$

The functions a_l are bounded on this compact neighborhhod and so are therefore the roots of $(**)$. It follows that the $\sigma_r^{(j)}$ are also bounded on this neighborhood. So, by Riemann's classic argument (using the Cauchy integral form on polydiscs and boundedness) we get an extension of $\sigma_r^{(j)}$ to *all* of $\mathbb{C}$ (as an entire function); $j = 1, 2$; $1 \leq r \leq m$. Now, we show that the $\sigma_r^{(j)}$, so extended, are really polynomials—this is a matter of how they grow. Write $\|z\| = \|(z_1, \ldots, z_n)\| = \max_{1 \leq i \leq n} |z_i|$. Then, from $(**)$, if $\xi \in U$, then (DX)

$$|\theta(\xi)| \leq 1 + \max\{|a_j(\pi(\xi))|\}.$$

If d is the maximum degree of the a_j's, then

$$|a_j(\pi(\xi))| \leq C \, \|\pi(\xi)\|^d \quad (\text{all } l).$$

Consequently, we can choose C so that

$$|\theta(\xi)| \leq C \, \|\pi(\xi)\|^d.$$

This works for all the roots $\theta_1, \ldots, \theta_m$ and since $\sigma_r^{(j)}$ is a polynomial of degree r in these roots, we deduce that

$$|\sigma_r^{(j)} \pi(\xi)| \leq D \, \|\pi(\xi)\|^{rd}.$$

Lemma 1.46 *Say $f(z_1, \ldots, z_n)$ is entire on $\mathbb{C}^n$ and $|f(z_1, \ldots, z_n)| \leq D \|z\|^q$. Then, $f(z_1, \ldots, z_n)$ is a polynomial of degree at most q.*

The proof of the lemma will finish the proof of Theorem C.

Proof. Write the MacLaurin series for f:

$$f(z_1, \ldots, z_n) = F_0 + F_1(z_1, \ldots, z_n) + \cdots + F_l(z_1, \ldots, z_n) + \cdots,$$

where $F_l(z_1, \ldots, z_n)$ is a homogeneous polynomial of degree l in $z_1, \ldots, z_n$. We must show $F_l \equiv 0$ if $t > q$. Say not, pick the minimal t with $t > q$ where $F_t \not\equiv 0$. As $F_t \not\equiv 0$, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ so that $F_t(\alpha_1, \ldots, \alpha_n) \neq 0$. Let ζ be a complex variable and write $z_j = \alpha_j \zeta$. Then,

$$\begin{aligned} g(\zeta) &= f(\alpha_1\zeta, \ldots, \alpha_n\zeta) - F_0 - F_1(\alpha_1\zeta, \ldots, \alpha_n\zeta) - \cdots - F_q(\alpha_1\zeta, \ldots, \alpha_n\zeta) \\ &= F_t(\alpha_1\zeta, \ldots, \alpha_n\zeta) + O(|\zeta|^{t+1}) \\ &= \zeta^t F_t(\alpha_1, \ldots, \alpha_n) + O(|\zeta|^{t+1}). \end{aligned}$$

If we divide by ζ^q, we get

$$\frac{g(\zeta)}{\zeta^q} = \zeta^{t-q} F_t(\alpha_1, \ldots, \alpha_n) + O(|\zeta|^{t-q+1}),$$

where the right hand side is an entire function of ζ. But, by the growth hypothesis on f, the left hand side is bounded on $\mathbb{C}$. By Liouville's theorem, this function is constant. Yet, the right hand side is not constant, since $F_t(\alpha_1, \ldots, \alpha_n) \neq 0$ and $t > q$, a contradiction. $\square$

Now that the lemma is proved, so is our theorem. $\square$

We now come back to some unfinished business regarding complex algebraic varieties in $\mathbb{C}^n$. We begin by proving a fact that was claimed without proof in the remark before Proposition 1.1.

Theorem 1.47 *Suppose V and W are affine varieties, with $W \subseteq \mathbb{C}^n$, and $\varphi\colon V \to W$ is a morphism. Then, there exist $F_1, \ldots, F_n \in A[V]$ so that*

$$\varphi(v) = (F_1(v), \ldots, F_n(v)), \quad \textit{for all } v \in V.$$

Proof. Since V is a variety it is quasi-compact, so V is covered by some affine opens, $V_{g_1}, \ldots, V_{g_t}$. For any $v \in V_{g_j}$, we have

$$\varphi(v) = \left(\frac{f_1^{(j)}(v)}{g_j^{\nu_1}(v)}, \ldots, \frac{f_n^{(j)}(v)}{g_j^{\nu_n}(v)}\right),$$

for some $f_1^{(j)}, \ldots, f_n^{(j)}, g_j \in A[V]$, with $g_j \neq 0$ on V_{g_j}. If ν is the maximum of the ν_i's, since $V_{g^\nu} = V_g$, we may assume that $\nu = 1$. Thus, on each V_{g_j}, we have

$$\varphi(v) = \left(\frac{f_1^{(j)}(v)}{g_j(v)}, \ldots, \frac{f_n^{(j)}(v)}{g_j(v)}\right). \tag{$*$}$$

Since φ is well-defined, the local definitions of φ must agree on $V_{g_i} \cap V_{g_j} = V_{g_i g_j}$, and we have

$$\frac{f_l^{(j)}(v)}{g_j(v)} = \frac{f_l^{(i)}(v)}{g_i(v)}$$

for all $v \in V_{g_i g_j}$ and all l, $1 \leq l \leq n$. As a consequence,

$$f_l^{(j)} g_i - f_l^{(i)} g_j = 0 \quad \text{on } V_{g_i g_j},$$

which implies that

$$f_l^{(j)} g_i - f_l^{(i)} g_j = 0 \quad \text{in } A[V]_{g_i g_j}.$$

Therefore, there are some integers n_{ijl} so that

$$(g_i g_j)^{n_{ijl}} (f_l^{(j)} g_i - f_l^{(i)} g_j) = 0 \quad \text{in } A[V].$$

Let $N = \max\{n_{ijl}\}$, where $1 \leq i, j, \leq t$, $1 \leq l \leq n$. We have

$$(g_i g_j)^N f_l^{(j)} g_i = (g_i g_j)^N f_l^{(i)} g_j, \tag{$**$}$$

for all i, j, l, with $1 \leq i, j, \leq t$, $1 \leq l \leq n$. Now, the V_{g_i} cover V. Hence, the g_i have no common zero, and neither do the g_i^{N+1} (since $V_{g^N} = V_g$). By the Nullstellensatz,

$$(g_1^{N+1}, \ldots, g_t^{N+1}) = (1),$$

the unit ideal in $A[V]$, and thus, there are some $h_i \in A[V]$ so that

$$1 = \sum_{i=1}^{t} h_i g_i^{N+1}.$$

But, we have

$$\begin{aligned} g_i^N f_l^{(i)} &= g_i^N f_l^{(i)} \left(\sum_{r=1}^t h_r g_r^{N+1} \right) \\ &= \sum_{r=1}^t h_r g_r^{N+1} g_i^N f_l^{(i)} \\ &= \sum_{r=1}^t h_r g_i^{N+1} g_r^N f_l^{(r)} \qquad \text{by } (**) \\ &= g_i^{N+1} \left(\sum_{r=1}^t h_r g_r^N f_l^{(r)} \right). \end{aligned}$$

Letting

$$F_l = \sum_{r=1}^t h_r g_r^N f_l^{(r)},$$

we have $F_l \in A[V]$, and

$$g_i^N f_l^{(i)} = g_i^{N+1} F_l \quad \text{in } A[V], \text{ for all } i \text{ with } 1 \leq i \leq t.$$

For any $v \in V_{g_i}$, we get

$$\frac{f_l^{(i)}(v)}{g_i(v)} = F_l(v),$$

and by $(*)$,

$$\varphi(v) = (F_1(v), \ldots, F_n(v)).$$

$\square$

Corollary 1.48 *Say X is a complex affine variety, then the ring of global holomorphic functions on X, i.e., the ring $\Gamma(X, \mathcal{O}_X)$, is exactly the coordinate ring, $A[X]$, of X.*

Proof. The ring $\Gamma(X, \mathcal{O}_X)$ is just $\mathrm{Hom}_{\mathbb{C}-\mathrm{vars}}(X, \mathbb{C}^1)$, essentially by definition. By Theorem 1.47, there is some $F \in A[X]$ so that if $\varphi \in \Gamma(X, \mathcal{O}_X)$, we have $\varphi(x) = F(x)$, for all $x \in X$. Therefore, $\Gamma(X, \mathcal{O}_X) = A[X]$. $\square$

Corollary 1.49 *The category of affine complex varieties is naturally anti-equivalent to the category of reduced (i.e., no nilpotent elements) finitely generated $\mathbb{C}$-algebras.*

Proof. Say A is a reduced f.g. $\mathbb{C}$-algebra, we can make a variety—it is denoted $\mathrm{Spec}\, A$—as follows: The underlying topological space (in the norm topology), $X = \mathrm{Spec}\, A$, is

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(A, \mathbb{C}) &= \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbb{C}[Z_1, \ldots, Z_n]/\mathfrak{A}, \mathbb{C}) \\ &= \{(z_1, \ldots, z_n) \mid f_1(z_1, \ldots, z_n) = \cdots = f_p(z_1, \ldots, z_n) = 0\}, \end{aligned}$$

where $\mathfrak{A} = (f_1, \ldots, f_p)$ is a radical ideal. The sheaf is as before, use the opens, X_g, where $g \neq 0$ (with $g \in A$) and on those, use as functions, h/g, with $g, h \in A$. This gives $\Gamma(X_g, \mathcal{O}_X)$.

Conversely, given an affine variety, X, make the f.g. reduced $\mathbb{C}$-algebra, $A = A[X]$.

We also need to show how maps of rings transform to morphisms and the other way around. Say A and B are reduced $\mathbb{C}$-algebras. Then, if $\theta\colon A \to B$, set $\widetilde{\theta}\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *via*: Pick $x \in \mathrm{Spec}\, B$, i.e., $x \in \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(B, \mathbb{C})$, then $\widetilde{\theta}(x) = x \circ \theta$. Localize $\widetilde{\theta}$ at the g's so that the Y_g's cover $Y = \mathrm{Spec}\, B$, then $B_g \longrightarrow A_{\widetilde{\theta}(g)}$ is just the locally defined map of sheaves. If $X = \mathrm{Spec}\, A$ and we have a morphism $Y \longrightarrow X$, then we get a map $\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(Y, \mathcal{O}_Y)$; but, Corollary 1.48 says this is a ring map from A to B. The rest of the checking is (DX). $\square$

Corollary 1.50 *A n.a.s.c. that $(V, \mathcal{O}_V)$ be a complex affine variety is that the canonical map*

$$\mathrm{Hom}_{\mathbb{C}-\mathrm{vars}}(T, V) \longrightarrow \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\Gamma(V, \mathcal{O}_V), \Gamma(T, \mathcal{O}_T))$$

is a bijection for every complex variety, T.

Proof. An affine variety, V, does have the above property. For, say V is affine and pick *any* T. What is $\mathrm{Hom}_{\mathbb{C}-\mathrm{vars}}(T, V)$? We can cover T by affines, T_α, and get morphisms $T_\alpha \longrightarrow V$, for all α; clearly, they agree on the overlaps $T_\alpha \cap T_\beta$. The $T_\alpha \cap T_\beta$ are affine (because V is separated), in fact, their coordinate rings are

$$\Gamma(T_\alpha, \mathcal{O}_T \restriction T_\alpha) \otimes_{\Gamma(T, \mathcal{O}_T)} \Gamma(T_\beta, \mathcal{O}_T \restriction T_\beta).$$

We get the commutative diagrams

$$\begin{tikzcd} & T_\alpha \arrow[dr] & \\ T_\alpha \cap T_\beta \arrow[ur] \arrow[dr] & & V \\ & T_\beta \arrow[ur] & \end{tikzcd} \quad . \tag{$\dagger$}$$

Apply Corollary 1.49, this gives ring morphisms

$$\begin{tikzcd} & \Gamma(T_\alpha, \mathcal{O}_{T_\alpha}) \arrow[dr] & \\ \Gamma(V, \mathcal{O}_V) \arrow[ur] \arrow[dr] & & T_\alpha \cap T_\beta, \mathcal{O}_T) \\ & \Gamma(T_\beta, \mathcal{O}_{T_\beta}) \arrow[ur] & \end{tikzcd} \quad . \tag{$*$}$$

By definition of sheaves, we get a map of sheaves,

$$\Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma(T, \mathcal{O}_T).$$

Conversely, a map $\Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma(T, \mathcal{O}_T)$ gives by restriction to $\Gamma(T_\alpha, \mathcal{O}_T)$ (resp. $\Gamma(T_\beta, \mathcal{O}_T)$) our commutative diagram ($*$), and by Corollary 1.49, it gives ($\dagger$) and this is a morphism $T \longrightarrow V$.

Now, given V with the property of the corollary, make $\widetilde{V} = \mathrm{Spec}(\Gamma(V, \mathcal{O}_V))$; then $\widetilde{V}$ is affine. By hypothesis, V has the property, $\widetilde{V}$ has it by the previous part of the proof, so the functors $T \rightsquigarrow \mathrm{Hom}_{\mathbb{C}-\mathrm{vars}}(T, V)$ and $T \rightsquigarrow \mathrm{Hom}_{\mathbb{C}-\mathrm{vars}}(T, \widetilde{V})$ are canonically isomorphic. Yoneda's Lemma implies $V \cong \widetilde{V}$. $\square$

Corollary 1.51 *Say $\varphi\colon X \to Y$ is a morphism of complex affine varieties and $\widetilde{\varphi}\colon A[Y] \to A[X]$ is the corresponding algebra map. Then,*

(1) The morphism φ is a closed immersion iff $\widetilde{\varphi}$ is surjective ($\mathrm{Ker}\, \widetilde{\varphi}$ *defines the image).*

(2) The algebra map $\widetilde{\varphi}$ is injective iff $\mathrm{Im}\, \varphi$ *is Z-dense in Y.*

Proof. (DX). $\square$

Chapter 2

Cohomology of (Mostly) Constant Sheaves and Hodge Theory

2.1 Real and Complex

Let X be a complex analytic manifold of (complex) dimension n. Viewed as a real manifold, X is a C^∞-manifold of dimension $2n$. For every $x \in X$, we know $T_{X,x}$ is a $\mathbb{C}$-vector space of complex dimension n, so, $T_{X,x}$ is a real vector space of dimension $2n$. Take local (complex) coordinates $z_1, \ldots, z_n$ at $x \in X$, then we get real local coordinates $x_1, y_1, \ldots, x_n, y_n$ on X (as an $\mathbb{R}$-manifold), where $z_j = x_j + iy_j$. (Recall, T_X is a complex holomorphic vector bundle). If we view $T_{X,x}$ as a real vector space of dimension $2n$, then we can complexify $T_{X,x}$, i.e., form

$$T_{X,x_{\mathbb{C}}} = T_{X,x} \otimes_{\mathbb{R}} \mathbb{C},$$

a complex vector space of dimension $2n$. A basis of $T_{X,x}$ at x (as $\mathbb{R}$-space) is just

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}.$$

These are a $\mathbb{C}$-basis for $T_{X,x_{\mathbb{C}}}$, too. We can make the change of coordinates to the coordinates z_j and $\overline{z}_j$, namely,

$$z_j = x_j + iy_j, \quad \overline{z}_j = x_j - iy_j,$$

and of course,

$$x_j = \frac{1}{2}(z_j + \overline{z}_j), \quad y_j = \frac{1}{2i}(z_j - \overline{z}_j).$$

So, $T_{X,x_{\mathbb{C}}}$ has a basis consisting of the $\partial/\partial z_j, \partial/\partial \overline{z}_j$; in fact, for $f \in C^\infty(\text{open})$, we have

$$\frac{\partial f}{\partial z_j} = \frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j} \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}_j} = \frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}.$$

More abstractly, let V be a $\mathbb{C}$-vector space of dimension n and view V as a real vector space of dimension $2n$. If $e_1, \ldots, e_n$ is a $\mathbb{C}$-basis for V, then $ie_1, \ldots, ie_n$ make sense. Say $e_j = f_j + ig_j$ (from $\mathbb{C}$-space to $\mathbb{R}$-space), then, $ie_j = if_j - g_j = -g_j + if_j$. Consequently, the map $(e_1, \ldots, e_n) \mapsto (ie_1, \ldots, ie_n)$ corresponds to the map

$$((f_1, g_1), \ldots, (f_n, g_n)) \xrightarrow{J} ((-g_1, f_1), \ldots, (-g_n, f_n))$$

where V is viewed as $\mathbb{R}$-space of dimension $2n$. The map J an endomorphism of V viewed as $\mathbb{R}$-space and obviously, it satisfies

$$J^2 = -\text{id}.$$

If, conversely, we have an $\mathbb{R}$-space, V, of even dimension, $2n$ and if an endomorphism $J \in \mathrm{End}_{\mathbb{R}}(V)$ with $J^2 = -\mathrm{id}$ is given, then we can give V a complex structure as follows:

$$(a + ib)v = av + bJ(v).$$

In fact, the different complex structures on the real vector space, V, of dimension $2n$ are in one-to-one correspondence with the homogeneous space $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$, *via*

$$\text{class}\, A \mapsto AJA^{-1}.$$

Definition 2.1 An *almost complex manifold* is a real C^∞-manifold together with a bundle endomorphism, $J\colon T_X \to T_X$, so that $J^2 = -\mathrm{id}$.

Proposition 2.1 *If $(X, \mathcal{O}_X)$ is a complex analytic manifold, then it is almost complex.*

Proof. We must construct J on T_X. It suffices to do this locally and check that it is independent of the coordinate patch. Pick some open, U, where $T_X \restriction U$ is trivial. By definition of a patch, we have an isomorphism $(U, \mathcal{O}_X \restriction U) \xrightarrow{\sim} (B_{\mathbb{C}}(0, \epsilon), \mathcal{O}_B)$ and we have local coordinates denoted $z_1, \ldots, z_n$ in both cases. On $T_X \restriction U$, we have $\partial/\partial z_1, \ldots, \partial/\partial z_n$ and $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n$, as before. The map J is given by

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right) \xrightarrow{J} \left(-\frac{\partial}{\partial y_1}, \ldots, -\frac{\partial}{\partial y_n}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

We need to show that this does not depend on the local trivialization. Go back for a moment to two complex manifolds, $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, of dimension $2n$ and consider a smooth map $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. For every $x \in X$, we have an induced map on tangent spaces, $df\colon T_{X,x} \to T_{Y,y}$, where $y = f(x)$ and if, as $\mathbb{R}$-spaces, we use local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ on $T_{X,x}$ and local coordinates $u_1, \ldots, u_n, v_1, \ldots, v_n$ on $T_{Y,y}$, then df is given by the Jacobian

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \left(\frac{\partial u_\alpha}{\partial x_j}\right) & \left(\frac{\partial u_\alpha}{\partial y_j}\right) \\ \left(\frac{\partial v_\alpha}{\partial x_j}\right) & \left(\frac{\partial v_\alpha}{\partial y_j}\right) \end{pmatrix}.$$

If f is holomorphic, the Cauchy-Riemann equations imply

$$\frac{\partial u_\alpha}{\partial x_j} = \frac{\partial v_\alpha}{\partial y_j} \quad \text{and} \quad \frac{\partial v_\alpha}{\partial x_j} = -\frac{\partial u_\alpha}{\partial y_j}.$$

Now, this gives

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \left(\frac{\partial v_\alpha}{\partial y_j}\right) & \left(\frac{\partial u_\alpha}{\partial y_j}\right) \\ \left(-\frac{\partial u_\alpha}{\partial y_j}\right) & \left(\frac{\partial v_\alpha}{\partial y_j}\right) \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Going back to our problem, if we have different trivializations, on the overlap, the transition functions are holomorphic, so $J_{\mathbb{R}}(f)$ is as above. Now J in our coordinates is of the form

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0 \end{pmatrix}$$

and we have $JJ_{\mathbb{R}}(f) = J_{\mathbb{R}}(f)J$ when f is holomorphic (DX). $\square$

So, an almost complex structure is a bundle invariant.

Question: Does S^6 possess a complex structure?

The usual almost complex structure from S^7 (= unit Cayley numbers = unit octonions) is not a complex structure. Borel and Serre proved that the only spheres with an almost complex structure are: S^0, S^2 and S^6.

Say we really have complex coordinates, $z_1, \ldots, z_n$ down in X. Then, on $T_X \otimes_{\mathbb{R}} \mathbb{C}$, we have the basis

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n},$$

and so, in this basis, if we write $f = (w_1, \ldots, w_n)$, where $w_\alpha = u_\alpha + iv_\alpha$, we get

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \left(\frac{\partial w_\alpha}{\partial z_j}\right) & \left(\frac{\partial w_\alpha}{\partial \overline{z}_j}\right) \\ \left(\frac{\partial \overline{w}_\alpha}{\partial z_j}\right) & \left(\frac{\partial \overline{w}_\alpha}{\partial \overline{z}_j}\right) \end{pmatrix},$$

and, again, if f is holomorphic, we get

$$\frac{\partial w_\alpha}{\partial \overline{z}_j} = \frac{\partial \overline{w}_\alpha}{\partial z_j} = 0,$$

which yields

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \left(\frac{\partial w_\alpha}{\partial z_j}\right) & 0 \\ 0 & \left(\frac{\partial \overline{w}_\alpha}{\partial \overline{z}_j}\right) \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}.$$

Write

$$J(f) = \left(\frac{\partial w_\alpha}{\partial z_j}\right)$$

and call it the *holomorphic Jacobian*. We get

(1) $J_{\mathbb{R}}(f) = \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix}$, so, $\mathbb{R}\text{-rank}\, J_{\mathbb{R}}(f) = 2\mathbb{C}\text{-rank}\, J(f)$.

(2) We have $\det(J_{\mathbb{R}}(f)) = |\det(J(f))|^2 \geq 0$, and $\det(J_{\mathbb{R}}(f)) > 0$ if f is a holomorphic isomorphism (in this case, $m = n =$ the common dimension of X, Y).

Hence, we get the first statement of

Proposition 2.2 *Holomorphic maps preserve the orientation of a complex manifold and each complex manifold possesses an orientation.*

Proof. We just proved the first statement. To prove the second statement, as orientations are preserved by holomorphic maps we need only give an orientation locally. But, locally, a patch is biholomorphic to a ball in $\mathbb{C}^n$. Therefore, it is enough to give $\mathbb{C}^n$ an orientation, i.e., to give $\mathbb{C}$ an orientation. However, $\mathbb{C}$ is oriented as (x, ix) gives the orientation. $\square$

Say we have a real vector space, V, of dimension $2n$ and look at $V \otimes_{\mathbb{R}} \mathbb{C}$. Say V also has a complex structure, J. Then, the extension of J to $V \otimes_{\mathbb{R}} \mathbb{C}$ has two eigenvalues, $\pm i$. On $V \otimes_{\mathbb{R}} \mathbb{C}$, we have the two eigenspaces, $(V \otimes_{\mathbb{R}} \mathbb{C})^{1,0} =$ the i-eigenspace and $(V \otimes_{\mathbb{R}} \mathbb{C})^{0,1} =$ the $-i$-eigenspace. Of course,

$$(V \otimes_{\mathbb{R}} \mathbb{C})^{0,1} = \overline{(V \otimes_{\mathbb{R}} \mathbb{C})^{1,0}}.$$

Now, look at $\bigwedge^l (V \otimes_{\mathbb{R}} \mathbb{C})$. We can examine

$$\bigwedge^{p,0}(V \otimes_{\mathbb{R}} \mathbb{C}) \stackrel{\mathrm{def}}{=} \bigwedge^{p}[(V \otimes_{\mathbb{R}} \mathbb{C})^{1,0}] \quad \text{and} \quad \bigwedge^{0,q}(V \otimes_{\mathbb{R}} \mathbb{C}) \stackrel{\mathrm{def}}{=} \bigwedge^{q}[(V \otimes_{\mathbb{R}} \mathbb{C})^{0,1}],$$

and also

$$\bigwedge^{p,q}(V \otimes_{\mathbb{R}} \mathbb{C}) \stackrel{\text{def}}{=} \bigwedge^{p,0}(V \otimes_{\mathbb{R}} \mathbb{C}) \otimes \bigwedge^{0,q}(V \otimes_{\mathbb{R}} \mathbb{C}).$$

Note that we have

$$\bigwedge^{l}(V \otimes_{\mathbb{R}} \mathbb{C}) = \coprod_{p+q=l} \bigwedge^{p,q}(V \otimes_{\mathbb{R}} \mathbb{C}).$$

Now, say X is an almost complex manifold and apply the above to $V = T_X, T_X^D$; we get bundle decompositions for $T_X \otimes_{\mathbb{R}} \mathbb{C}$ and $T_X^D \otimes_{\mathbb{R}} \mathbb{C}$. Thus,

$$\bigwedge^{\bullet}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) = \coprod_{l=1}^{2n} \coprod_{p+q=l} \bigwedge^{p,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}).$$

Note that J on $\bigwedge^{p,q}$ is multiplication by $(-1)^q i^{p+q}$. Therefore, J does not act by scalar multiplication in general on $\bigwedge^l(V \otimes_{\mathbb{R}} \mathbb{C})$.

Say X is now a complex manifold and $f\colon X \to Y$ is a C^∞-map to another complex manifold, Y. Then, for every $x \in X$, we have the linear map

$$Df\colon T_{X,x} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow T_{Y,f(x)} \otimes_{\mathbb{R}} \mathbb{C}.$$

The map f won't in general preserve the decomposition $T_{X,x} \otimes_{\mathbb{R}} \mathbb{C} = T_{X,x}^{1,0} \coprod T_{X,x}^{0,1}$.

However, f is holomorphic iff for every $x \in X$, we have $Df\colon T_{X,x}^{1,0} \to T_{Y,f(x)}^{1,0}$.

Let us now go back to a real manifold, X. We have the usual exterior derivative

$$d\colon \bigwedge^{l} T_{X,x}^D \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \bigwedge^{l+1} T_{X,x}^D \otimes_{\mathbb{R}} \mathbb{C},$$

namely, if $\xi_1, \ldots, \xi_{2n}$ are real coordinates at x, we have

$$\sum_{|I|=l} a_I d\xi_I \mapsto \sum_{|I|=l} da_I \wedge d\xi_I.$$

here, the a_I are $\mathbb{C}$-valued function on X near x and $d\xi_I = d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_l}$, with $I = \{i_1 < i_2 < \cdots < i_l\}$.

In the *almost complex* case, we have the p,q-decomposition of $T_X^D \otimes_{\mathbb{R}} \mathbb{C}$ and consequently

$$\bigwedge^{p,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \stackrel{i_{p,q}}{\hookrightarrow} \bigwedge^{l}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \stackrel{d}{\longrightarrow} \bigwedge^{l+1}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) = \coprod_{r+s=l+1} \bigwedge^{r,s}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}).$$

We let

$$\partial = \{\partial_{p,q} = pr_{p+1,q} \circ d \circ i_{p,q}\colon \bigwedge^{p,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \bigwedge^{p+1,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C})\}_{p,q}$$

and

$$\overline{\partial} = \{\overline{\partial}_{p,q} = pr_{p,q+1} \circ d \circ i_{p,q}\colon \bigwedge^{p,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \bigwedge^{p,q+1}(T_X^D \otimes_{\mathbb{R}} \mathbb{C})\}_{p,q}.$$

Note that $d = \partial + \overline{\partial} +$ other stuff. Let us take a closer look in local coordinates. We can pick $\xi_1, \ldots, \xi_n$, some coordinates for $T_X^{1,0}$, then $\overline{\xi}_1, \ldots, \overline{\xi}_n$ are coordinates for $T_X^{0,1}$ (say $x_1, \ldots, x_{2n}$ are local coordinates in the base). Then, any $\omega \in \bigwedge^{p,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C})$ has the form

$$\omega = \sum_{\substack{|I|=p \\ |\widetilde{I}|=q}} a_{I,\widetilde{I}}\, d\xi_I \wedge d\overline{\xi}_{\widetilde{I}},$$

and so

$$d\omega = \sum_{\substack{|I|=p \\ |\widetilde{I}|=q}} da_{I,\widetilde{I}} \wedge d\xi_I \wedge d\overline{\xi}_{\widetilde{I}} + \sum_{\substack{|I|=p \\ |\widetilde{I}|=q}} a_{I,\widetilde{I}}\, d(d\xi_I \wedge d\overline{\xi}_{\widetilde{I}}) = \partial\omega + \overline{\partial}\omega + \text{stuff}.$$

If we are on a complex manifold, then we can choose the ξ_j so that $\xi_j = \partial/\partial z_j$ and $\overline{\xi}_j = \partial/\partial \overline{z}_j$, constant over our neigborhood and then,

$$\begin{aligned} d\omega &= \sum_{\substack{|I|=p \\ |\widetilde{I}|=q}} da_{I,\widetilde{I}}\, d\xi_I \wedge d\overline{\xi}_{\widetilde{I}} \\ &= \sum_{\substack{|I|=p \\ |\widetilde{I}|=q}} \sum_{j=1}^{n} \left(\frac{\partial a_{I,\widetilde{I}}}{\partial z_j} dz_j \wedge dz_I \wedge d\overline{z}_{\widetilde{I}} + \frac{\partial a_{I,\widetilde{I}}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_{\widetilde{I}} \right) \\ &= \partial\omega + \overline{\partial}\omega = (\partial + \overline{\partial})\omega. \end{aligned}$$

Consequently, on a complex manifold, $d = \partial + \overline{\partial}$.

On an almost complex manifold, $d^2 = 0$, yet, $\partial^2 \neq 0$ and $\overline{\partial}^2 \neq 0$ in general.

However, suppose we are lucky and $d = \partial + \overline{\partial}$. Then,

$$0 = d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2,$$

and we deduce that $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$, *in this case*.

Definition 2.2 The almost complex structure on X is *integrable* iff near every $x \in X$, there exist real coordinates, $\xi, \ldots, \xi_n$ in $T_X^{1,0}$ and $\overline{\xi}_1, \ldots, \overline{\xi}_n$ in $T_X^{0,1}$, so that $d = \partial + \overline{\partial}$.

By what we just did, a complex structure is integrable. A famous theorem of Newlander-Nirenberg (1957) shows that if X is an almost complex C^∞-manifold whose almost complex structure is integrable, then there exists a unique complex structure (i.e., complex coordinates everywhere) inducing the almost complex one.

Remark: Say V has a complex structure given by J. We have

$$V = V \otimes_{\mathbb{R}} \mathbb{R} \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C} \stackrel{pr_{1,0}}{\longrightarrow} V^{1,0}.$$

The vector space $V^{1,0}$ also has a complex structure, namely, multiplication by i. So, we have an isomorphism $V \cong V^{1,0}$, as $\mathbb{R}$-spaces, but also an isomorphism $V \cong V^{1,0}$, as $\mathbb{C}$-spaces, where the complex structure on V is J and the complex structure on $V^{0,1}$ is multiplication by i. Therefore, we also have an isomorphism $V \cong V^{1,0}$, where the complex structure on V is $-J$ and the complex structure on $V^{0,1}$ is multiplication by $-i$.

For tangent spaces, $T_X^{1,0}$ is spanned by $\partial/\partial z_1, \ldots, \partial/\partial z_n$, the space $T_X^{0,1}$ is spanned by $\partial/\partial \overline{z}_1, \ldots, \partial/\partial \overline{z}_n$; also, $T_X^{D\,1,0}$ is spanned by $dz_1, \ldots, dz_n$ and $T_X^{D\,0,1}$ is spanned by $d\overline{z}_1, \ldots, d\overline{z}_n$.

2.2 Cohomology, de Rham, Dolbeault

Let X be a real $2n$-dimensional C^∞-manifold and let d be the exterior derivative, then we get the complex

$$T_X^D \xrightarrow{d} \bigwedge^2 T_X^D \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} T_X^D,$$

$(d^2 = 0)$. The same is true for complex-valued forms, we have the complex

$$T_X^D \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{d} \bigwedge^2 T_X^D \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} T_X^D \otimes_{\mathbb{R}} \mathbb{C},$$

$(d^2 = 0)$. Here, there is an abuse of notation: T_X^D denotes a sheaf, so we should really use a notation such as $\mathcal{T}_X^D$. To alleviate the notation, we stick to T_X^D, as the context makes it clear that it is a sheaf. These maps induce maps on global C^∞-sections, so we get the complexes

$$\Gamma(X, T_X^D) \xrightarrow{d} \bigwedge^2 \Gamma(X, T_X^D) \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} \Gamma(X, T_X^D)$$

and

$$\Gamma(X, T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{d} \bigwedge^2 \Gamma(X, T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} \Gamma(X, T_X^D \otimes_{\mathbb{R}} \mathbb{C}).$$

Define

$$\begin{aligned}
Z^l_{\mathrm{DR}}(X) &= \mathrm{Ker}\, d, \quad \text{where} \quad d\colon \bigwedge^l \Gamma(X, T_X^D) \longrightarrow \bigwedge^{l+1} \Gamma(X, T_X^D) \\
Z^l_{\mathrm{DR}}(X)_{\mathbb{C}} &= \mathrm{Ker}\, d, \quad \text{where} \quad d\colon \bigwedge^l \Gamma(X, T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \bigwedge^{l+1} \Gamma(X, T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \\
B^l_{\mathrm{DR}}(X) &= \mathrm{Im}\, d, \quad \text{where} \quad d\colon \bigwedge^{l-1} \Gamma(X, T_X^D) \longrightarrow \bigwedge^{l} \Gamma(X, T_X^D) \\
B^l_{\mathrm{DR}}(X)_{\mathbb{C}} &= \mathrm{Ker}\, d, \quad \text{where} \quad d\colon \bigwedge^{l-1} \Gamma(X, T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \bigwedge^{l} \Gamma(X, T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \\
H^l_{\mathrm{DR}}(X) &= Z^l_{\mathrm{DR}}(X)/B^l_{\mathrm{DR}}(X) \\
H^l_{\mathrm{DR}}(X)_{\mathbb{C}} &= Z^l_{\mathrm{DR}}(X)_{\mathbb{C}}/B^l_{\mathrm{DR}}(X)_{\mathbb{C}}.
\end{aligned}$$

Note: $H^l_{\mathrm{DR}}(X)_{\mathbb{C}} = H^l_{\mathrm{DR}}(X) \otimes_{\mathbb{R}} \mathbb{C}$. These are the *de Rham cohomology groups*. For Dolbeault cohomology, take X, a complex manifold of dimension n, view it as a real manifold of dimension $2n$, consider the complexified cotangent bundle, $T_X^D \otimes_{\mathbb{R}} \mathbb{C}$, and decompose its wedge powers as

$$\bigwedge^l (T_X^D \otimes_{\mathbb{R}} \mathbb{C}) = \coprod_{p+q=l} \bigwedge^{p,q} (T_X^D \otimes_{\mathbb{R}} \mathbb{C}).$$

Since X is a complex manifold, $d = \partial + \overline{\partial}$ and so, $\partial^2 = \overline{\partial}^2 = 0$. Therefore, we get complexes by fixing p or q:

(a) Fix q: $\bigwedge^{0,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{\partial} \bigwedge^{1,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bigwedge^{n,q}(T_X^D \otimes_{\mathbb{R}} \mathbb{C})$.

(b) Fix p: $\bigwedge^{p,0}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{\overline{\partial}} \bigwedge^{p,1}(T_X^D \otimes_{\mathbb{R}} \mathbb{C}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \bigwedge^{p,n}(T_X^D \otimes_{\mathbb{R}} \mathbb{C})$.

The above are the *Dolbeault complexes* and we have the corresponding cohomology groups $H^{p,q}_{\overline{\partial}}(X)$ and $H^{p,q}_{\partial}(X)$. Actually, the $H^{p,q}_{\overline{\partial}}(X)$ are usually called the *Dolbeault cohomology groups*. The reason for that is if $f\colon X \to Y$ is holomorphic, then df and $(df)^D$ respect the p,q-decomposition. Consequently,

$$(df)^D\colon \bigwedge^{p,q}(T^D_{Y,f(x)} \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \bigwedge^{p,q}(T^D_{X,x} \otimes_{\mathbb{R}} \mathbb{C})$$

for all $x \in X$ and

$$(df)^D \circ \overline{\partial}_Y = \overline{\partial}_X \circ (df)^D$$

imply that $(df)^D$ induces maps $H_{\overline{\partial}}^{p,q}(Y) \longrightarrow H_{\overline{\partial}}^{p,q}(X)$.

The main local fact is the *Poincaré lemma.*

Lemma 2.3 *(Poincaré Lemma) If X is a real C^∞-manifold and is actually a star-shaped manifold (or particularly, a ball in $\mathbb{R}^n$), then*

$$H_{\mathrm{DR}}^p(X) = (0), \quad \textit{for all} \quad p \geq 1.$$

If X is a complex analytic manifold and is a polydisc ($PD(0, r)$), then

(a) $H_{\overline{\partial}}^{p,q}(X) = (0)$, for all $p \geq 0$ and all $q \geq 1$.

(b) $H_{\partial}^{p,q}(X) = (0)$, for all $q \geq 0$ and all $p \geq 1$.

Proof. Given any form $\omega \in \bigwedge^{p,q}(PD(0, r))$ with $\overline{\partial}\omega = 0$, we need to show that there is some $\eta \in \bigwedge^{p,q-1}(PD(0, r))$ so that $\overline{\partial}\eta = \omega$. There are three steps to the proof.

Step I. Reduction to the case $p = 0$.

Say the lemma holds is $\omega \in \bigwedge^{0,q}(PD(0, r))$. Then, our ω is of the form

$$\omega = \sum_{\substack{|I|=p \\ |J|=q}} a_{I,J}\, dz_I \wedge d\overline{z}_J.$$

Write

$$\omega_I = \sum_{|J|=q} a_{I,J}\, d\overline{z}_J \in \bigwedge^{0,q}(PD(0, r)).$$

Claim: $\overline{\partial}\omega_I = 0$.

We have $\omega = \sum_{|I|=p} dz_I \wedge \omega_I$ and

$$0 = \overline{\partial}\omega = \sum_{|I|=p} \overline{\partial}(dz_I \wedge \omega_I) = \sum_{|I|=p} \pm dz_I \wedge \overline{\partial}\omega_I.$$

These terms are in the span of

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_j \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$$

and by linear independence of these various wedges, we must have $\overline{\partial}\omega_I = 0$, for all I. Then, by the assumption, there is some $\eta_I \in \bigwedge^{0,q-1}(PD(0, r))$, so that $\overline{\partial}\eta_I = \omega_I$. It follows that

$$\omega = \sum_{|I|=p} dz_I \wedge \overline{\partial}\eta_I = \sum_{|I|=p} \pm\overline{\partial}(dz_I \wedge \eta_I) = \overline{\partial}(\sum_{|I|=p} \pm dz_I \wedge \eta_I),$$

with $\sum_{|I|=p} \pm dz_I \wedge \eta_I \in \bigwedge^{p,q-1}(PD(0, r))$, which concludes the proof of Step I.

Step II: Interior part of the proof.

We will prove that for every $\epsilon > 0$, there is some $\eta \in \bigwedge^{0,q-1}(PD(0, r))$ so that $\overline{\partial}\eta = \omega$ *in* $PD(0, r - \epsilon)$.

Let us say that η *depends on* $d\overline{z}_1, \ldots, d\overline{z}_s$ if the terms $a_J d\overline{z}_J$ in η where $J \not\subseteq \{1, \ldots, s\}$ are all zero, i.e., in η, only terms $a_J d\overline{z}_J$ appear for $J \subseteq \{1, \ldots, s\}$.

Claim: If ω depends on $d\overline{z}_1, \ldots, d\overline{z}_s$, then there is some $\eta \in \bigwedge^{0,q-1}(PD(0,r))$ so that $\omega - \overline{\partial}\eta$ depends only on $d\overline{z}_1, \ldots, d\overline{z}_{s-1}$ in $PD(0, r-\epsilon)$.

Clearly, if the claim is proved, the interior part is done by a trivial induction. In ω, isolate the terms depending on $d\overline{z}_1, \ldots, d\overline{z}_{s-1}$, call these ω_2 and ω_1 the rest. Now, $\omega_1 = \theta \wedge d\overline{z}_s$, so $\omega = \theta \wedge d\overline{z}_s + \omega_2$ and we get

$$0 = \overline{\partial}\omega = \overline{\partial}(\theta \wedge d\overline{z}_s) + \overline{\partial}\omega_2. \tag{$*$}$$

Examine the terms

$$\frac{\partial a_J}{\partial \overline{z}_l}\, d\overline{z}_s \wedge d\overline{z}_J, \quad \text{where} \quad l > s.$$

Linear independence and $(*)$ imply

$$\frac{\partial a_J}{\partial \overline{z}_l} = 0 \quad \text{if} \quad J \subseteq \{1, 2, \ldots, s-1\} \quad \text{and} \quad l > s.$$

If $s \in J$, write $\widetilde{J} = J - \{s\}$. Look at the function

$$\eta_J(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{|\xi| \leq r-\epsilon} a_J(z_1, \ldots, z_{s-1}, \xi, z_{s+1}, \ldots, z_n)\, \frac{d\xi \wedge d\overline{\xi}}{\xi - z_s}. \tag{$**$}$$

We have the basic complex analysis lemma:

Lemma 2.4 *Say $g(\xi) \in C^\infty(\Delta_r)$ (where Δ_r is the open disc of radius r), then the function*

$$f(z) = \frac{1}{2\pi i} \int_{|\xi| \leq r-\epsilon} g(\xi)\, \frac{d\xi \wedge d\overline{\xi}}{\xi - z}$$

is in $C^\infty(\Delta^r)$ and $\dfrac{\partial f}{\partial \overline{z}} = g$ on $\Delta_{r-\epsilon}$.

By this lemma, we have

$$a_J(z_1, \ldots, z_n) = \frac{\partial \eta_J}{\partial \overline{z}_s} \quad \text{on} \quad \Delta_{r-\epsilon}(z\text{'s})$$

and if $l > s$,

$$\frac{\partial \eta_J}{\partial \overline{z}_l} = \frac{1}{2\pi i} \int_{|\xi| \leq r-\epsilon} \frac{\partial a_J}{\partial \overline{z}_l}\, \frac{d\xi \wedge d\overline{\xi}}{\xi - z_s} = 0,$$

by the above. So, if we set $\eta = \sum_J \eta_J dz_{\widetilde{J}}$, then $\omega - \overline{\partial}\eta$ depends only on $d\overline{z}_1, \ldots, d\overline{z}_{s-1}$ in $PD(0, r-\epsilon)$.

Step III: Exhaustion.

Pick a sequence, $\{\epsilon_t\}$, with ϵ_t monotonically decreasing to 0 and examine $PD(0, r-\epsilon_t)$. Write $r_t = r - \epsilon_t$, then the sequence $\{r_t\}$ monotonically increases to r.

Claim. We can find a sequence, $\eta_t \in \bigwedge^{0,q-1}(PD(0,r))$, such that

(1) η_t has compact support in $PD(0, r_{t+1})$.

(2) $\eta_t = \eta_{t-1}$ on $PD(0, r_{t-1})$.

(3) $\overline{\partial}\eta_t = \omega$ on $PD(0, r_t)$.

We proceed by induction on q, here is the induction step. Pick a sequence of cutoff C^∞-functions, γ_t, so that

(i) γ_t has compact support in $PD(0, r_{t+1})$.

(ii) $\gamma_t \equiv 1$ on $\overline{PD(0, r_t)}$.

Having chosen η_t, we will find η_{t+1}. First, by the interior part of the proof, there is some $\widetilde{\eta}_{t+1} \in \bigwedge^{0,q-1}(PD(0,r))$ with $\overline{\partial}\widetilde{\eta}_{t+1} = \omega$ in $PD(0, r_{t+1})$. Examine $\widetilde{\eta}_{t+1} - \eta_t$ on $PD(0, r_t)$, then

$$\overline{\partial}(\widetilde{\eta}_{t+1} - \eta_t) = \overline{\partial}\widetilde{\eta}_{t+1} - \overline{\partial}\eta_t = \omega - \omega = 0.$$

By the induction hypothesis, there is some $\beta \in \bigwedge^{0,q-2}(PD(0,r))$ with

$$\overline{\partial}\beta = \widetilde{\eta}_{t+1} - \eta_t \quad \text{on} \quad PD(0, r_t).$$

Let $\eta_{t+1} = \gamma_{t+1}(\widetilde{\eta}_{t+1} - \overline{\partial}\beta) = \gamma_{t+1}\eta_t$. We have

(1) $\eta_{t+1} \in C_0^\infty(\bigwedge^{0,q-1}(PD(0, r_{t+2})))$.

(3) As $\gamma_{t+1} \equiv 1$ on $PD(0, r_{t+1})$, we have $\eta_{t+1} = \widetilde{\eta}_{t+1} - \overline{\partial}\beta$ and so, $\overline{\partial}\eta_{t+1} = \overline{\partial}\widetilde{\eta}_{t+1} = \omega$ on $PD(0, r_{t+1})$.

(2) $\eta_{t+1} - \eta_t = \widetilde{\eta}_{t+1} - \overline{\partial}\beta - \eta_t = 0$ on $PD(0, r_t)$.

Now, for any compact subset, K, in $PD(0,r)$, there is some t so that $K \subseteq PD(0, r_t)$. It follows that the η_t's stabilize on K and our sequence converges uniformly on compacta. Therefore,

$$\eta = \lim_{t \mapsto \infty} \eta_t = \overline{\partial}\eta \quad \text{and} \quad \overline{\partial}\eta = \lim_{t \mapsto \infty} \overline{\partial}\eta_t = \omega.$$

Finally, we have to deal with the case $q = 1$. Let $\omega \in \bigwedge^{0,1}(PD(0,r))$, with $\overline{\partial}\omega = 0$. Again, we need to find some functions, η_t, with compact support on $PD(0, r_{t+1})$, so that

(α) $\overline{\partial}\eta_t = \omega$ on $PD(0, r_t)$.

(β) η_t converges uniformly on compacta to η, with $\overline{\partial}\eta = \omega$. Here, $\eta_t, \eta \in C^\infty(PD(0,r))$.

Say we found η_t with

$$\|\eta_t - \eta_{t-1}\|_{\infty, \overline{PD(0, r_{t-2})}} \leq \frac{1}{2^{t-1}}.$$

Pick $\widetilde{\eta}_{t+1} \in C^\infty(PD(0,r))$, with $\overline{\partial}\widetilde{\eta}_{t+1} = \omega$ on $PD(0, r_{t+1})$. Then, on $PD(0, r_t)$, we have

$$\overline{\partial}(\widetilde{\eta}_{t+1} - \eta_t) = \overline{\partial}\widetilde{\eta}_{t+1} - \overline{\partial}\eta_t = \omega - \omega = 0.$$

So, $\widetilde{\eta}_{t+1} - \eta_t$ is holomorphic in $PD(0, r_t)$. Take the MacLaurin series for it and truncate it to the polynomial θ so that on the compact $\overline{PD(0, r_{t-1})}$, we have

$$\|\widetilde{\eta}_{t+1} - \eta_t - \theta\|_{\infty, \overline{PD(0, r_{t-1})}} \leq \frac{1}{2^t}.$$

Take $\eta_{t+1} = \gamma_{t+1}(\widetilde{\eta}_{t+1} - \theta)$. Now, η_{t+1} has compact support on $PD(0, r_{t+2})$ and on $PD(0, r_{t+1})$, we have $\gamma_{t+1} \equiv 1$. This implies that $\eta_{t+1} = \widetilde{\eta}_{t+1} - \theta$, so

$$\|\eta_{t+1} - \eta_t\|_{\infty, \overline{PD(0, r_{t-1})}} \leq \frac{1}{2^t}$$

and

$$\overline{\partial}\eta_{t+1} = \overline{\partial}\widetilde{\eta}_{t+1} + \overline{\partial}\theta = \overline{\partial}\widetilde{\eta}_{t+1} = \omega \quad \text{on} \quad PD(0, r_{t+1}),$$

as θ is a polynomial. Therefore, the η_t's converge uniformly on compacta and if $\eta = \lim_{t \mapsto \infty} \eta_t$, we get $\overline{\partial}\eta = \omega$. $\square$

Corollary 2.5 *($\partial\overline{\partial}$ Poincaré) Say $\omega \in \bigwedge^{p,q}(U)$, where $U \subseteq X$ is an open subset of a complex manifold, X, and assume $d\omega = 0$. Then, for all $x \in U$, there is a neighborhood, $V \ni x$, so that $\omega = \partial\overline{\partial}\eta$ on V, for some $\eta \in \bigwedge^{p-1,q-1}(V)$.*

Proof. The statement is local on X, therefore we may assume $X = \mathbb{C}^n$. By ordinary d-Poincaré, for every $x \in X$, there is some open, $V_1 \ni x$, and some $\zeta \in \bigwedge^{p+q-1}(V_1)$, so that $\omega = d\zeta$. Now,

$$\bigwedge^{p+q-1}(V_1) = \coprod_{r+s=p+q-1} \bigwedge^{r,s}(V_1),$$

so, $\zeta = (\zeta_{r,s})$, where $\zeta_{r,s} \in \bigwedge^{r,s}(V_1)$. We have

$$\omega = d\zeta = \sum_{r,s} d\zeta_{r,s} = \sum_{r,s} (\partial + \overline{\partial})\zeta_{r,s}.$$

Observe that if $(r,s) \neq (p-1,q)$ or $(r,s) \neq (p,q-1)$, then the $\zeta_{r,s}$'s have $d\zeta_{r,s} \notin \bigwedge^{p,q}(V_1)$. It follows that $\zeta_{r,s} = 0$ and we can delete these terms from ζ; we get $\zeta = \zeta_{p-1,q} + \zeta_{p,q-1}$ with $d\zeta = 0$. We also have

$$\omega = d\zeta = (\partial + \overline{\partial})\zeta = \partial\zeta_{p-1,q} + \overline{\partial}\zeta_{p,q-1} + \overline{\partial}\zeta_{p-1,q} + \partial\zeta_{p,q-1} = \omega + \overline{\partial}\zeta_{p-1,q} + \partial\zeta_{p,q-1},$$

that is, $\overline{\partial}\zeta_{p-1,q} + \partial\zeta_{p,q-1} = 0$. Yet, $\overline{\partial}\zeta_{p-1,q}$ and $\partial\zeta_{p,q-1}$ belong to different bigraded components, so $\overline{\partial}\zeta_{p-1,q} = \partial\zeta_{p,q-1} = 0$. We now use the $\overline{\partial}$ and ∂-Poincaré lemma to get a polydisc, $V \subseteq V_1$ and some forms η_1 and η_2 in $\bigwedge^{p-1,q-1}(V)$, so that $\zeta_{p-1,q} = \overline{\partial}\eta_1$ and $\zeta_{p,q-1} = \partial\eta_2$. We get

$$\partial\overline{\partial}(\eta_1) = \partial\zeta_{p-1,q} \quad \text{and} \quad \partial\overline{\partial}(\eta_2) = -\overline{\partial}\partial(\eta_2) = -\overline{\partial}\zeta_{p,q-1}$$

and so,

$$\partial\overline{\partial}(\eta_1 - \eta_2) = \partial\zeta_{p-1,q} + \overline{\partial}\zeta_{p,q-1} = \omega,$$

which concludes the proof. $\square$

Remark: Take $\mathcal{C}^\infty$ = the sheaf of germs of real-valued C^∞-functions on X, then

$$\mathcal{H} = \mathrm{Ker}\left(\partial\overline{\partial}\colon \mathcal{C}^\infty \longrightarrow \bigwedge^{1,1}(X)\right)$$

is called the *sheaf of germs of pluri-harmonic functions*.

Corollary 2.6 *With X as in Corollary 2.5, the sequences*

$$0 \longrightarrow \Omega^p_X \hookrightarrow \bigwedge^{p,0} X \xrightarrow{\overline{\partial}} \bigwedge^{p,1} X \xrightarrow{\overline{\partial}} \cdots$$

(when $p = 0$, it is $0 \longrightarrow \mathcal{O}_X \hookrightarrow \bigwedge^{0,0} X \xrightarrow{\overline{\partial}} \bigwedge^{0,1} X \xrightarrow{\overline{\partial}} \cdots$),

$$0 \longrightarrow \overline{\Omega}^q_X \hookrightarrow \bigwedge^{0,q} X \xrightarrow{\partial} \bigwedge^{1,q} X \xrightarrow{\partial} \cdots$$

and

$$0 \longrightarrow \mathcal{H} \hookrightarrow \mathcal{C}^\infty X \xrightarrow{\partial\overline{\partial}} \bigwedge^{1,1} X \xrightarrow{d} \bigwedge^{2,1} X \coprod \bigwedge^{1,2} X \xrightarrow{d} \cdots$$

are resolutions (i.e., exact sequences of sheaves) of Ω^p_X, $\overline{\Omega}^q_X$, $\mathcal{H}$, respectively.

Proof. These are immediate consequences of $\overline{\partial}$, ∂, $\partial\overline{\partial}$ and d-Poincaré. $\square$

In Corollary 2.6, the sheaf Ω_X^p is the *sheaf of holomorphic p-forms* (locally, $\omega = \sum_I a_I dz_I$, where the a_I are holomorphic functions), $\overline{\Omega}_X^q$ is the *sheaf of anti-holomorphic q-forms* ($\omega = \sum_I a_I d\overline{z}_I$, where the a_I are anti-holomorphic functions) and $\mathcal{H}$ is the *sheaf of pluri-harmonic functions*.

If $\mathcal{F}$ is a sheaf of abelian groups, by cohomology, we mean derived functor cohomology, i.e., we have

$$\Gamma\colon \mathcal{F} \mapsto \mathcal{F}(X) = \Gamma(X, \mathcal{F}),$$

a left-exact functor and

$$H^p(X, \mathcal{F}) = (R^p\Gamma)(\mathcal{F}) \in \mathcal{A}\mathrm{b}.$$

We know that this cohomology can be computed using flasque (= flabby) resolutions

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_1 \longrightarrow \cdots \longrightarrow \mathcal{G}_n \longrightarrow \cdots,$$

where the $\mathcal{G}_i$'s are *flasque*, i.e., for every open, $U \subseteq X$, for every section $\sigma \in \mathcal{G}(U)$, there is a global section, $\tau \in \mathcal{G}(X)$, so that $\sigma = \tau \upharpoonright U$. If we apply Γ, we get a complex of (abelian) groups

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}_0) \longrightarrow \Gamma(X, \mathcal{G}_1) \longrightarrow \cdots \longrightarrow \Gamma(X, \mathcal{G}_n) \longrightarrow \cdots, \qquad (*)$$

and then $H^p(X, \mathcal{F})$ = the pth cohomology group of $(*)$.

Unfortunately, the sheaves arising naturally (from forms, etc.) are *not* flasque; they satisfy a weaker condition. In order to describe this condition, given a sheaf, $\mathcal{F}$, we need to make sense of $\mathcal{F}(S)$, where $S \subseteq X$ is a *closed* subset. Now, remember (see Appendix A on sheaves, Section A4) that for any subspace, Y of X, if $j\colon Y \hookrightarrow X$ is the inclusion map, then for any sheaf, $\mathcal{F}$, on X, the sheaf $j^*\mathcal{F} = \mathcal{F} \upharpoonright Y$ is the restriction of $\mathcal{F}$ to Y. For every $x \in Y$, the stalk of $\mathcal{F} \upharpoonright Y$ at x is equal to $\mathcal{F}_x$. Consequently, if S is any subset of X, we have $\sigma \in \mathcal{F}(S)$ iff there is an open cover, $\{U_\alpha\}$, of S and a family of sections, $\sigma_\alpha \in \mathcal{F}(U_\alpha)$, so that for every α, we have

$$\sigma \upharpoonright S \cap U_\alpha = \sigma_\alpha \upharpoonright S \cap U_\alpha.$$

Remark: (Inserted by J.G.) If X is *paracompact*, then for any closed subset, $S \subseteq X$, we have

$$\mathcal{F}(S) = \varinjlim_{U \supseteq S} \mathcal{F}(U),$$

where U ranges over all open subsets of S (see Godement[5] , Chapter 3, Section 3.3, Corollary 1). [Recall that for any cover, $\{U_\alpha\}_\alpha$, of X, we say that that $\{U_\alpha\}_\alpha$ is *locally finite* iff for every $x \in X$, there is some open subset, $U_x \ni x$, so that U_x meets only finitely many U_α. A topological space, X, is *paracompact* iff it is Hausdorff and if every open cover possesses a locally finite refinement.]

Now, we want to consider sheaves, $\mathcal{F}$, such that for every closed subset, S, the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(S)$ is onto.

Definition 2.3 Let X be a paracompact topological space. A sheaf, $\mathcal{F}$, is *soft* (*mou*) iff for every *closed* subset, $S \subseteq X$, the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F} \upharpoonright S(S)$ is onto. A sheaf, $\mathcal{F}$, is *fine* iff for all locally finite open covers, $\{U_\alpha \longrightarrow X\}$, there exists a family, $\{\eta_\alpha\}$, with $\eta_\alpha \in \mathrm{End}(\mathcal{F})$, so that

(1) $\eta_\alpha \upharpoonright \mathcal{F}_x = 0$, for all x in some neighborhood of U_α^c, i.e., $\mathrm{supp}(\eta_\alpha) \subseteq U_\alpha$.

(2) $\sum_\alpha \eta_\alpha = \mathrm{id}$.

We say that the family $\{\eta_\alpha\}$ is a *sheaf partition of unity subordinate to the given cover* $\{U_\alpha \longrightarrow X\}$ *for* $\mathcal{F}$.

Remark: The following sheaves are fine on any complex or real C^∞-manifold:

(1) $\mathcal{C}^\infty$

(2) $\bigwedge^p$

(3) $\bigwedge^{p,q}$

(4) Any locally-free $\mathcal{C}^\infty$-bundle (= C^∞-vector bundle).

For, any open cover of our manifold has a locally finite refinement, so we may assume that our open cover is locally finite (recall, a manifold is locally compact and second-countable, which implies paracompactness). Then, take a C^∞-partition of unity subordinate to our cover, $\{U_\alpha \longrightarrow X\}$, i.e., a family of C^∞-functions, φ_α, so that

(1) $\varphi_\alpha \geq 0$.

(2) $\mathrm{supp}(\varphi) << U_\alpha$ (this means $\mathrm{supp}(\varphi)$ is compact and contained in U_α).

(3) $\sum_\alpha \varphi_\alpha = 1$.

Then, for η_α, use multiplication by φ_α

Remark: If we know a sheaf of rings, $\mathcal{A}$, on X is fine, then every $\mathcal{A}$-module is also fine and the same with soft.

Proposition 2.7 *Let X be a paracompact space. Every fine sheaf is soft. Say*

$$0 \longrightarrow \mathcal{F}' \stackrel{\lambda}{\longrightarrow} \mathcal{F} \stackrel{\mu}{\longrightarrow} \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of sheaves and $\mathcal{F}'$ is soft. Then,

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0 \quad \textit{is exact.}$$

Again, if

$$0 \longrightarrow \mathcal{F}' \stackrel{\lambda}{\longrightarrow} \mathcal{F} \stackrel{\mu}{\longrightarrow} \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of sheaves and if $\mathcal{F}'$ and $\mathcal{F}$ are soft, so is $\mathcal{F}''$. Every soft sheaf is cohomologically trivial ($H^p(X, \mathcal{F}) = (0)$ if $p > 0$).

Proof. Take $\mathcal{F}$ fine, S closed and $\tau \in \mathcal{F}(S)$. There is an open cover of S and sections, $\tau_\alpha \in \mathcal{F}(U_\alpha)$, so that $\tau_\alpha \restriction U_\alpha \cap S = \tau \restriction U_\alpha \cap S$. Let $U_0 = X - S$, an open, so that U_0 and the U_α cover X. By paracompactness, we may asume that the cover is locally finite. Take the $\eta_\alpha \in \mathrm{Aut}(\mathcal{F})$ guaranteed as $\mathcal{F}$ is fine. Now, we have $\eta_\alpha(\tau_\alpha) = 0$ near the boundary of U_α, so $\eta_\alpha(\tau_\alpha)$ extends to all of X (as section) by zero, call it σ_α. We have $\sigma_\alpha \in \mathcal{F}(X)$ and

$$\sigma = \sum_\alpha \sigma_\alpha \quad \text{exists (by local finiteness).}$$

As $\sigma_\alpha \restriction U_\alpha \cap S = \tau_\alpha \restriction U_\alpha \cap S$, we get

$$\sigma_\alpha = \eta_\alpha(\tau_\alpha) = \eta_\alpha(\tau) \quad \text{on } U_\alpha \cap S$$

and we deduce that

$$\sigma = \sum_\alpha \sigma_\alpha = \sum_\alpha \eta_\alpha(\tau_\alpha) = \sum_\alpha \eta_\alpha(\tau) = \Big(\sum_\alpha \eta_\alpha\Big)(\tau) = \tau; \quad \text{on } S.$$

Therefore, σ is a lift of τ to X from S.

Exactness of the sequence

$$0 \longrightarrow \mathcal{F}' \stackrel{\lambda}{\longrightarrow} \mathcal{F} \stackrel{\mu}{\longrightarrow} \mathcal{F}'' \longrightarrow 0$$

implies that for every $\sigma \in \mathcal{F}''(X)$, there is an open cover, $\{U_\alpha \longrightarrow X\}$, and a family of sections, $\tau_\alpha \in \mathcal{F}(U_\alpha)$, so that $\mu(\tau_\alpha) = \sigma \upharpoonright U_\alpha$. By paracompactness, we may replace the U_α's by a locally finite family of closed sets, S_α. Consider the set

$$\mathcal{S} = \left\{ (\tau, S) \;\middle|\; \begin{array}{l} (1)\ S = \bigcup S_\alpha, \quad \text{for some of our } S_\alpha \\ (2)\ \tau \in \mathcal{F}(S),\ \tau \upharpoonright S_\alpha = \tau_\alpha, \quad \text{for each } S_\alpha \text{ as in (1).} \end{array} \right\}$$

The set $\mathcal{S}$ is, as usual, partially ordered and it is inductive (DX). By Zorn's lemma, $\mathcal{S}$ possesses a maximal element, (τ, S). I claim that $X = S$.

If $S \neq X$, then there is some S_β with $S_\beta \not\subseteq S$. On $S \cap S_\beta$, we have

$$\mu(\tau - \tau_\beta) = \sigma - \sigma = 0,$$

where $\mu(\tau) = \sigma$, by (2), and $\mu(\tau_\beta) = \sigma$, by definition. By exactness, there is some $\zeta \in \mathcal{F}'(S \cap S_\beta)$ so that $\lambda(\zeta) = \tau - \tau_\beta$ on $S \cap S_\beta$. Now, as $\mathcal{F}'$ is soft, ζ extends to a global section of $\mathcal{F}'$, say, z. Define ω by

$$\omega = \begin{cases} \tau & \text{on } S \\ \tau_\beta + \lambda(z) & \text{on } S_\beta. \end{cases}$$

On $S \cap S_\beta$, we have $\omega = \tau = \tau_\beta + \lambda(z) = \tau_\beta + \lambda(\zeta) = \tau$, so ω and τ agree. But then, $(\omega, S \cup S_\beta) \in \mathcal{S}$ and $(\omega, S \cup S_\beta) > (\tau, S)$, a contradiction. Therefore, the sequence

$$0 \longrightarrow \mathcal{F}' \stackrel{\lambda}{\longrightarrow} \mathcal{F} \stackrel{\mu}{\longrightarrow} \mathcal{F}'' \longrightarrow 0$$

has globally exact sections.

Now, assume that $\mathcal{F}'$ and $\mathcal{F}$ are soft and take $\tau \in \mathcal{F}''$, with S closed. Apply the above to $X = S$; as $\mathcal{F}'$ is soft, we deduce that $\mathcal{F}(S) \longrightarrow \mathcal{F}''(S)$ is onto. As $\mathcal{F}$ and $\mathcal{F}'$ are soft, the commutative diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}''(X) \arrow[d] & \\ \mathcal{F}(S) \arrow[r] \arrow[d] & \mathcal{F}''(S) \arrow[r] & 0 \\ 0 & & \end{tikzcd}$$

implies that $\mathcal{F}''(X) \longrightarrow \mathcal{F}''(S)$ is surjective.

For the last part, we use induction. The induction hypothesis is: If $\mathcal{F}$ is soft, then $H^p(X, \mathcal{F}) = (0)$, for $0 < p \leq n$. When $n = 1$, we can embed $\mathcal{F}$ in a flasque sheaf, Q, and we have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow Q \longrightarrow \mathrm{cok} \longrightarrow 0. \tag{$\dagger$}$$

If we apply cohomology we get

$$0 \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, Q) = (0),$$

since Q is flasque, so $H^1(X, \mathcal{F}) = (0)$.

For the induction step, use (†) and note that cok is soft because $\mathcal{F}$and Q are soft (Q is flasque and flasque sheaves are soft over a paracompact space, see Homework). When we apply cohomology, we get

$$(0) = H^j(X, Q) \longrightarrow H^j(X, \mathrm{cok}) \longrightarrow H^{j+1}(X, \mathcal{F}) \longrightarrow H^{j+1}(X, Q) = (0), \quad (j \geq 1)$$

so $H^j(X, \mathrm{cok}) \cong H^{j+1}(X, \mathcal{F})$. As cok is soft, by the induction hypothesis, $H^j(X, \mathrm{cok}) = (0)$, so $H^{j+1}(X, \mathcal{F}) = (0)$. □

Corollary 2.8 *Each of the resolutions ($p > 0$)*

$$0 \longrightarrow \Omega^p_X \longrightarrow \bigwedge^{p,0} X \xrightarrow{\overline{\partial}} \bigwedge^{p,1} X \xrightarrow{\overline{\partial}} \cdots,$$

(for $p = 0$, a resolution of $\mathcal{O}_X$),

$$0 \longrightarrow \overline{\Omega}^q_X \longrightarrow \bigwedge^{0,q} X \xrightarrow{\partial} \bigwedge^{1,q} X \xrightarrow{\partial} \cdots,$$

and

$$0 \longrightarrow \begin{matrix} \mathbb{R} \\ \mathbb{C} \end{matrix} \longrightarrow \bigwedge^{0} X = \mathcal{C}^\infty \xrightarrow{d} \bigwedge^{1} X \xrightarrow{d} \cdots,$$

is an acyclic resolution (i.e., the cohomology of $\bigwedge^{p,q} X$, $\bigwedge^p X$ vanishes).

Proof. The sheaves $\bigwedge^{p,q} X$, $\bigwedge^p X$ are fine, therefore soft, by Proposition 2.7. □

Recall the spectral sequence of Čech cohomology ($\check{SS}$):

$$E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(\mathcal{F})) \Longrightarrow H^\bullet(X, \mathcal{F}),$$

where

(1) $\mathcal{F}$ is a sheaf of abelian groups on X

(2) $\mathcal{H}^q(\mathcal{F})$ is the *presheaf* defined by $U \rightsquigarrow H^q(U, \mathcal{F})$.

Now, we have the following vanishing theorem (see Godement [5]):

Theorem 2.9 *(Vanishing Theorem) Say X is paracompact and $\mathcal{F}$ is a presheaf on X so that $\mathcal{F}^\sharp$(= associated sheaf to $\mathcal{F}$) is zero. Then,*

$$\check{H}^p(X, \mathcal{F}) = (0), \quad \textit{all } p \geq 0.$$

Putting the vanishing theorem together with the spectral sequence ($\check{SS}$), we get:

Theorem 2.10 *(Isomorphism Theorem) If X is a paracompact space, then for all sheaves, $\mathcal{F}$, the natural map*

$$\check{H}^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

is an isomorphism for all $p \geq 0$.

Proof. The natural map $\check{H}^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ is just the edge homomorphism from ($\check{SS}$). By the handout on cohomology,

$$\mathcal{H}^q(\mathcal{F})^\sharp = (0), \quad \text{all} \quad q \geq 1.$$

Thus, the vanishing says

$$E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(\mathcal{F})) = (0), \quad \text{all} \quad p \geq 0,\ q \geq 1,$$

which implies that the spectral sequence ($\check{SS}$) degenerates and we get our isomorphism. □

Comments: How to get around the spectral sequence ($\check{SS}$).

(1) Look at the presheaf $\mathcal{F}$ and the sheaf $\mathcal{F}^\sharp$. There is a map of presheaves, $\mathcal{F} \longrightarrow \mathcal{F}^\sharp$, so we get a map, $\check{H}^p(X,\mathcal{F}) \longrightarrow \check{H}^p(X,\mathcal{F}^\sharp)$. Let $K = \mathrm{Ker}\,(\mathcal{F} \longrightarrow \mathcal{F}^\sharp)$ and $C = \mathrm{Coker}\,(\mathcal{F} \longrightarrow \mathcal{F}^\sharp)$. We have the short exact sequences of presheaves

$$0 \longrightarrow K \longrightarrow \mathcal{F} \longrightarrow \mathrm{Im} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathrm{Im} \longrightarrow \mathcal{F}^\sharp \longrightarrow C \longrightarrow 0,$$

where Im is the presheaf image $\mathcal{F} \longrightarrow \mathcal{F}^\sharp$. The long exact sequence of Čech cohomology for presheaves gives

$$\cdots \longrightarrow \check{H}^p(X,K) \longrightarrow \check{H}^p(X,\mathcal{F}) \longrightarrow \check{H}^p(X,\mathrm{Im}) \longrightarrow \check{H}^{p+1}(X,K) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \check{H}^{p-1}(X,C) \longrightarrow \check{H}^p(X,\mathrm{Im}) \longrightarrow \check{H}^p(X,\mathcal{F}^\sharp) \longrightarrow \check{H}^p(X,C) \longrightarrow \cdots,$$

and as $K^\sharp = C^\sharp = (0)$, by the vanishing theorem, we get

$$\check{H}^p(X,\mathcal{F}) \cong \check{H}^p(X,\mathrm{Im}) \cong \check{H}^p(X,\mathcal{F}^\sharp).$$

Therefore, on a paracompact space, $\check{H}^p(X,\mathcal{F}) \cong \check{H}^p(X,\mathcal{F}^\sharp)$.

(2) Čech cohomology is a δ-functor on the *category of sheaves* for paracompact X.

Say

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact as sheaves. Then, if we write Im for $\mathrm{Im}(\mathcal{F} \longrightarrow \mathcal{F}'')$ as presheaves, we have the short exact sequence of presheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathrm{Im} \longrightarrow 0$$

and $\mathrm{Im}^\sharp = \mathcal{F}''$. Then, for presheaves, we have

$$\cdots \longrightarrow \check{H}^p(X,\mathcal{F}) \longrightarrow \check{H}^p(X,\mathrm{Im}) \longrightarrow \check{H}^{p+1}(X,\mathcal{F}') \longrightarrow \cdots$$

and by (1), $\check{H}^p(X,\mathcal{F}) \cong \check{H}^p(X,\mathcal{F}^\sharp)$, so we get (2).

(3) One knows, for soft $\mathcal{F}$ on a paracompact space, X, we have $\check{H}^p(X,\mathcal{F}) = (0)$, for all $p \geq 1$. Each $\mathcal{F}$ embeds in a flasque sheaf; flasque sheaves are soft, so $\{\check{H}^\bullet\}$ is an effaceable δ-functor on the category of sheaves and it follows that $\{\check{H}^\bullet\}$ is universal. By homological algebra, we get the isomorphism theorem, again.

In fact, instead of (3), one can prove the following proposition:

Proposition 2.11 *Say X is paracompact and $\mathcal{F}$ is a fine sheaf. Then, for a locally finite cover, $\{U_\alpha \longrightarrow X\}$, we have*

$$\check{H}^p(\{U_\alpha \longrightarrow X\},\mathcal{F}) = (0), \quad \textit{if } p \geq 1.$$

Proof. Take $\{\eta_\alpha\}$, the sheaf partition of unity of $\mathcal{F}$ subordinate to our cover, $\{U_\alpha \longrightarrow X\}$. Pick $\tau \in Z^p(\{U_\alpha \longrightarrow X\},\mathcal{F})$, with $p \geq 1$. So, we have $\tau = \tau(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$. Write

$$\omega = \sum_\beta \eta_\beta(\tau(U_\beta \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})).$$

Observe that ω exists as section over $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ as η_β is zero near the boundary of U_β; so ω can be extended from $U_\beta \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ to $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ by zero. You check (usual computation): $d\omega = \tau$. $\square$

Corollary 2.12 *If $\mathcal{F}$ is fine (over a paracompact, X), then*

$$\check{H}^p(X,\mathcal{F}) = (0), \quad \textit{for all } p \geq 1.$$

Figure 2.1: A triangulated manifold

Theorem 2.13 *(P. Dolbeault) If X is a complex manifold, then we have the isomorphisms*

$$H^q(X, \Omega_X^p) \cong H_{\overline{\partial}}^{p,q}(X) \cong \check{H}^q(X, \Omega_X^p).$$

Proof. The middle cohomology is computed from the resolution of sheaves

$$0 \longrightarrow \Omega_X^p \longrightarrow \bigwedge^{p,0} X \xrightarrow{\overline{\partial}} \bigwedge^{p,1} X \xrightarrow{\overline{\partial}} \bigwedge^{p,2} X \xrightarrow{\overline{\partial}} \cdots.$$

Moreover, the $\bigwedge^{p,q} X$ are acyclic for $H^\bullet(X, -)$ and for $\check{H}^\bullet(X, -)$. Yet, by homological algebra, we can compute $H^q(X, \Omega_X^p)$ and $\check{H}^q(X, \Omega_X^p)$ by *any* acyclic resolution (they are δ-functors), $\square$

To prove de Rham's theorem, we need to look at singular cohomology.

Proposition 2.14 *If X is a real or complex manifold and $\mathcal{F}$ is a constant sheaf (sheaf associated with a constant presheaf), then there is a natural isomorphism*

$$\check{H}^p(X, \mathcal{F}) \cong H_{\mathrm{sing}}^p(X, \mathcal{F}),$$

provided $\mathcal{F}$ is torsion-free.

Proof. The space, X, is triangulable, so we get a singular simplicial complex, $\mathcal{K}$ (see Figure 2.1). Pick a vertex, v, of $\mathcal{K}$ and set

$$\mathrm{St}(v) = \bigcup \{\overset{\circ}{\sigma} \in \mathcal{K} \mid v \in \sigma\},$$

the open star of v (i.e., the union of the interiors of the simplices having v as a vertex). If $v_0, \ldots, v_p$ are vertices, consider

$$\mathrm{St}(v_0) \cap \cdots \cap \mathrm{St}(v_p) = U_{v_0, \ldots, v_p}.$$

We have

$$U_{v_0, \ldots, v_p} = \begin{cases} \emptyset & \text{if } v_0, \ldots, v_p \text{ are not the vertices of a } p\text{-simplex} \\ \text{a connected set} & \text{if } v_0, \ldots, v_p \text{ are the vertices of a } p\text{-simplex.} \end{cases}$$

Observe that $\{U_v \longrightarrow X\}_{v \in \mathrm{vert}(\mathcal{K})}$ is an open cover of X and as $\mathcal{F}$ is a constant sheaf, we get

$$\mathcal{F}(U_{v_0, \ldots, v_p}) = \begin{cases} 0 & \text{if } (v_0, \ldots, v_p) \notin \mathcal{K} \\ \mathcal{F} & \text{if } (v_0, \ldots, v_p) \in \mathcal{K}. \end{cases}$$

Let τ be a Čech p-cochain, then $\tau(U_{v_0, \ldots, v_p}) \in \mathcal{F}$ and let

$$\Theta(\tau)((v_0, \ldots, v_p)) = \tau(U_{v_0, \ldots, v_p}),$$

where $(v_0, \ldots, v_p) \in \mathcal{K}$. Note that $\Theta(\tau)$ is a p-simplicial cochain and the map $\tau \mapsto \Theta(\tau)$ is an isomorphism

$$C^p(\{U_v \longrightarrow X\}, \mathcal{F}) \cong C_{\mathrm{sing}}^p(X, \mathcal{F})$$

that commutes with the coboundary operators on both sides. So, we get the isomorphism

$$\check{H}^p(\{U_v \longrightarrow X\}, \mathcal{F}) \cong H_{\mathrm{sing}}^p(X, \mathcal{F}).$$

We can subdivide $\mathcal{K}$ simplicially and we get refinements of our cover and those are arbitrarily fine. Subdivision does not change the right hand side and if we take right limits we get

$$\check{H}^p(X, \mathcal{F}) \cong H_{\mathrm{sing}}^p(X, \mathcal{F}).$$

As a consequence, we obtain

Theorem 2.15 *(de Rham) On a real or complex manifold, we have the isomorphisms*

$$H^p\left(X, \begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}\right) \cong \check{H}^p\left(X, \begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}\right) \cong H^p_{\mathrm{sing}}\left(X, \begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}\right) \cong H^p_{\mathrm{DR}}\left(X, \begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}\right)$$

Proof. The isomorphism of singular cohomology with Čech cohomology follows from Proposition 2.14. The isomorphism of derived functor cohomology with Čech cohomology follows since X is paracompact. Also de Rham cohomology is the cohomology of the resolution

$$0 \longrightarrow \begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix} \longrightarrow \mathcal{C}^\infty \xrightarrow{d} \bigwedge^1 X \xrightarrow{d} \bigwedge^2 X \xrightarrow{d} \cdots,$$

and the latter is an acyclic resolution, so it computes H^p or $\check{H}^p$. $\square$

Explicit Connection: de Rham $\rightsquigarrow$ Singular.

Take a singular p-chain, $\sum_j a_j \Delta_j$, where $\Delta_j = f_j(\Delta)$; $f_j \in \mathcal{C}(\Delta)$; $\Delta =$ the usual p-simplex ($a_j \in \mathbb{Z}$, or $a_j \in \mathbb{R}$, or $a_j \in \mathbb{C}$, ...) We say that this p-chain is *piecewise smooth*, for short, *ps*, iff the f_j's actually are C^∞-functions on a small neighborhood around Δ. By the usual C^∞-approximation (using convolution), each singular p-chain is approximated by a ps p-chain in such a way that cocycles are approximated by ps cocycles and coboundaries, too. In fact, the inclusion

$$C_p^{\mathrm{ps}}(X, \mathbb{R}) \hookrightarrow C_p^{\mathrm{sing}}(X, \mathbb{R})$$

is a chain map and induces an isomorphism

$$H_p^{\mathrm{ps}}(X, \mathbb{R}) \hookrightarrow H_p^{\mathrm{sing}}(X, \mathbb{R}).$$

Say $\omega \in \bigwedge^p X$, a de Rham p-cochain, i.e., a p-form. If $\sigma \in C_p^{\mathrm{ps}}(X, \mathbb{R})$, say $\sigma = \sum_j a_j f_j(\Delta)$ (with $a_j \in \mathbb{R}$), then define $\Phi(\omega)$ *via*:

$$\Phi(\omega)(\sigma) = \int_\sigma \omega \overset{\mathrm{def}}{=} \sum_j a_j \int_{f_j(\Delta)} \omega \overset{\mathrm{def}}{=} \sum_j a_j \int_\Delta f_j^* \omega \in \mathbb{R}.$$

The map $\Phi(\omega)$ is clearly a linear map on $C_p^{\mathrm{ps}}(X, \mathbb{R})$, so we have $\Phi(\omega) \in C^p_{\mathrm{ps}}(X, \mathbb{R})$. Also, observe that

$$\Phi(d\omega)(\tau) = \int_\tau \omega = \int_{\partial\tau} \omega \text{ (by Stokes) } = \Phi(\omega)(\partial\tau),$$

from which we conclude that $\Phi(d\omega)(\tau) = (\partial\Phi)(\omega)(\tau)$, and thus, $\Phi(d\omega) = \partial\Phi(\omega)$. This means that

$$\int : \bigwedge^p (X, \mathbb{R}) \longrightarrow C^p_{\mathrm{ps}}(X, \mathbb{R})$$

is a cochain map and so, we get our map

$$H^p_{\mathrm{DR}}(X, \mathbb{R}) \longrightarrow H^p_{\mathrm{sing}}(X, \mathbb{R}).$$

2.3 Hodge I, Analytic Preliminaries

Let X be a complex analytic manifold. An *Hermitian metric* on X is a C^∞-section of the vector bundle $(T_X^{1,0} \otimes \overline{T_X^{1,0}})^D$, which is Hermitian symmetric and positive definite. This means that for each $z \in X$, we have a map $(-,-)_z \colon T^{1,0}_{X,z} \otimes T^{1,0}_{X,z} \longrightarrow \mathbb{C}$ which is linear in its first argument, Hermitian symmetric and positive definite, that is:

(1) $(v, u)_z = \overline{(u, v)_z}$ (Hermitian symmetric)

(2) $(u_1 + u_2, v)_z = (u_1, v)_z + (u_2, v)_z$ and $(u, v_1 + v_2)_z = (u, v_1)_z + (u, v_2)_z$.

(3) $(\lambda u, v)_z = \lambda(u, v)_z$ and $(u, \mu v)_z = \overline{\mu}(u, v)_z$.

(4) $(u, u)_z \geq 0$, for all u, and $(u, u)_z = 0$ iff $u = 0$ (positive definite).

(5) $z \mapsto h(z) = (-, -)_z$ is a C^∞-function.

Remark: Note that (2) and (3) is equivalent to saying that we have a $\mathbb{C}$-linear map, $T^{1,0}_{X,z} \otimes T^{0,1}_{X,z} \longrightarrow \mathbb{C}$.

In local coordinates, since $(T^{1,0}_X)^D = \bigwedge^{1,0} T^D_X$ and $\overline{T^{1,0}_X} = T^{0,1}_X$ and since $\{dz_j\}, \{d\overline{z}_j\}$ are bases for $\bigwedge^{1,0} T^D_{X,z}$ and $\bigwedge^{0,1} T^D_{X,z}$, we get

$$h(z) = \sum_{k,l} h_{kl}(z) dz_k \otimes d\overline{z}_l,$$

for some matrix $(h_{kl}) \in M_n(\mathbb{C})$. Now, $(-, -)_z$ is an Hermitian inner product, so locally on a trivializing cover for $T^{1,0}_X, T^{0,1}_X$, by Gram-Schmidt, we can find $(1, 0)$-forms, $\varphi_1, \ldots, \varphi_n$, so that

$$(-, -)_z = \sum_{j=1}^n \varphi_j(z) \otimes \overline{\varphi_j(z)}.$$

The collection $\varphi_1, \ldots, \varphi_n$ is called a *coframe* for $(-, -)$ (on the respective open of the trivializing cover). Using a partition of unity subordinate to a trivializing cover, we find all these data exist on any complex manifold.

Consider $\Re(-, -)_z$ and $\Im(-, -)_z$. For $\lambda \in \mathbb{R}$, (1), (2), (3), (4), imply that $\Re(-, -)_z$ is a positive definite bilinear form, C^∞ as a function of z, i.e, as $T_{X,z}$ real tangent space $\cong T^{1,0}_{X,z}$, we see that $\Re(-, -)_z$ is a C^∞-Riemannian metric on X. Hence, we have concepts such as length, area, volume, curvature, etc., associated to an Hermitian metric, namely, those concepts for the real part of $(-, -)_z$, i.e., the associated Riemannian metric.

If we look at $\Im(-, -)_z$, then (1), (2), (3) and (5) imply that for $\lambda \in \mathbb{R}$, we have an alternating real bilinear nondegenerate form on $T^{1,0}_{X,z}$, C^∞ in z. That is, we get an element of $(T^{1,0}_{X,z} \wedge T^{1,0}_{X,z})^D \subseteq \bigwedge^2 (T^D_{X,z} \otimes \mathbb{C})$. In fact, this is a $(1, 1)$-form. Look at $\Im(-, -)_z$ in a local coframe. Say $\varphi_k = \alpha_k + i\beta_k$, where $\alpha_k, \beta_k \in T^D_{X,z}$. We have

$$\begin{aligned}
\sum_k \varphi_k(z) \otimes \overline{\varphi_k(z)} &= \sum_k (\alpha_k(z) + i\beta_k(z)) \otimes (\alpha_k(z) - i\beta_k(z)) \\
&= \sum_k (\alpha_k(z) \otimes \alpha_k(z) + \beta_k(z) \otimes \beta_k(z)) + i \sum_k (\beta_k(z) \otimes \alpha_k(z) - \alpha_k(z) \otimes \beta_k(z)).
\end{aligned}$$

Now, a symmetric bilinear form yields a linear form on $S^2 T_{X,z} = S^2 T^{1,0}_{X,z}$; consequently, the real part of the Hermitian inner product is $\Re(-, -)_z = \sum_k (\alpha_k(z)^2 + \beta_k(z)^2)$. We usually write ds^2 for $\sum_k \varphi_k \otimes \overline{\varphi_k}$ and $\Re(ds^2)$ is the associated Riemannian metric. For $\Im(ds^2)$, we have a form in $\bigwedge^2 (T^{1,0}_{X,z})^D$:

$$\Im(ds^2) = -2 \sum_{k=1}^n \alpha_k \wedge \beta_k.$$

We let

$$\omega_{ds^2} = \omega = -\frac{1}{2} \Im(ds^2)$$

and call it the *associated* $(1,1)$*-form to the Hermitian* ds^2. If we write $\varphi_k = \alpha_k + i\beta_k$, we have

$$\sum_{k=1}^{n} \varphi_k \wedge \overline{\varphi_k} = \sum_{k=1}^{n} (\alpha_k + i\beta_k) \wedge (\alpha_k - i\beta_k) = -2i \sum_{k=1}^{n} \alpha_k \wedge \beta_k.$$

Therefore,

$$\omega = \sum_{k=1}^{n} \alpha_k \wedge \beta_k = \frac{i}{2} \sum_{k=1}^{n} \varphi_k \wedge \overline{\varphi_k},$$

which shows that ω is a $(1,1)$-form.

Remark: The expession for ω in terms of $\Im(ds^2)$ given above depends on the definition of $\wedge$. In these notes,

$$\alpha \wedge \beta = \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha),$$

but in some books, one finds

$$\omega = -\Im(ds^2).$$

Conversely, suppose we are given a real $(1,1)$-form. This means, ω is a $(1,1)$-form and for all ξ,

$$\omega(\overline{\xi}) = \overline{\omega(\xi)} \quad \text{(reality condition)}.$$

Define an "inner product" *via*

$$H(v,w) = \omega(v \wedge i\overline{w}).$$

We have

$$\begin{aligned} H(w,v) &= \omega(w \wedge i\overline{v}) \\ &= -\omega(i\overline{v} \wedge w) \\ &= \omega(\overline{iv \wedge \overline{w}}) \\ &= \overline{\omega(iv \wedge \overline{w})} \\ &= \overline{\omega(v \wedge i\overline{w})} \\ &= \overline{H(v,w)}. \end{aligned}$$

(Note we could also set $H(v,w) = -\omega(v \wedge \overline{iw})$.) Consequently, $H(v,w)$ will be an inner product provided $H(v,v) > 0$ iff $v \neq 0$. So, we need $\omega(v \wedge i\overline{v}) = -i\omega(v \wedge \overline{v}) > 0$, for all $v \neq 0$. Therefore, we say ω *is positive definite* iff

$$-i\omega(v \wedge \overline{v}) > 0, \quad \text{for all } v \neq 0.$$

Thus, $\omega = -(1/2)\Im(ds^2)$ recaptures all of ds^2. You check (DX) that ω is positive definite iff in local coordinates

$$\omega = \frac{i}{2} \sum_{k,l} h_{kl}(z) dz_k \wedge d\overline{z}_l,$$

where (h_{kl}) is a Hermitian positive definite matrix.

Example 1. Let $X = \mathbb{C}^n$, with $ds^2 = \sum_{k=1}^{n} dz_k \otimes d\overline{z}_k$. As usual, if $z_k = x_k + iy_k$, we have

(a) $\Re(ds^2) = \sum_{k=1}^{n} (dx_k^2 + dy_k^2)$, the ordinary Euclidean metric.

(b) $\omega = -(1/2)\,\Im(ds^2) = (i/2)\, \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k$, a positive definite $(1,1)$-form.

Remark: Assume that $f\colon Y \to X$ is a complex analytic map and that we have an Hermitian metric on X. Then, $Df\colon T_Y \to T_X$ maps $T^{1,0}_{Y,y}$ to $T^{1,0}_{X,f(y)}$, for all $y \in Y$. We define an "inner product" on Y *via*

$$\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\right)_y = \left(Df\,\frac{\partial}{\partial y_k}, Df\,\frac{\partial}{\partial y_l}\right)_{f(y)}.$$

We get a Hermitian symmetric form on Y. If we assume that Df is everywhere an injection, then our Hermitian metric, ds^2, on X induces one on Y; in particular, this holds if $Y \hookrightarrow X$.

Assume Df is injective everywhere. We have the dual map, $f^*\colon T_X^D \to T_Y^D$, i.e., $f^*\colon \bigwedge^{1,0} X \to \bigwedge^{1,0} Y$. Pick U small enough in Y so that

(1) $T_Y \restriction U$ is trivial

(2) $T_X \restriction f(U)$ is trivial.

(3) We have a local coframe, $\varphi_1, \ldots, \varphi_n$, on $T_X \restriction f(U)$ and $f^*(\varphi_{m+1}) = \cdots = f^*(\varphi_n) = 0$, where $m = \dim(Y)$ and $n = \dim(X)$.

Then,

$$f^*\omega_X = f^*\left(\frac{i}{2}\sum_{k=1}^n \varphi_k \wedge \overline{\varphi_k}\right) = \frac{i}{2}\sum_{k=1}^m f^*(\varphi_k) \wedge f^*(\overline{\varphi_k}) = \omega_Y.$$

Hence, the $(1,1)$-form of the induced metric on Y (from X) is the pullback of the $(1,1)$-form of the metric on X.

Consequently (Example 1), on an affine variety, we get an induced metric and an induced form computable from the embedding in some $\mathbb{C}^N$.

Example 2: *Fubini-Study Metric on* $\mathbb{P}^n$. Let π be the canonical projection, $\pi\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$, let $z_0, \ldots, z_n$ be coordinates on $\mathbb{C}^{n+1}$ and let $(Z_0\colon \cdots \colon Z_n)$ be homogeneous coordinates on $\mathbb{P}^n$. For a small open , U, pick some *holomorphic* section, $F\colon U \to \mathbb{C}^{n+1} - \{0\}$, of π (so that $\pi \circ F = \mathrm{id}_U$). For any $p \in U$, consider

$$\|F(p)\|^2 = \sum_{j=0}^n F_j(p)\overline{F_j(p)} \neq 0.$$

Pick U small enough so that $\log\|F\|^2$ is defined. Now, set

$$\omega_F = \frac{i}{2\pi}\,\partial\overline{\partial}\log\|F\|^2.$$

We need to show that this definition does not depend on the choice of the holomorphic section, F. So, let S be another holomorphic section of π over U. As $\pi \circ S = \pi \circ F = \mathrm{id}$ on U, we have

$$(S_0(p)\colon \cdots \colon S_n(p)) = (F_0(p)\colon \cdots \colon F_n(p)), \quad \text{for all } p \in U,$$

so, there is a holomorphic function, λ, on U, so that

$$\lambda(p)S(p) = F(p), \quad \text{for all } p \in U.$$

We have

$$\|F\|^2 = F\overline{F} = \lambda\overline{\lambda}S\overline{S} = \lambda\overline{\lambda}\,\|S\|^2,$$

so we get

$$\log\|F\|^2 = \log\lambda + \log\overline{\lambda} + \log\|F\|^2.$$

Consequently,

$$\omega_F = \frac{i}{2\pi}\,\partial\overline{\partial}(\log\lambda + \log\overline{\lambda}) + \omega_S = \omega_S,$$

since λ is holomorphic, $\overline{\lambda}$ is anti-holomorphic, $\overline{\partial}(\text{holo}) = 0$, $\partial\overline{\partial} = -\overline{\partial}\partial$ and $\partial(\text{anti-holo}) = 0$. Clearly, our ω_F are $(1,1)$-forms. Now, cover $\mathbb{P}^n$ by opens, as above; pick any section on each such open, use a partition of unity and get a *global* $(1,1)$-form on $\mathbb{P}^n$ which is C^∞. We still need to check positivity, but since the unitary group, $U(n+1)$, acts transitively on $\mathbb{C}^{n+1}$, we see that $\mathbb{PU}(n)$ acts transitively on $\mathbb{P}^n$ and our form is invariant. Therefore, it is enough to check positivity at one point, say $(1\colon 0\colon \cdots : 0)$. This point lies in the open $Z_0 \neq 0$. Lift Z_0 to $\mathbb{C}^{n+1} - \{0\}$ *via*

$$F\colon (Z_0\colon \cdots : Z_n) \mapsto (1, z_1, \ldots, z_n), \quad \text{where} \quad z_j = \frac{Z_j}{Z_0}.$$

Thus, $\|F\|^2 = 1 + \sum_{k=1}^n z_k\overline{z}_k$, and we get

$$\begin{aligned}
\partial\overline{\partial}\log\Big(1 + \sum_{k=1}^n z_k\overline{z}_k\Big) &= \partial\left(\frac{\sum_{k=1}^n z_k d\overline{z}_k}{1 + \sum_{k=1}^n z_k\overline{z}_k}\right)\\
&= \frac{\Big(\sum_{k=1}^n dz_k \wedge d\overline{z}_k\Big)\Big(1 + \sum_{k=1}^n z_k\overline{z}_k\Big) - \Big(\sum_{k=1}^n \overline{z}_k dz_k\Big) \wedge \Big(\sum_{l=1}^n z_l d\overline{z}_l\Big)}{\Big(1 + \sum_{k=1}^n z_k\overline{z}_k\Big)^2}.
\end{aligned}$$

When we evaluate the above at $(1\colon 0\colon \cdots : 0)$, we get $\sum_{k=1}^n dz_k \wedge d\overline{z}_k$ and so

$$\omega_F(1\colon 0\colon \cdots : 0) = \frac{i}{2\pi}\sum_{k=1}^n dz_k \wedge d\overline{z}_k,$$

which is positive. Therefore, we get a Hermitian metric on $\mathbb{P}^n$, this is the *Fubini-Study metric*. As a consequence, every projective manifold inherits an Hermitian metric from the Fubini-Study metric.

From now on, assume that X is *compact* manifold (or each object has compact support). Look at the bundles $\bigwedge^{p,q}$ and choose once and for all an Hermitian metric on X and let ω be the associated positive $(1,1)$-form. So, locally in a coframe,

$$\omega = \frac{i}{2}\sum_{k=1}^n \varphi_k \wedge \overline{\varphi_k}.$$

At each z, a basis for $\bigwedge_z^{p,q}$ is just $\{\varphi_I \wedge \overline{\varphi}_J\}$, where $I = \{i_1 < \cdots < i_p\}$, $J = \{j_1 < \cdots < j_q\}$ and

$$\varphi_I \wedge \overline{\varphi}_J = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} \wedge \overline{\varphi}_{j_1} \wedge \cdots \wedge \overline{\varphi}_{j_q}.$$

We can define an orthonormal basis of $\bigwedge_z^{p,q}$ if we decree that the $\varphi_I \wedge \overline{\varphi}_J$ are pairwise orthogonal, and we set

$$\|\varphi_I \wedge \overline{\varphi}_J\|^2 = (\varphi_I \wedge \overline{\varphi}_J, \varphi_I \wedge \overline{\varphi}_J) = 2^{p+q}.$$

This gives $\bigwedge_z^{p,q}$ a C^∞-varying Hermitian inner product. To understand where 2^{p+q} comes from, look at $\mathbb{C}$. Then, near z, we have $\varphi = dz$, $\overline{\varphi} = d\overline{z}$, so

$$dz \wedge d\overline{z} = (dx + idy) \wedge (dx - idy) = -i(dx \wedge dy + dx \wedge dy) = -2i\, dx \wedge dy.$$

Therefore, $\|dz \wedge d\overline{z}\| = 2$ and $\|dz \wedge d\overline{z}\|^2 = 4 = 2^{1+1}$ (here, $p = 1$ and $q = 1$).

Let us write $\bigwedge^{p,q}(X)$ for the set of global C^∞-sections, $\Gamma_{C^\infty}(X, \bigwedge^{p,q})$. Locally, on an open, U, we have

$$\omega = \frac{i}{2}\sum_{k=1}^n \varphi_k \wedge \overline{\varphi_k} \in \bigwedge^{1,1}(U)$$

and so, we deduce that

$$\omega^n = \left(\frac{i}{2}\right)^n n!\,(-1)^{\binom{n}{2}}\,\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi_1} \wedge \cdots \wedge \overline{\varphi_n}.$$

We call $\Phi(z) = \omega^n(z)/n! = C_n\,\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi_1} \wedge \cdots \wedge \overline{\varphi_n}$ the *volume form* and $C_n = (\frac{i}{2})^n (-1)^{\binom{n}{2}}$ the *twisting constant*. We can check that Φ is a real, positive form, so we can integrate w.r.t. to it. For $\xi, \eta \in \bigwedge^{p,q}(X)$, set

$$(\xi, \eta) = \int_X (\xi, \eta)_z\, \Phi(z) \in \mathbb{C}.$$

This makes $\bigwedge^{p,q}(X)$ a complex (infinite-dimensional) inner-product space. We have

$$\overline{\partial}\colon\ \bigwedge^{p,q-1}(X) \to \bigwedge^{p,q}(X)$$

and say (as in the finite dimensional case) $\overline{\partial}$ is a closed operator (i.e., $B_{\overline{\partial}}^{p,q}$ is closed in $\bigwedge^{p,q}(X)$). Pick some $\xi \in Z_{\overline{\partial}}^{p,q}$, i.e., with $\overline{\partial}(\xi) = 0$. All the cocyles representing the class of ξ (an element of $H_{\overline{\partial}}^{p,q}$) form the translates $\xi + B_{\overline{\partial}}^{p,q} \subseteq \bigwedge^{p,q}(X)$. This translate is a closed and convex subset of $\bigwedge^{p,q}(X)$.

Does there exist a smallest (in the norm we've just defined) cocycle in this cohomology class—if so, how to find it?

Now, we can ask if $\overline{\partial}$ has an adjoint. If so, call it $\overline{\partial}^*$ and then, $\overline{\partial}^*\colon \bigwedge^{p,q}(X) \to \bigwedge^{p,q-1}(X)$ and

$$(\overline{\partial}^*(\xi), \eta) = (\xi, \overline{\partial}(\eta)), \quad \text{for all } \xi, \eta.$$

Then, Hodge observed the

Proposition 2.16 *The cocycle, ξ, is of smallest norm in its cohomology class iff $\overline{\partial}^*(\xi) = 0$.*

Proof.

($\Leftarrow$). Compute

$$\|\xi + \overline{\partial}\eta\|^2 = (\xi + \overline{\partial}\eta, \xi + \overline{\partial}\eta) = \|\xi\|^2 + \|\eta\|^2 + 2\Re(\xi, \overline{\partial}\eta).$$

But, $(\xi, \overline{\partial}\eta) = (\overline{\partial}^*(\xi), \eta) = 0$, by hypothesis, so

$$\|\xi + \overline{\partial}\eta\|^2 = \|\xi\|^2 + \|\eta\|^2,$$

which shows the minimality of $\|\xi\|$ in $\xi + B_{\overline{\partial}}^{p,q}$ and the uniqneness of such a ξ.

($\Rightarrow$). We know that $\|\xi + \overline{\partial}\eta\|^2 \geq \|\xi\|^2$, for all our η's. Make

$$f(t) = (\xi + t\overline{\partial}\eta, \xi + t\overline{\partial}\eta).$$

The function $f(t)$ has a global minimum at $t = 0$ and by calculus, $f'(t) \restriction_{t=0} = 0$. We get

$$\Big((\overline{\partial}\eta, \xi + t\overline{\partial}\eta) + (\xi + t\overline{\partial}\eta, \overline{\partial}\eta)\Big)_{t=0} = 0,$$

that is, $\Re(\xi, \overline{\partial}\eta) = 0$. But, $i\eta$ is another element of $\bigwedge^{p,q-1} X$. So, let

$$g(t) = (\xi + it\overline{\partial}\eta, \xi + it\overline{\partial}\eta).$$

Repeating the above argument, we get $\Im(\xi, \overline{\partial}\eta) = 0$. Consequently, we have $(\xi, \overline{\partial}\eta) = 0$, for all η. Since $(\overline{\partial}^*(\xi), \eta) = (\xi, \overline{\partial}(\eta))$, we conclude that $(\overline{\partial}^*(\xi), \eta) = 0$, for all η, so $\overline{\partial}^*(\xi) = 0$, as required. $\square$

If the reasoning can be justified, then

(1) In each cohomology class of $H^{p,q}_{\overline{\partial}}$, there is a unique (minimal) representative.

(2)
$$H^{p,q}_{\overline{\partial}}(X) \cong \left\{ \xi \in \bigwedge^{p,q} X \;\middle|\; \begin{array}{l} (a)\ \overline{\partial}\xi = 0 \\ (b)\ \overline{\partial}^*\xi = 0 \end{array} \right\}.$$

We know from previous work that $H^{p,q}_{\overline{\partial}}(X) \cong H^q(X, \Omega^p_X)$.

Making $\overline{\partial}^*$. First, we make the Hodge $*$ operator:
$$*\colon \bigwedge^{p,q} X \to \bigwedge^{n-p,n-q} X$$
by *pure algebra*. We want
$$(\xi(z), \eta(z))_z\, \Phi(z) = \xi(z) \wedge *\,\eta(z) \quad \text{for all } \xi.$$
We need to define $*$ on basis elements, $\xi = \varphi_I \wedge \overline{\varphi}_J$. We want
$$(\varphi_I \wedge \overline{\varphi}_J, \sum_{K,L} \eta_{K,L}\, \varphi_K \wedge \overline{\varphi}_L)\, C_n\, \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi_1} \wedge \cdots \wedge \overline{\varphi_n} = \varphi_I \wedge \overline{\varphi}_J \wedge \sum_{\substack{|M|=n-p \\ |N|=n-q}} a_{M,N}\, \varphi_M \wedge \overline{\varphi}_N,$$
where $|I| = |K| = p$ and $|J| = |L| = q$. The left hand side is equal to
$$2^{p+q}\, \overline{\eta}_{I,J}\, C_n\, \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi_1} \wedge \cdots \wedge \overline{\varphi_n}$$
and the right hand side is equal to
$$\sum_{\substack{|M|=n-p \\ |N|=n-q}} a_{M,N}\, \varphi_I \wedge \overline{\varphi}_J \wedge \varphi_M \wedge \overline{\varphi}_N = a_{I^0,J^0}\, \varphi_I \wedge \overline{\varphi}_J \wedge \varphi_{I^0} \wedge \overline{\varphi}_{J^0},$$
where $I^0 = \{1, \ldots, n\} - I$ and $J^0 = \{1, \ldots, n\} - J$. The right hand side has $\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi_1} \wedge \cdots \wedge \overline{\varphi_n}$ in scrambled order. Consider the permutation
$$(1, 2, \ldots, n; \widetilde{1}, \widetilde{2}, \ldots, \widetilde{n}) \mapsto (i_1, \ldots, i_p, \widetilde{j}_1, \ldots, \widetilde{j}_q, i^0_1, \ldots, i^0_{n-p}, \widetilde{j}^0_1, \ldots, \widetilde{j}^0_{n-q}).$$
If we write $\mathrm{sgn}_{I,J}$ for the sign of this permutation, we get
$$a_{I^0,J^0} = 2^{p+q-n}\, i^n (-1)^{\binom{n}{2}}\, \overline{\eta}_{I,J}\, \mathrm{sgn}_{I,J}.$$
Therefore,
$$*\eta = *\sum_{K,L} \eta_{K,L}\, \varphi_K \wedge \overline{\varphi}_L = 2^{p+q-n}\, i^n (-1)^{\binom{n}{2}} \sum_{\substack{|K^0|=n-p \\ |L^0|=n-q}} \mathrm{sgn}_{K,L}\, \overline{\eta}_{K,L}\, \varphi_{K^0} \wedge \overline{\varphi}_{L^0}.$$
Now, set
$$\overline{\partial}^* = -*\circ \overline{\partial} \circ *,$$
where $\overline{\partial}^*\colon \bigwedge^{p,q} X \xrightarrow{*} \bigwedge^{n-p,n-q} X \xrightarrow{\overline{\partial}} \bigwedge^{n-p,n-q+1} X \xrightarrow{*} \bigwedge^{p,q-1} X$.

I claim that $-*\circ \overline{\partial} \circ *$ is the formal adjoint, $\overline{\partial}^*$, we seek. Consider
$$(\overline{\partial}\xi, \eta) = \int_X (\overline{\partial}\xi, \eta)_z \Phi(z) = \int_X \overline{\partial}\xi \wedge *\eta,$$

where $\xi \in \bigwedge^{p,q-1}(X)$ and $\eta \in \bigwedge^{p,q}(X)$. Now, $\overline{\partial}(\xi \wedge *\eta) = \overline{\partial}\xi \wedge *\eta + (-1)^{p+q}\xi \wedge \overline{\partial}(*\eta)$, so we get

$$\int_X \overline{\partial}(\xi \wedge *\eta) = (\overline{\partial}\xi, \eta) + (-1)^{p+q} \int_X \xi \wedge \overline{\partial}(*\eta).$$

Also, $\xi \wedge *\eta \in \bigwedge^{p,q-1}(X) \wedge \bigwedge^{n-p,n-q}(X)$, i.e., $\xi \wedge *\eta \in \bigwedge^{n,n-1}(X)$. But, $d = \partial + \overline{\partial}$, so

$$d(\xi \wedge *\eta) = \partial(\xi \wedge *\eta) + \overline{\partial}(\xi \wedge *\eta) = \overline{\partial}(\xi \wedge *\eta),$$

and we deduce that

$$\int_X \overline{\partial}(\xi \wedge *\eta) = \int_X d(\xi \wedge *\eta) = \int_{\partial X} \xi \wedge *\eta = 0,$$

if either X is compact (in which case $\partial X = \emptyset$), or the forms have compact support (and hence, vanish on ∂X). So, we have

$$(\overline{\partial}\xi, \eta) = (-1)^{p+q} \int_X \xi \wedge \overline{\partial}(*\eta).$$

Check (DX): For $\eta \in \bigwedge^{p,q}(X)$, we have

$$** \, \eta = (-1)^{p+q}\eta.$$

As $*\eta \in \bigwedge^{n-p,n-q}(X)$, we have $\overline{\partial}(*\eta) \in \bigwedge^{n-p,n-q+1}(X)$, and so,
$** \, \overline{\partial}(*\eta) = (-1)^{2n-p-q+1}\overline{\partial}(*\eta) = (-1)^{p+q-1}\overline{\partial}(*\eta)$. We conclude that

$$\begin{aligned} (\overline{\partial}\xi, \eta) &= -\int_X \xi \wedge ** \, \overline{\partial}(*\eta) \\ &= \int_X \xi \wedge *(- * \overline{\partial} * (\eta)) \\ &= (\xi, - * \overline{\partial} * (\eta)). \end{aligned}$$

Therefore, $\overline{\partial}^* = - * \overline{\partial} \, *$, as contended.

Now, we define the *Hodge Laplacian*, or *Laplace-Beltrami operator*, $\square$, by:

$$\square = \overline{\partial}^* \, \overline{\partial} + \overline{\partial} \, \overline{\partial}^* \colon \bigwedge^{p,q}(X) \longrightarrow \bigwedge^{p,q}(X).$$

You check (DX) that $\square$ is formally self-adjoint.

Claim: $\square(\varphi) = 0$ iff both $\overline{\partial}\varphi = 0$ and $\overline{\partial}^*\varphi = 0$.

First, assume $\square(\varphi) = 0$ and compute $(\varphi, \square(\varphi))$. We get

$$\begin{aligned} (\varphi, \square(\varphi)) = &= (\varphi, \overline{\partial}^* \, \overline{\partial}\varphi) + (\varphi, \overline{\partial} \, \overline{\partial}^* \varphi) \\ &= \overline{(\overline{\partial}^* \, \overline{\partial}\varphi, \varphi)} + (\overline{\partial}^* \varphi, \overline{\partial}^* \varphi) \\ &= \overline{(\overline{\partial}\varphi, (\overline{\partial}\varphi)} + \|\overline{\partial}^* \varphi\|^2 \\ &= \|\overline{\partial}\varphi\|^2 + \|\overline{\partial}^* \varphi\|^2. \end{aligned}$$

Therefore, if $\square(\varphi) = 0$, then $\overline{\partial}\varphi = 0$ and $\overline{\partial}^*\varphi = 0$. The converse is obvious by definition of $\square(\varphi)$.

Consequently, our minimality is equivalent to $\square(\varphi) = 0$, where $\square$ is a second-order differential operator.

To understand better what the operator $\square$ does, consider the special case where $X = \mathbb{C}^n$ (use compactly supported "gadgets"), with the standard inner product, and $\bigwedge^{0,0}(X) = \mathcal{C}_0^\infty$. Pick $f \in \mathcal{C}_0^\infty$, then again, $\square(f) \in \mathcal{C}_0^\infty$ and on those f, we have $\overline{\partial}^* f = 0$. Consequently,

$$\square(f) = \overline{\partial}^* \, \overline{\partial} f = \overline{\partial}^* \left(\sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} \, d\overline{z}_j \right).$$

We also have

$$
\begin{aligned}
*\left(\sum_{j=1}^{n}\frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j\right) &= 2^{1-n}i^n(-1)^{\binom{n}{2}}\sum_{j=1}^{n}\overline{\left(\frac{\partial f}{\partial \overline{z}_j}\right)}\, dz_1\wedge\cdots\wedge dz_n\wedge d\overline{z}_{\{j\}^0}\,\mathrm{sgn}_{\emptyset,\{j\}}\\
&= 2^{1-n}i^n(-1)^{\binom{n}{2}}\sum_{j=1}^{n}\frac{\partial \overline{f}}{\partial z_j}\,\mathrm{sgn}_{\emptyset,\{j\}}\, dz_1\wedge\cdots\wedge dz_n\wedge d\overline{z}_{\{j\}^0}.
\end{aligned}
$$

Taking $\overline{\partial}$ of the above expression, we get

$$
\begin{aligned}
&2^{1-n}i^n(-1)^{\binom{n}{2}}\sum_{k,j=1}^{n}\frac{\partial^2 \overline{f}}{\partial \overline{z}_k\partial z_j}\,\mathrm{sgn}_{\emptyset,\{j\}}\, d\overline{z}_k\wedge d\overline{z}_j\wedge dz_1\wedge\cdots\wedge dz_n\wedge d\overline{z}_{\{j\}^0}\\
&\qquad = 2^{1-n}i^n(-1)^{\binom{n}{2}}(-1)^n\sum_{j=1}^{n}\frac{\partial^2 \overline{f}}{\partial z_j\partial \overline{z}_j}\,\mathrm{sgn}_{\emptyset,\{j\}}\, dz_1\wedge\cdots\wedge dz_n\wedge d\overline{z}_j\wedge d\overline{z}_{\{j\}^0}.
\end{aligned}
$$

Taking $-*$ of the above, we get

$$
-2i^{2n}(-1)^{\binom{n}{2}}(-1)^n\sum_{j=1}^{n}\frac{\partial^2 f}{\partial z_j\partial \overline{z}_j} = -2\sum_{j=1}^{n}\frac{\partial^2 f}{\partial z_j\partial \overline{z}_j}.
$$

But,

$$
\frac{4\partial^2 f}{\partial z_j\partial \overline{z}_j} = \frac{\partial^2 f}{\partial x_j^2} + \frac{\partial^2 f}{\partial y_j^2},
$$

and this implies that on $\bigwedge^{0,0}(X)$, $\square(f)$ up to a constant $(-1/2)$ is just the usual Laplacian.

Write $\mathcal{H}^{p,q}(X)$ for the kernel of $\square$ on $\bigwedge^{p,q}(X)$, the space of *harmonic forms*. Here is Hodge's theorem.

Theorem 2.17 *(Hodge, (1941)) Let X be a complex manifold and assume that X is compact. Then,*

(1) The space $\mathcal{H}^{p,q}(X)$ is finite-dimensional.

(2) There exist a projection, $\mathcal{H}\colon \bigwedge^{p,q}(X)\to\mathcal{H}^{p,q}(X)$, so that we have the **orthogonal decomposition** *(Hodge decomposition)*

$$
\bigwedge^{p,q}(X) = \mathcal{H}^{p,q}(X)\overset{\perp}{\coprod}\overline{\partial}\bigwedge^{p,q-1}(X)\overset{\perp}{\coprod}\overline{\partial}^*\bigwedge^{p,q+1}(X).
$$

(3) There exists a **parametrix** *(= pseudo-inverse), G, (Green's operator) for $\square$, and it is is uniquely determined by*

(a) $\mathrm{id} = \mathcal{H} + \square G = \mathcal{H} + G\square$, and

(b) $G\overline{\partial} = \overline{\partial}G$, $G\overline{\partial}^* = \overline{\partial}^*G$ and $G\restriction\mathcal{H}^{p,q}(X) = 0$.

Remarks: (1) If a decomposition "à la Hodge" exists, it must be an orthogonal decomposition. Say $\xi\in\overline{\partial}\bigwedge^{p,q-1}(X)$ and $\eta\in\overline{\partial}^*\bigwedge^{p,q+1}(X)$, then

$$
(\xi,\eta) = (\overline{\partial}\xi_0, \overline{\partial}^*\eta_0) = (\overline{\partial}\,\overline{\partial}\xi_0,\eta_0) = 0,
$$

and so, $\overline{\partial}\bigwedge^{p,q-1}(X)\perp\overline{\partial}^*\bigwedge^{p,q+1}(X)$. Observe that we can write the Hodge decomposition as

$$
\bigwedge^{p,q}(X) = \mathcal{H}^{p,q}(X)\overset{\perp}{\coprod}\square\bigwedge^{p,q}(X).
$$

For, if $\xi \in \Box\bigwedge^{p,q}(X)$, then $\xi = \overline{\partial}(\overline{\partial}^*\xi_0) + \overline{\partial}^*(\overline{\partial}\xi_0)$, and this implies

$$\Box\overset{p,q}{\bigwedge}(X) \subseteq \overline{\partial}\overset{p,q-1}{\bigwedge}(X) + \overline{\partial}^*\overset{p,q+1}{\bigwedge}(X).$$

However, the right hand side is an orthogonal decomposition and it follows that

$$\mathcal{H}^{p,q}(X) + \Box\overset{p,q}{\bigwedge}(X) = \mathcal{H}^{p,q}(X) + \overline{\partial}\overset{p,q-1}{\bigwedge}(X) \overset{\perp}{\coprod} \overline{\partial}^*\overset{p,q+1}{\bigwedge}(X) = \overset{p,q}{\bigwedge}(X).$$

For perpendicularity, as $\Box$ is self-adjoint, for $\xi \in \mathcal{H}^{p,q}(X)$, we have

$$(\xi, \Box(\eta)) = (\Box(\xi), \eta) = 0,$$

since $\Box(\xi) = 0$.

(2) We can give a n.a.s.c. that $\Box(\xi) = \eta$ has a solution, given η. Namely, by (3a),

$$\eta = \mathcal{H}(\eta) + \Box(G(\eta)).$$

If $\mathcal{H}(\eta) = 0$, then $\eta = \Box(G(\eta))$ and we can take $\xi = G(\eta)$. Conversely, orhogonality implies that if $\eta = \Box(\xi)$, then $\mathcal{H}(\eta) = 0$. Therefore, $\mathcal{H}(\eta)$ is the obstruction to solving $\Box(\xi) = \eta$.

How many solutions does $\Box(\xi) = \eta$ have?

The solutions of $\Box(\xi) = \eta$ are in one-to-one correspondence with $\xi_0 + \mathcal{H}^{p,q}(X)$, where ξ_0 is a solution and if we take $\xi_0 \in \mathrm{Ker}\ \mathcal{H}$, then ξ_0 is unique, given by $G(\eta)$.

(3) Previous arguments, once made correct, give us the isomorphisms

$$\mathcal{H}^{p,q}(X) \cong H^{p,q}_{\overline{\partial}} \cong H^q(X, \Omega^p_X).$$

Therefore, $H^q(X, \Omega^p_X)$ is a finite-dimensional vector space, for X a compact, complex manifold.

For the proof of Hodge's theorem, we need some of the theory of distributions. At first, restrict to $\mathcal{C}^\infty_0(U)$ (smooth functions of compact support) on some open, $U \subseteq \mathbb{C}^n$. One wants to understand the dual space, $(\mathcal{C}^\infty_0(U))^D$. Consider $g \in L^2(U)$, then for any $\varphi \in \mathcal{C}^\infty_0(U)$, we set

$$\lambda_g(\varphi) = \int_U \varphi\overline{g}d\mu.$$

(Here, μ is the Lebesgue measure on $\mathbb{C}^n$.) So, we have $\lambda_g \in \mathcal{C}^\infty_0(U)^D$. Say $\lambda_g(\varphi) = 0$, for all φ. Take E, a measurable subset of U of finite measure with $\overline{E}$ compact. Then, as χ_E is L^2, the function χ_E is L^2-approximable by $\mathcal{C}^\infty_0(U)$-functions. So, there is some $\varphi \in \mathcal{C}^\infty_0(U)$ so that

$$\|\varphi - \chi_E\|_2 < \epsilon.$$

As $\chi_E = \chi_E - \varphi + \varphi$, we get

$$\int_E \overline{g}d\mu = \int_U \chi_E\overline{g}d\mu = \int_U (\chi_E - \varphi)\overline{g}d\mu + \int_U \varphi\overline{g}d\mu = \int_U (\chi_E - \varphi)\overline{g}d\mu$$

(by hypothesis, $\lambda_g(\varphi) = 0$). Therefore,

$$\left|\int_E \overline{g}d\mu\right| \leq \|\chi_E - \varphi\|_2 \|\overline{g}\|_2 < \|g\|_2\, \epsilon,$$

which implies that $g \equiv 0$ almost everywhere. It follows that $L^2(U) \hookrightarrow (\mathcal{C}_0^\infty(U))^D$. The same argument applies for $g \in \mathcal{C}(U)$ and uniform approximations by $\mathcal{C}_0^\infty$-functions, showing that $\mathcal{C}(U) \hookrightarrow (\mathcal{C}_0^\infty(U))^D$.

Notation. Set

$$D_j = \frac{1}{i}\frac{\partial}{\partial X_j} = -i\frac{\partial}{\partial X_j},$$

where $X_1, \ldots, X_n$ are real coordinates in $\mathbb{C}^n$, and if $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_j \in \mathbb{Z}$ and $\alpha_j \geq 0$, set $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ and $|\alpha| = \sum_{j=1}^n \alpha_j$. Also, for any n-tuple $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, we let $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ and $|\xi|^\alpha = |\xi_1|^{\alpha_1} \cdots |\xi_n|^{\alpha_n}$. The reason for the factor $1/i$ is this: Say v is a function and look at $D_j(\overline{v}) = -i\partial\overline{v}/\partial X_j$. But,

$$\overline{D_j(v)} = \overline{-i\frac{\partial v}{\partial X_j}} = i\frac{\partial \overline{v}}{\partial X_j} = -D_j(\overline{v}).$$

Therefore,

$$D_j(u\overline{v}) = (D_j u)\overline{v} + uD_j\overline{v} = (D_j u)\overline{v} - u\overline{D_j v}.$$

Consider $u, v \in \mathcal{C}_0^\infty(U)$; then,

$$(D_j u, v) = \int_U (D_j u)\overline{v} = \int_U D_j(u\overline{v}) + \int_U u\,\overline{D_j(v)}.$$

The first term on the right hand side is zero as u and v have compact support, so we get

$$(D_j u, v) = \int_U u\,\overline{D_j(v)} = (u, D_j v),$$

which says that the D_j's are formally self-adjoint. Repeated application of the above gives

$$(D^\alpha u, v) = (u, D^\alpha v)$$

and also

$$\int_U (D^\alpha u)\overline{v} = \int_U u\overline{(D^\alpha v)}.$$

Definition 2.4 Let $\widetilde{\mathcal{D}}(U) = \mathcal{C}_0^\infty(U)^{\mathrm{alg}\,D}$ be the set of (complex-valued) linear functionals on $\mathcal{C}_0^\infty(U)$. Now define, $\mathcal{D}(U)$, the *space of distributions* on U, so that $\lambda \in \mathcal{D}(U)$ iff $\lambda \in \mathcal{C}_0^\infty(U)^{\mathrm{alg}\,D}$ and λ is "continuous", i.e., there is some $k \geq 0$ and some C_λ, so that for all $\varphi \in \mathcal{C}_0^\infty(U)$,

$$|\lambda(\varphi)| \leq C_\lambda \max_{|\alpha| \leq k} \|D^\alpha \varphi\|_\infty. \tag{$*$}$$

As an example of a distribution, if $g \in \mathcal{C}_0(U)$, so g is bounded (all we need is boundedness and intergrability), then

$$\lambda_g(\varphi) = \int_U \varphi\overline{g}d\mu.$$

Then, we have

$$|\lambda_g(\varphi)| \leq \|\varphi\|_\infty \; \|g\|_1\,,$$

so we can take $C_{\lambda_g} = \|g\|_1$ and we get a distribution. The intuition in $(*)$ is that the bigger k is, the "worse" λ is as a distribution (k indicates how many derivatives we need to control).

We can differentiate distributions: Take $g \in \mathcal{C}^1$, we have

$$\lambda_g(\varphi) = \int_U \varphi\overline{g}d\mu$$

and so,

$$\lambda_{D_j g}(\varphi) = \int_U \varphi\overline{D_j g}d\mu = \int_U (D_j\varphi)\overline{g}d\mu = \lambda_g(D_j\varphi).$$

This gives the reason behind the

Definition 2.5 If $\lambda \in \mathcal{D}(U)$, let $D^\alpha\lambda \in \widetilde{\mathcal{D}}(U)$, defined by

$$(D^\alpha\lambda)(\varphi) = \lambda(D^\alpha\varphi).$$

Claim: If $\lambda \in \mathcal{D}(U)$, then $D^\alpha\lambda \in \mathcal{D}(U)$.

Indeed, we have

$$|(D^\alpha\lambda)(\varphi)| = |\lambda(D^\alpha\varphi)| \leq C_\lambda \max_{|\beta|\leq k} \|D^{\alpha+\beta}(\varphi)\|_\infty \leq C_\lambda \max_{|\gamma|\leq k+|\alpha|} \|D^\gamma(\varphi)\|_\infty.$$

Therefore, $D^\alpha\lambda$ is again a distribution. Given a multi-index, α, write

$$\sigma(\alpha) = |\alpha| + \left\lceil \frac{n}{2} \right\rceil + 1.$$

This is the *Sobolev number of* α ($n =$ dimension of the underlying space). Now, we can define the Sobolev norm and the Sobolev spaces, H_s ($s \in \mathbb{Z}, s \geq 0$). If $\varphi \in \mathcal{C}_0^\infty(U)$, set

$$\|\varphi\|_s^2 = \sum_{|\alpha|\leq s} \|D^\alpha\varphi\|_{L^2}^2.$$

This is the *Sobolev s-norm*. It comes from an inner product

$$(\varphi, \psi)_s = \sum_{|\alpha|\leq s} (D^\alpha\varphi, D^\alpha\psi).$$

If we complete $\mathcal{C}_0^\infty(U)$ in this norm, we get a Hilbert space, the *Sobolev space*, H_s.

Say $s > r$, then for all $\varphi \in \mathcal{C}_0^\infty(U)$, we have

$$\|\varphi\|_r^2 \leq \|\varphi\|_s^2.$$

Hence, if $\{\varphi_i\}$ is a Cauchy sequence in the s-norm, it is also a Cauchy sequence in the r-norm and we get a continuous embedding

$$H_s \subseteq H_r \quad \text{if} \quad s > r.$$

Let $H_\infty = \bigcap_{s\geq 0} H_s$.

Theorem 2.18 *(Sobolev Inequality and Embedding Theorem) For all $\varphi \in \mathcal{C}_0^\infty(U)$, for all α, we have*

$$\|D^\alpha\varphi\|_\infty \leq K_\alpha \|\varphi\|_{\sigma(\alpha)} \quad \textit{and} \quad H_s(U) \subseteq \mathcal{C}^m(\overline{U}),$$

provided U has finite measure, $m \geq 0$ and $\sigma(m) \leq s$. Furthermore, $H_s(U) \subseteq L^{\frac{2n}{n-2}}(U)$ if $n > 2s$.

(We have $\sigma(m) \leq s$ iff $m < s - \lceil \frac{n}{2} \rceil$.)

Theorem 2.19 *(Rellich Lemma) The continuous embedding, $\rho_s^r\colon H_s \hookrightarrow H_r$, (for $s > r$) is a compact operator. That is, for any bounded set, B, the image $\rho_s^r(B)$ has a compact closure. Alternatively, if $\{\varphi_j\}$ is a bounded sequence in H_s, then $\{\rho_s^r(\varphi_j)\}$ possesses a converging subsequence in H_r.*

To connect with distributions, we use the Fourier Transform. If $\varphi \in \mathcal{C}_0(U)$, we set

$$\widehat{\varphi}(\theta) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{C}^n} \varphi(x) e^{-i\overline{(x,\theta)}}\, dx,$$

where $(x,\theta) = \sum_{j=1}^n x_j\overline{\theta}_j$. (Recall that over $\mathbb{R}$, we are in $\mathbb{R}^{2n}$.) The purpose of the fudge factor in front of the integral is to insure that Fourier transform of the Gaussian

$$\varphi(x) = e^{-\frac{\|x\|^2}{2}}$$

is itself. As

$$\int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{2}}\, dx = \left(\sqrt{2\pi}\right)^n,$$

we determine that the "fudge factor" is $(2\pi)^{-n}$. It is also interesting to see what $\widehat{D_j\varphi}(\theta)$ is. We have

$$\widehat{D_j\varphi}(\theta) = \int_{\mathbb{C}^n} (D_j\varphi)(x)e^{-i\overline{(x,\theta)}}\, dx = \int_{\mathbb{C}^n} \varphi(x)\overline{D_j e^{i(x,\theta)}}\, dx.$$

Now,

$$\frac{\partial}{\partial x_j}\, e^{i\sum x_k\overline{\theta}_k} = i\overline{\theta}_k e^{i\sum x_k\overline{\theta}_k}$$

and

$$D_j e^{i\sum x_k\overline{\theta}_k} = -i\frac{\partial}{\partial x_j}\, e^{i\sum x_k\overline{\theta}_k} = \overline{\theta}_k e^{i\sum x_k\overline{\theta}_k}.$$

It follows that

$$\widehat{D_j\varphi}(\theta) = \theta_j\widehat{\varphi}(\theta),$$

that is, D_j turns into multiplication by θ_j by the Fourier transform. We also get

Theorem 2.20 *(Plancherel) If $\varphi\in\mathcal{C}_0^\infty$, then*

$$\|\varphi\|_{L^2} = \|\widehat{\varphi}\|_{L^2}.$$

As a consequence, we can compute the Sobolev norm using the Fourier transform:

$$\|\widehat{\varphi}\|_s^2 = \sum_{|\alpha|\leq s} \|\widehat{D^\alpha\varphi}\|_{L^2}^2$$

and

$$\begin{aligned}
\sum_{|\alpha|\leq s} \|\widehat{D^\alpha\varphi}\|_{L^2}^2 &= \sum_{|\alpha|\leq s}\int_{\mathbb{C}^n} \theta^\alpha\widehat{\varphi}(\theta)\,\overline{\theta^\alpha}\,\overline{\widehat{\varphi}(\theta)}\, d\theta \\
&= \int_{\mathbb{C}^n}\sum_{|\alpha|\leq s} |\theta|^{2\alpha}\,|\widehat{\varphi}(\theta)|^2\, d\theta \\
&\leq \int_{\mathbb{C}^n} (1+|\theta|^2)^s\,|\widehat{\varphi}(\theta)|^2\, d\theta \\
&\leq \text{Const}\int_{\mathbb{C}^n} |\theta|^{2\alpha}\,|\widehat{\varphi}(\theta)|^2\, d\theta = \text{Const}\,\|\varphi\|_s^2.
\end{aligned}$$

(Using Plancherel in the last step.) Therefore, the norm

$$\|\widehat{\varphi}\|_s^2 = \int_{\mathbb{C}^n} (1+|\theta|^2)^s\,|\widehat{\varphi}(\theta)|^2\, d\theta$$

satisfies

$$\|\varphi\|_s^2 \leq \|\widehat{\varphi}\|_s^2 \leq \text{Const}\ \|\varphi\|_s^2,$$

that is, these norms are equivalent and we can measure φ by the Sobolev norm on the Fourier transform.

Observe that we can define H_{-s} $(s > 0)$ *via* the completion of $\mathcal{C}_0^\infty$ in the norm $\int_{\mathbb{C}^n} (1+|\theta|^2)^{-s} |\widehat{\varphi}(\theta)|^2 \, d\theta$. Clearly, we get the chain of inclusions

$$\cdots \supseteq H_{-n} \supseteq H_{-n+1} \supseteq \cdots \supseteq H_{-1} \supseteq H_0 = L^2 \supseteq H_1 \cdots \supseteq H_{n-1} \supseteq H_n \supseteq \cdots H_\infty.$$

This suggests defining $H_{-\infty}$ by

$$H_{-\infty} = \bigcup_{n\in\mathbb{Z}} H_n.$$

The Sobolev embedding lemma implies $H_\infty \subseteq \mathcal{C}^\infty(\overline{U})$ and $\mathcal{C}_0^\infty(U) \subseteq H_\infty$. Now, H_{-s} defines linear functionals on H_s; say $\psi \in H_{-s}$ and $\varphi \in H_s$. Consider

$$\psi(\varphi) := \int (\varphi\overline{\psi})(\theta)\, d\theta = \int \sqrt{(1+|\theta|^2)^s}\varphi \, \frac{1}{\sqrt{(1+|\theta|^2)^s}} \, \overline{\psi} \, d\theta.$$

By Cauchy-Schwarz,

$$|\psi(\varphi)| = |(\varphi, \psi)| = \int (\varphi\overline{\psi})(\theta)\, d\theta \leq \|\varphi\|_s \, \|\psi\|_{-s}\,.$$

Therefore, we have a map $H_{-s} \mapsto H_s^D$ and it follows that $H_{-s} \cong H_s^D$, up to conjugation.

Remark: If $\varphi \in \mathcal{C}_0^\infty(U)$ and $\lambda \in \mathcal{D}(U)$, then

$$|\lambda(\varphi)| \leq C_\lambda \max_{|\alpha|\leq k} \|D^\alpha \varphi\|_\infty, \quad \text{for some } k.$$

By Sobolev's inequality,

$$|\lambda(\varphi)| \leq C_\lambda K_\alpha \|\varphi\|_{\sigma(\alpha)},$$

for some suitable α so that $|\alpha| \leq k$. Thus, if $\lambda \in \mathcal{D}(U)$, then there exist some α such that λ is a continuous functional on $\mathcal{C}_0^\infty(U)$ in the $\sigma(\alpha)$-norm. But then, λ extends to an element of $H_{\sigma(\alpha)}^D$ (by completion) and we conclude that $\mathcal{D}(U) = H_{-\infty}$.

Proof of Theorem 2.19 (Rellich Lemma). Given a bounded sequence, $\{\varphi_k\}_{k=1}^\infty$, there is some $C > 0$ so that, for every k,

$$\int_{\mathbb{R}^n} (1+|\theta|^2)^s |\widehat{\varphi}_k(\theta)|^2 d\theta \leq C.$$

Thus, for every θ, the sequence of $(1+|\theta|^2)^s|\widehat{\varphi}_k(\theta)|^2$ is a bounded sequence of complex numbers. Therefore, for every θ, we have a Cauchy subsequence in $\mathbb{C}$. As there exists a countable dense subset of θ's in $\mathbb{R}^n$, the $\aleph_0$-diagonalization procedure yields a subsequence of the φ_k's so that *this* subsequence is Cauchy at every θ (i.e., $(1+|\theta|^2)^s|\widehat{\varphi}_k(\theta)|^2$ is Cauchy at every θ) and, of course, we replace the φ_k's by this subsequence. Now, pick $\epsilon > 0$, and write U_0 for the set of all θ's such that

$$\frac{1}{(1+|\theta|^2)^{s-r}} \geq \epsilon.$$

Look at

$$\begin{aligned}
\|\varphi_k - \varphi_l\|_r^2 &= \int_{\mathbb{R}^n} (1+|\theta|^2)^r |(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 d\theta \\
&= \int_{U_0} (1+|\theta|^2)^r |(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 d\theta + \int_{\mathbb{R}^n - U_0} (1+|\theta|^2)^r |(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 d\theta.
\end{aligned}$$

But, as $\{(1+|\theta|^2)^s|\widehat{\varphi}_k(\theta)|^2\}$ is Cauchy, there is some large N so that for all $k, l \geq N$,

$$(1+|\theta|^2)^r|(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 \leq (1+|\theta|^2)^s|(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 < \epsilon/\mu(U_0)$$

for all θ. Then, the first integral is at most ϵ. In the second integral,

$$(1+|\theta|^2)^r|(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 = \frac{(1+|\theta|^2)^s}{(1+|\theta|^2)^{s-r}}\,|(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 < \epsilon\,\text{numerator}.$$

But then,

$$\int_{\mathbb{R}^n - U_0}(1+|\theta|^2)^r|(\widehat{\varphi}_k - \widehat{\varphi}_l)(\theta)|^2 d\theta < \epsilon\int_{\mathbb{R}^n}\text{numerator} < C\epsilon.$$

Therefore, $\{\varphi_k\}$ is Cauchy in H_r, and since H_r is complete, the sequence $\{\varphi_k\}$ converges in H_r. $\square$

Proof of Theorem 2.18 (Sobolev's Theorem). Pick $\varphi \in \mathcal{C}_0^\infty(U)$ and take $s = 1$. Then, for every j, as

$$|\varphi(x)| \leq \int_\infty^\infty |D_j\varphi(x)|dx_j,$$

we get

$$|\varphi(x)|^n \leq \prod_{j=1}^n \left(\int_\infty^{-\infty} |D_j\varphi(x)|dx_j\right).$$

Thus, we have

$$|\varphi(x)|^{n/(n-1)} \leq \prod_{j=1}^n \left(\int_{-\infty}^{\infty} |D_j\varphi(x)|dx_j\right)^{1/(n-1)}. \tag{$*$}$$

We will use the generalized Hölder inequality: If

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1,$$

and if $\varphi_j \in L^{p_j}$, for $j = 1, \ldots, m$, then $\varphi_1 \cdots \varphi_m \in L^1$ and

$$\|\varphi_1 \cdots \varphi_m\|_{L^1} \leq \|\varphi_1\|_{L^{p_1}} \cdots \|\varphi_m\|_{L^{p_m}}.$$

Assume that $n \geq 2$ and set $p_j = n-1$, for $1 \leq j \leq n-1$. Integrate $(*)$ w.r.t. $x_1, x_2, \ldots, x_n$, but in between integration, use the Hölder inequality:

$$\begin{aligned}
\int_{-\infty}^{\infty} |\varphi(x)|^{n/(n-1)}dx_1 &\leq \int_{-\infty}^{\infty}\left(\left[\int_{-\infty}^{\infty}|D_1(\varphi)|dx_1\right]^{1/(n-1)}\prod_{j=2}^n\left[\int_{-\infty}^{\infty}|D_j(\varphi)|dx_j\right]^{1/(n-1)}\right)dx_1 \\
&\leq \left[\int_{-\infty}^{\infty}|D_1(\varphi)|dx_1\right]^{1/(n-1)}\left[\prod_{j=2}^n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|D_j(\varphi)|dx_j dx_1\right]^{1/(n-1)}.
\end{aligned}$$

If we repeat this procedure, we get

$$\int_U |\varphi(x)|^{n/(n-1)}dx \leq \left[\prod_{j=1}^n \int_U |D_j(\varphi)|dx\right]^{1/(n-1)}.$$

Raising the above to the power $(n-1)/n$, we get

$$\|\varphi\|_{L^{n/(n-1)}} \leq \left(\prod_{j=1}^{n}\int_U |D_j(\varphi)|dx\right)^{1/n} \leq \frac{1}{n}\left(\sum_{j=1}^{n}\int_U |D_j(\varphi)|dx\right),$$

by the arithmetic-geometric mean inequality. Apply this to φ^γ, for some approrpiate choice of γ. For the rest of this argument, we need $n > 2$ and we choose γ to satisfy

$$\gamma\left(\frac{n}{n-1}\right) = 2(\gamma - 1).$$

We deduce that

$$\gamma = \frac{2(n-1)}{n-2} > 0,$$

as $n > 2$. We plug φ^γ in the above and we get

$$\begin{aligned}
\|\varphi^\gamma\|_{L^{\frac{n}{n-1}}} &\leq \frac{1}{n}\sum_{j=1}^{n}\int_U |D_j(\varphi^\gamma)|\,dx \\
&= \frac{\gamma}{n}\sum_{j=1}^{n}\int_U |\varphi^{\gamma-1}|\,|D_j(\varphi)|\,dx \\
&= \frac{\gamma}{n}\sum_{j=1}^{n}\left\|\varphi^{\gamma-1}\right\|_{L^2}\|D_j(\varphi)\|_{L^2},
\end{aligned}$$

by Cauchy-Schwarz. The left hand side is equal to

$$\left(\int_U |\varphi^{\frac{\gamma n}{n-1}}|dx\right)^{\frac{n-1}{n}} = \left(\int_U |\varphi^{2(\gamma-1)}|dx\right)^{\frac{n-1}{n}}.$$

On the right hand side, the term $\|\varphi^{\gamma-1}\|_{L^2}$ is common to the summands, so pull it out. This factor is

$$\left(\int_U |\varphi^{2(\gamma-1)}|dx\right)^{\frac{1}{2}}.$$

When we divide both sides by this factor, we get

$$\left(\int_U |\varphi^{2(\gamma-1)}|\,dx\right)^{\frac{n-1}{n}-\frac{1}{2}} \leq \frac{\gamma}{n}\sum_{j=1}^{n}\|D_j(\varphi)\|_{L^2}.$$

But,

$$2(\gamma-1) = \frac{\gamma n}{n-1} = \frac{2n}{n-2}$$

and

$$\frac{n-1}{n} - \frac{1}{2} = \frac{n-2}{2n}.$$

We obtain

$$\left(\int_U \left|\varphi^{\frac{2n}{n-2)}}\right|\,dx\right)^{\frac{n-2}{2n}} \leq \frac{2(n-1)}{n(n-2)}\sum_{j=1}^{n}\|D_j(\varphi)\|_{L^2}.$$

Therefore, we get the Sobolev inequality for the case $s = 1$ and $n > 2$: For every $\varphi \in \mathcal{C}_0^\infty(U)$, we have

$$\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq K(n)\,\|\varphi\|_1, \tag{$*$}$$

where $K(n) = \frac{2(n-1)}{n(n-2)}$.

Now, say $\psi \in H_1$, then there is a sequence, $\{\varphi_q\}$, converging to ψ in the $\| \, \|_1$-norm, with $\varphi_q \in \mathcal{C}_0^\infty(U)$. Consequently, this is a Cauchy sequence in the $\| \, \|_1$-norm and so,

$$\|\varphi_q - \varphi_r\|_1 < \epsilon \quad \text{for all } q, r \text{ sufficiently large}$$

which implies that

$$\|\varphi_q - \varphi_r\|_{L^{\frac{2n}{n-2}}} < \epsilon \quad \text{for all } q, r \text{ sufficiently large.}$$

Therefore, the φ_q converge to a limit, $\psi_0 \in L^{\frac{2n}{n-2}}$.

(a) The map $\psi \mapsto \psi_0$ does not depend on the choice of the Cauchy sequence.

(b) This map is an injection.

As a consequence, we get the Sobolev embedding when $s = 1$:

$$H_1 \hookrightarrow L^{\frac{2n}{n-2}}, \quad \text{if } n > 2.$$

If we pass to the limit in $(*)$, we get: For every $\psi \in H_1$,

$$\|\psi\|_{L^{\frac{2n}{n-2}}} \leq K(n) \|\psi\|_1 . \tag{$*$}$$

Now, we want the Sobolev inequality on $\|D^\alpha \varphi\|_\infty$ when $s = 1$. In this case, $\sigma(\alpha) \leq s$ implies $|\alpha| + 1 + \left\lceil \frac{n}{2} \right\rceil \leq 1$. Thus, $n = 1$ and $\alpha = 0$. Therefore, we have to prove

$$\|\varphi\|_\infty \leq K \|\varphi\|_1 .$$

In the present case, $U \subseteq \mathbb{R}$ and $\varphi \in \mathcal{C}_0^\infty(U)$. Then, we have

$$\varphi(x) = \int_{-\infty}^x \varphi'(t) dt,$$

so

$$|\varphi(x)| \leq \int_{-\infty}^x |\varphi'(t)| \, dt \leq \|1\|_{L^2} \|D\varphi\|_{L^2} \leq \sqrt{\mu(U)} \, \|\varphi\|_1 ,$$

where we used Cauchy Schwarz in the first inequality. If we take sup's, we get the following Sobolev inequality for the case $s = n = 1$:

$$\|\varphi\|_\infty \leq K \|\varphi\|_1 . \tag{$**$}$$

Next, consider the embedding property. Here, we have $0 \leq m \leq s - \left\lceil \frac{n}{2} \right\rceil$, so $m = 0$. Take $\psi \in H_1$ and, as before, approximate ψ by some sequence, $\{\varphi_q\}$, where $\varphi_q \in \mathcal{C}_0^\infty(U)$. Then, $(**)$ implies that

$$\|\varphi_q - \varphi_r\|_\infty \leq K \|\varphi_q - \varphi_r\|_1 .$$

As the right hand side is smaller than ϵ for all $q, r \geq N$ (for some large N), we deduce that the φ_q converge uniformly to some $\psi_0 \in \mathcal{C}^0(\overline{U})$. Then, again, the map $\psi \mapsto \psi_0$ is well-define and an embedding. Therefore,

$$H_1(U) \subseteq \mathcal{C}^0(\overline{U}),$$

which is the Sobolev embedding in the case $s = n = 1$.

To prove the general case, we use induction on s and iterate the argument. The induction hypothesis is

(a) If $n > 2s$, then for all $\varphi \in \mathcal{C}^\infty(U)$,

$$\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq K(n)\,\|\varphi\|_s\,. \tag{$*$}$$

(a′) There is an embedding, $H_s(U) \hookrightarrow L^{\frac{2n}{n-2}}$, so $(*)$ holds for all $\psi \in H_2$.

(b) If $0 \leq m \leq s - \left\lceil \frac{n}{2} \right\rceil$ ($\sigma(m) \leq s$), then

$$\|D^\alpha\varphi\|_\infty \leq K\,\|\varphi\|_{\sigma(\alpha)} \leq K\,\|\varphi\|_s\,. \tag{$**$}$$

(Here, $\sigma(\alpha) \leq s$.)

(b′) There is an embedding, $H_s(U) \hookrightarrow \mathcal{C}^m(\overline{U})$, i.e., $(**)$ holds for all $\psi \in H_s$.

(a) Actually, this part does not require induction. As the case $s = 1$ has been settled, we may assume $s > 1$ (and $n > 2s$). We need to show that for any $\varphi \in \mathcal{C}_0^\infty(U)$,

$$\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq \|\varphi\|_s.$$

We have

$$\|\varphi\|_1 \leq \|\varphi\|_s$$

and as $n > 2s > 2$, by the $s = 1$ case,

$$\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq \|\varphi\|_1.$$

We conclude immediately that

$$\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq \|\varphi\|_s.$$

Note that (a′) is a consequence of (a) in the same way as before.

(b) Assume $0 \leq m < s+1-\left\lceil \frac{n}{2} \right\rceil$, i.e., $m-1 < s - \left\lceil \frac{n}{2} \right\rceil$. Pick $\varphi \in \mathcal{C}_0^\infty(U)$ and look at $D_j\varphi$ and $\sigma(\beta) \leq s$. Observe that $m-1$ is such a $|\beta|$. By $(**)$,

$$\left\|D^\beta D_j\varphi\right\|_\infty \leq K\,\|D_j\varphi\|_{\sigma(\beta)}\,, \quad \text{for all } j.$$

But all $D^\alpha\varphi$ are of this form, for some β with $\sigma(\beta) \leq s$. Therefore,

$$\|D^\alpha\varphi\|_\infty \leq K\,\|D_j\varphi\|_{\sigma(\beta)} \leq K\,\|\varphi\|_{s+1}\,, \quad \text{by } (\dagger)$$

which is exactly $(**)$. By the induction hypothesis, each $D_j\varphi \in \mathcal{C}^{m-1}(\overline{U})$ and we conclude that $\varphi \in \mathcal{C}^m(\overline{U})$. $\square$

Notion of a Weak Solution to, say $\Box\varphi = \psi$.

Definition 2.6 Given $\psi \in \mathcal{D}(U)$ (but, usually, $\psi \in \mathcal{C}^\infty(U)$), we call $\varphi \in \mathcal{D}(U)$ a *weak solution* of $\Box\varphi = \psi$ iff for every $\eta \in \mathcal{C}_0^\infty(U)$, we have

$$\varphi(\Box\eta) = \psi(\eta).$$

Motivation: We know that $\Box\varphi$ is defined by

$$(\Box\varphi)(\eta) = \varphi(\Box\eta).$$

Therefore, $\Box\varphi = \psi$ in $\mathcal{D}(U)$ when and only when φ is a weak solution.

2.4 Hodge II, Globalization & Proof of Hodge's Theorem

Let X be a manifold (real, not necessarily complex) and assume that we also have a C^∞-complex vector bundle, V, on X. We are interested in the sets of sections, $\Gamma(U, V)$, where U is an open in X. The spaces T_X and T_X^D are real vector bundles and we write $T_{X,\mathbb{C}}$ and $T_{X,\mathbb{C}}^D$ for their complexification. We need to differentiate sections of V. For this, we introduce connections. For any open subset, U, of X, the space of (C^∞)-sections, $\Gamma(U, \bigwedge^p T_X^D \otimes V)$, of the bundle $\bigwedge^p T_X^D \otimes_{\mathcal{C}^\infty(U)} V$ is denoted by $\mathcal{A}_V^p(U)$ (with $\mathcal{A}_V^0(U) = \Gamma(U, V)$). Elements of $\mathcal{A}_V^p(U)$ are called *differential p-forms over U with coefficients in V*. The space of (C^∞)-sections, $\Gamma(U, \bigwedge^p T_X^D)$, of the bundle $\bigwedge^p T_X^D$ is denoted by $\mathcal{A}_X^p(U)$ (with $\mathcal{A}_X^0(U) = \mathcal{C}^\infty(U)$). For short, we usually write $\mathcal{A}_V^p$ for $\mathcal{A}_V^p(X)$ (global sections).

Definition 2.7 A *connection*, ∇, on V over X is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{A}_V^0 \to \mathcal{A}_V^1,$$

so that for every section $s \in \Gamma(X, V)$ and every $f \in \mathcal{C}^\infty(X)$, the *Leibnitz rule* holds:

$$\nabla(fs) = df \otimes s + f\nabla s.$$

From now on, when we write T_X (or T_X^D), it is always understood that we mean $T_{X,\mathbb{C}}$ (or $T_{X,\mathbb{C}}^D$). We can be more general and require Leibnitz in this case: Say $\xi \in \mathcal{A}_X^p$ and $\eta \in \mathcal{A}_V^q$ (Note, the above case corresponds to $p = 0$, $q = 1$). We require

$$\nabla(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^{\deg \xi}\, \xi \wedge \nabla\eta.$$

Note that we are extending ∇ to a $\mathbb{C}$-linear map $\mathcal{A}_V^p \longrightarrow \mathcal{A}_V^{p+1}$.

Look locally over an open U where V is trivial. Pick a frame, $e_1, \ldots, e_n$, for V over U (this means that we have n sections, $e_1, \ldots, e_n$, over U, such that for every $x \in U$, the vectors $e_1(x), \ldots, e_n(x)$ form a basis of the fibre of V over $x \in U$). Then, we can write

$$\nabla e_i = \sum_{j=1}^n \theta_{ij} \otimes e_j,$$

where (θ_{ij} is a matrix of 1-forms over U). The matrix (θ_{ij}) is called the *connection matrix* of ∇ w.r.t. the frame $e_1, \ldots, e_n$. Conversely, if (θ_{ij}) is given, we can use Leibnitz to determine ∇ (over U). Say $s \in \Gamma(U, V)$, then,

$$s = \sum_{i=1}^n s_i e_i,$$

with $s_i \in \Gamma(U, \mathcal{C}^\infty)$. We have

$$\begin{aligned}
\nabla s &= \sum_{i=1}^n \nabla(s_i e_i) \\
&= \sum_{i=1}^n (ds_i \otimes e_i + s_i \nabla e_i) \\
&= \sum_{j=1}^n ds_j \otimes e_j + \sum_{i=1}^n s_i \sum_{j=1}^n \theta_{ij} \otimes e_j \\
&= \sum_{j=1}^n \Big(ds_j \otimes e_j + \Big(\sum_{i=1}^n s_i \theta_{ij} \Big) \otimes e_j \Big),
\end{aligned}$$

which can be written

$$\nabla s = ds + s \cdot \theta.$$

In the general case where $s \in \mathcal{A}_V^p$, we get (DX)

$$\nabla s = ds + (-1)^p\, s \wedge \theta, \quad \text{where } p = \deg s.$$

Given a connection, ∇, we can differentiate sections. Given $v \in \Gamma(U, T_X)$, i.e., a vector field over U, then for any section, $s \in \Gamma(U, V)$, we define $\nabla_v(s)$ = *covariant derivative w.r.t. v of s* (= directional derivative w.r.t. v) by:

$$\nabla_v(s) = (\nabla s)(v),$$

where $\nabla s \in \Gamma(U, T_X^D \otimes V)$, and we use the pairing, $\Gamma(U, T_X^D) \otimes \Gamma(U, T_X) \longrightarrow \mathcal{C}^\infty$, namely, evaluation (= contraction).

What happens when we change local frame (gauge transformtaion)? Let $\widetilde{e}_1, \ldots, \widetilde{e}_n$ be a new frame, say

$$\widetilde{e}_i = \sum_{j=1}^n g_{ij} e_j$$

which can be written $\widetilde{e} = g \cdot e$, in matrix form (where the g_{ij} are functions). We know

$$\nabla \widetilde{e}_i = \sum_{j=1} \widetilde{\theta}_{ij} \otimes \widetilde{e}_j$$

and

$$\begin{aligned}
\nabla \widetilde{e}_i &= \sum_{j=1}^n \nabla(g_{ij} e_j) \\
&= \sum_{j=1}^n (dg_{ij} \otimes e_j + g_{ij} \nabla e_j) \\
&= \sum_{k=1}^n (dg_{ik} \otimes e_k) + \sum_{j=1}^n \sum_{k=1}^n g_{ij} \theta_{jk} \otimes e_k \\
&= \sum_{k=1}^n \left(dg_{ik} \otimes e_k) + \Big(\sum_{j=1}^n g_{ij} \theta_{jk} \Big) \otimes e_k \right),
\end{aligned}$$

which, in matrix form, says

$$\nabla \widetilde{e} = dg\, e + g\theta e.$$

But, $e = g^{-1}\widetilde{e}$, so

$$\nabla \widetilde{e} = dg\, g^{-1}\, e + g\theta g^{-1}\widetilde{e},$$

and finally, we have the change of basis formula (gauge transformation)

$$\widetilde{\theta} = dg\, g^{-1} + g\theta g^{-1}.$$

For the general Leibnitz rule

$$\nabla(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^{\deg \xi}\, \xi \wedge \nabla \eta,$$

(with $\xi \in \mathcal{A}_X^p$, $\eta \in \mathcal{A}_V^q$), note that $d\xi \wedge \eta \in \mathcal{A}_X^{p+1} \wedge \mathcal{A}_V^q$ and $\xi \wedge \nabla\eta \in \mathcal{A}_X^p \wedge \mathcal{A}_V^{q+1}$, so we can concatenate ∇, that is, take ∇ again. We have

$$\nabla^2(\xi \wedge \eta) = \nabla(d\xi \wedge \eta) + (-1)^{\deg \xi}\, \nabla(\xi \wedge \nabla \eta),$$

so $\nabla^2 \colon \mathcal{A}_V^{p+q} \to \mathcal{A}_V^{p+q+2}$. The operator, ∇^2 (really, its part $\nabla^2 \colon \mathcal{A}_V^0 \longrightarrow \mathcal{A}_V^2$) is the *curvature operator of* ∇ (a $\mathbb{C}$-linear map).

Definition 2.8 A connection ∇ is *flat* iff ∇ vanishes, i.e., $\nabla^2 = 0$. That is, the infinite sequence

$$\mathcal{A}_V^0 \xrightarrow{\nabla} \mathcal{A}_V^1 \xrightarrow{\nabla} \mathcal{A}_V^2 \xrightarrow{\nabla} \mathcal{A}_V^3 \xrightarrow{\nabla} \cdots$$

is a complex.

Not only is ∇^2 a $\mathbb{C}$-linear map, it is C^∞-linear. This is the lemma

Lemma 2.21 *The curvature operator of a connection is C^∞-linear. That is, for any $f \in C^\infty(U)$ and $s \in \Gamma(U, V)$, we have*

$$\nabla^2(f\, s) = f(\nabla^2(s)).$$

Proof. We have

$$\begin{aligned}
\nabla^2(f\, s) &= \nabla(\nabla(f\, s)) \\
&= \nabla(df \wedge s + f \wedge \nabla s) \\
&= d(df) \wedge s - df \wedge \nabla s + df \wedge \nabla s + f \wedge \nabla^2 s \\
&= f \wedge \nabla^2 s \\
&= f \nabla^2 s,
\end{aligned}$$

which proves the lemma. $\square$

Consequently, the curvature, ∇^2, is induced by a bundle map,

$$\Theta \colon V \longrightarrow \bigwedge^2 T_X^D \otimes V,$$

and the latter is a global section of

$$\mathcal{H}om(V, \bigwedge^2 T_X^D \otimes V) \cong \bigwedge^2 T_X^D \otimes \mathcal{E}nd(V) \cong \bigwedge^2 T_X^D \otimes V^D \otimes V.$$

We will need to know how to compute the curvature in a local frame. Say $e_1, \ldots, e_n$ is a frame for V over U and $e_1^D, \ldots, e_n^D$ is the dual frame in V^D (over U). As the $e_i^D \otimes e_j$ form a frame for $V^D \otimes V$ over U, we see that over this frame, Θ is given by a matrix of 2-forms, also denoted Θ. Thus, we can write

$$\nabla^2(e_i) = \sum_{i=1}^n \Theta_{ij} \otimes e_j,$$

where Θ_{ij} is a matrix of 2-forms over U, called the *curvature matrix* w.r.t. local frame, $e_1, \ldots, e_n$. Say we change basis to $\widetilde{e}_1, \ldots, \widetilde{e}_n$. We have

$$\nabla^2(\widetilde{e}_i) = \sum_{i=1}^n \widetilde{\Theta}_{ij} \otimes \widetilde{e}_j,$$

with $\widetilde{e}_i = \sum_{m=1}^n G_{im}\, e_m$, i.e., $\widetilde{e} = G\, e$. Then, we get

$$\begin{aligned}
\nabla^2(\widetilde{e}_i) &= \nabla^2(\sum_{m=1}^n G_{im} e_m) = \sum_{m=1}^n G_{im} \nabla^2(e_m) \\
&= \sum_{m=1}^n G_{im} \sum_{j=1}^n \Theta_{ml} \otimes e_j \\
&= \sum_{j=1}^n \Big(\sum_{m=1}^n G_{im} \Theta_{ml} \Big) \otimes e_j.
\end{aligned}$$

The left hand side is $\widetilde{\Theta}\,\widetilde{e} = G\Theta e$, but $e = G^{-1}\widetilde{e}$, so $\widetilde{\Theta}\,\widetilde{e} = G\Theta G^{-1}\widetilde{e}$, i.e.,

$$\widetilde{\Theta} = G\Theta G^{-1}.$$

If θ is the connection matrix of ∇ w.r.t. some local frame for V over U, for any section, $s \in \Gamma(U, V)$, we compute $\nabla^2(s)$ as follows: First, we have

$$\nabla^2(s) = \nabla(\nabla(s)) = \nabla(ds + s \wedge \theta)) = \nabla(ds) + \nabla(s \wedge \theta).$$

Now, as ds has degree 1 (since s has degree 0), we have

$$\nabla(ds) = d(ds) - ds \wedge \theta$$

and

$$\nabla(s \wedge \theta) = ds \wedge \theta - s \wedge \nabla\theta.$$

As θ has degree 1 (θ is a 1-form),

$$\nabla\theta = d\theta - \theta \wedge \theta,$$

so we obtain

$$\nabla^2(s) = d(ds) - ds \wedge \theta + ds \wedge \theta + s \wedge (d\theta - \theta \wedge \theta),$$

i.e.,

$$\nabla^2(s) = s \wedge (d\theta - \theta \wedge \theta).$$

Therefore, we have the *Maurer-Cartan's equation* (in matrix form):

$$\Theta = d\theta - \theta \wedge \theta.$$

Say X is a complex manifold and V is a *holomorphic* vector bundle over X. Then, while ∇ is not unique on V, we can however uniquely extend $\overline{\partial}$ to V.

Proposition 2.22 *If V is a holomorphic vector bundle on the C^∞-manifold, X, and if we define for a local holomorphic frame, $e_1, \ldots, e_n$, of V and for $s \in \mathcal{A}^p_V$,*

$$\overline{\partial}(s) = \overline{\partial}\Big(\sum_{i=1}^{n} \omega_i \otimes e_i\Big) = \sum_{i=1}^{n} \overline{\partial}(\omega_i) \otimes e_i,$$

then $\overline{\partial}$ defined this way is independent of the local holomorphic frame. Hence, $\overline{\partial}$ is well defined on $\mathcal{A}^p_V$.

Proof. Let $\{\widetilde{e}_1, \ldots, \widetilde{e}_n\}$ be another holomorphic frame for V over U. We have

$$\widetilde{e}_i = \sum_{j=1}^{n} g_{ij} e_j,$$

where the g_{ij} are holomorphic functions on U. Then,

$$s = \sum_{i=1}^{n} \widetilde{\omega}_i \otimes \widetilde{e}_i = \sum_{i=1}^{n} \widetilde{\omega}_i \otimes \sum_{j=1}^{n} g_{ij} e_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \widetilde{\omega}_i g_{ij}\Big) \otimes e_j.$$

Now, $\overline{\partial}s$ (according to the $\widetilde{e}_j$'s) $= \sum_{i=1}^{n} \overline{\partial}\,\widetilde{\omega}_i \otimes \widetilde{e}_i$, while
$\overline{\partial}s$ (according to the e_j's) $= \sum_{j=1}^{n} \overline{\partial}\Big(\sum_{i=1}^{n} \widetilde{\omega}_i g_{ij}\Big) \otimes e_j$. The second term is equal to

$$\sum_{j=1}^{n} \Big(\sum_{i=1}^{n} (\overline{\partial}\,\widetilde{\omega}_i g_{ij} + \widetilde{\omega}_i \overline{\partial} g_{ij})\Big) \otimes e_j.$$

But, $\overline{\partial} g_{ij} = 0$, as the g_{ij} are holomorphic. Thus,
$\overline{\partial} s$ (according to the e_j's) $= \sum_{j=1}^n \left(\sum_{i=1}^n \overline{\partial}\, \widetilde{\omega}_i g_{ij} \right) \otimes e_j$ and
$\overline{\partial} s$ (according to the $\widetilde{e}_j$'s) $= \sum_{i=1}^n \left(\overline{\partial}\, \widetilde{\omega}_i \otimes \sum_{j=1}^n g_{ij} e_j \right) = \sum_{j=1}^n \left(\sum_{i=1}^n \overline{\partial}\, \widetilde{\omega}_i g_{ij} \right) \otimes e_j$. These expressions are identical, which shows that $\overline{\partial}$ is well defined. $\square$

As a corollary, the Dolbeault theorem is valid for vector bundles, that is

Theorem 2.23 *(Dolbeault) If V is a holomorphic vector bundle over a complex manifold, X, then there is an isomorphism*

$$H^{p,q}_{\overline{\partial}}(X, V) \xrightarrow{\sim} H^q(X, \Omega^p_X \otimes \mathcal{O}_X(V)).$$

Suppose now that X is a complex manifold and V is a C^∞-bundle which possesses an Hermitian metric. That is, for all x, we have $(\xi_x, \eta_x) \in \mathbb{C}$, for all $\xi_x, \eta_x \in V_x$ and (ξ_x, η_x) varies C^∞ with x and is an Hermitian inner product for all x.

Definition 2.9 If ∇ is a connection on a vector bundle, V, as above, then ∇ is a *unitary connection* (i.e., comptable with given metric) iff

$$d(\xi_x, \eta_x) = (\nabla \xi_x, \eta_x) + (\xi_x, \nabla \eta_x).$$

If instead X is a complex manifold but V is not holomorphic (yet), then the splitting $T^D_X = T^{D\,(0,1)}_X \coprod T^{D\,(1,0)}_X$ yields the splitting

$$\mathcal{A}^0_V \xrightarrow{\nabla} \mathcal{A}^1_V \longrightarrow \Gamma(X, T^{D\,(0,1)}_X \otimes V) \coprod \Gamma(X, T^{D\,(1,0)}_X \otimes V),$$

which gives the splitting $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

Definition 2.10 If ∇ is a connection on a holomorphic vector bundle, V, over a complex manifold, X, then ∇ is a *holomorphic connection* (i.e., comptable with the complex structure) iff

$$\nabla^{0,1} = \overline{\partial}.$$

Call a vector bundle, V, on a complex manifold, X, an *Hermitian bundle* iff it is both holomorphic and possesses an Hermitian metric.

Theorem 2.24 *Given an Hermitian vector bundle, V, on a complex manifold, X, there exists a unique connection (the uniholo connection, also known as "Chern connection") which is both holomorphic and unitary. Denote it by ∇_V.*

Proof. Look locally at a holomorphic frame and take θ, the connection matrix of some connection, ∇. Then

$$\nabla(e_i) = \sum_{j=1}^n \theta_{ij} \otimes e_j.$$

The connection ∇ will be holomorphic iff

$$\nabla^{0,1} \restriction U = \overline{\partial} \quad \text{on } U,$$

i.e., $\nabla^{0,1}(\sum_{i=1}^n s_i \otimes e_i) = \overline{\partial} s_i \otimes e_i$. Therefore (as $\overline{\partial} 1 = 0$), we must have $\nabla^{0,1}(e_i) = 0$, for every e_i. So, in the frame, θ is a $(1,0)$-matrix and the converse is clear.

For this ∇ to be unitary, we need

$$db_{ij} = d(e_i, e_j) = (\nabla e_i, e_j) + (e_i, \nabla e_j).$$

On the right hand side, we have

$$\begin{aligned}\Big(\Big(\sum_{k=1}^{n}\theta_{ik}e_k\Big), e_j\Big) + \Big(e_i, \Big(\sum_{r=1}^{n}\theta_{jr}e_r\Big)\Big) &= \sum_{k=1}^{n}\theta_{ik}(e_k, e_j) + \sum_{r=1}^{n}\overline{\theta}_{jr}(e_i, e_r)\\ &= \sum_{k=1}^{n}\theta_{ik}b_{kj} + \sum_{r=1}^{n}\overline{\theta}_{jr}b_{ir}\\ &= (\theta B + \overline{\theta}^{\top} B)_{ij}.\end{aligned}$$

(Here $B = (b_{ij}) = (e_i, e_j)$.) Now, $dB = \partial B + \overline{\partial} B = \theta B + \theta^* B$, from which we deduce (by equating the $(1, 0)$ pieces and the $(0, 1)$ pieces) that

$$\partial B = \theta B \quad \text{and} \quad \overline{\partial} B = \theta^* B.$$

(Here, $\theta^* = \overline{\theta}^{\top}$). The first equation has the unique solution

$$\theta = \partial B \cdot B^{-1}.$$

A simple calculation shows that it also solves the second equation (DX). By uniqueness, this solution is independent of the frame and so, it patches to give the uniholo connection. □

Corollary 2.25 *(of the proof) For a holomorphic vector bundle, V, over a complex manifold, X, a connection ∇ is holomorphic iff in every holomorphic local frame its connection matrix is a matrix of $(1, 0)$-forms.*

Corollary 2.26 *For a vector bundle, V, with Hermitian metric a connection ∇ is unitary iff in each every unitary local frame its connection matrix is skew-Hermitian.*

Proof. By the proof, for a unitary frame, the connection ∇ is unitary iff

$$0 = dI = \theta I + \theta^* I = \theta + \theta^*.$$

Therefore, $\theta^* = -\theta$, as claimed. □

Say $s \in \mathcal{A}^p_V(U)$, $t \in \mathcal{A}^q_V(U)$ and V is a unitary bundle with an Hermitian metric. For a local frame over U, set

$$\{s, t\} = \sum_{\mu,\nu} s_\mu \wedge t_\nu \,(e_\mu, e_\nu),$$

called the *Poisson bracket*, where $s = \sum_{\mu=1}^{n} s_\mu e_\mu$ and $t = \sum_{\nu=1}^{n} t_\nu e_\nu$. Then, we have

Corollary 2.27 *A connection, ∇, on a unitary bundle is a unitary connection iff for all s, t (as above), locally*

$$d\{s, t\} = \{\nabla s, t\} + (-1)^{\deg s}\, \{s, \nabla t\}.$$

Proof. (DX).

Corollary 2.28 *If ∇ is a unitary connection on the unitary bundle, V, then the local curvature matrix, Θ, in a unitary frame is skew Hermitian.*

Proof. We know that $\Theta = d\theta - \theta \wedge \theta$ and θ is skew Hermitian (DX). □

Proposition 2.29 *Say V and $\widetilde{V}$ are unitary and holomorphic vector bundles over a complex manifold, X, and W is a holomorphic subbundle of V. Write, as usual, $\nabla_V, \nabla_{\widetilde{V}}$, for the uniholo connections on V and $\widetilde{V}$. Then,* *(1) W is also an Hermitian vector bundle and if $V = W \coprod^{\perp} W^{\perp}$ in the metric decomposition for V, we have*

$$\nabla_W = pr_W \circ \nabla_V.$$

That is, the diagram

$$\begin{tikzcd} \mathcal{A}_V^0 \arrow[r, "\nabla_V"] \arrow[d, shift left, "pr_W"] & \mathcal{A}_V^1 \arrow[d, "pr_W"] \\ \mathcal{A}_W^0 \arrow[u, shift left, hook, "\mathrm{incl}"] \arrow[r, "\nabla_W"'] & \mathcal{A}_W^1 \end{tikzcd}$$

commutes.

(2) $V \otimes \widetilde{V}$ is again an Hermitian bundle in a canonical way. And for this structure,

$$\nabla_{V \otimes \widetilde{V}} = \nabla_V \otimes 1 + 1 \otimes \nabla_{\widetilde{V}}.$$

(3) Write V^D for the dual bundle of V. Then, V^D is again an Hermitian bundle in a canonical way and ∇_{V^D} is related to ∇_V by the following local equation in a local frame:

$$d(e_i, e_j^D) = (\nabla_V e_i, e_j^D) + (e_i, \nabla_{V^D} e_j^D).$$

Proof. In each case, make a candidate connection satisfying the appropriate formula. Check it is both holomorphic and unitary and use uniqueness.

(1) Write $\nabla = pr_W \circ \nabla_V$. It is clear that W inherits a metric from V. Check that ∇ is holomorphic. Pick $s \in \Gamma(U, V)$ and embed s in $\Gamma(U, V)$. As $\nabla^{0,1} = pr_W \circ \nabla_V^{0,1}$, we have

$$\nabla^{0,1} s = pr_W(\nabla_V^{0,1} s) = pr_W(\overline{\partial} s) = \overline{\partial} s,$$

by the way $\overline{\partial}$ is defined ($\overline{\partial}$ does not touch basis vectors) and $s = \sum_{i=1}^k \omega_i \otimes e_i$. We need to check that $\nabla = pr_W \circ \nabla_V$ is the uniholo connection. As the metric on W is the restriction of the metric on V, we have

$$\begin{aligned} d(s,t)_W &= d(s,t)_V = (\nabla_V s, t) + (s, \nabla_V s, t) \\ &= (pr_W \circ \nabla_V s, t) + (pr_{W^\perp} \circ \nabla_V s, t) + (s, pr_W \circ \nabla_V t) + (s, pr_{W^\perp} \circ \nabla_V t). \end{aligned}$$

As W is perpendicular to $W^{\perp}$, the second and fourth terms in the right hand side are 0. Therefore, we get

$$d(s,t)_W = (\nabla s, t) + (s, \nabla t).$$

This shows ∇ is also unitary. By uniqueness, $\nabla = \nabla_W$.

(2) Say V and $\widetilde{W}$ are hermitian bundles. Metrize $V \otimes \widetilde{V}$ as follows: Consider $s \otimes \widetilde{s}$ and $t \otimes \widetilde{t}$, for some sections smt of V and sections $\widetilde{s}, \widetilde{s}$ of $\widetilde{V}$ and define

$$(s \otimes \widetilde{s}, t \otimes \widetilde{t}) = (s,t)(\widetilde{s}, \widetilde{t})$$

and extend by complex sesquilinearity. We get a hermitian form on $V \otimes \widetilde{V}$ and (DX), it is positive definite. The space $V \otimes \widetilde{V}$ is also holomorphic as each is. Let

$$\nabla = \nabla_V \otimes 1 + 1 \otimes \nabla_{\widetilde{V}}.$$

In order to check that ∇ is unitary, it suffices to check this on the special inner product $(s\otimes\widetilde{s}, t\otimes\widetilde{t})$. We have

$$\begin{aligned}
d(s\otimes\widetilde{s}, t\otimes\widetilde{t}) &= d((s,t)(\widetilde{s},\widetilde{t}))\\
&= d(s,t)\,(\widetilde{s},\widetilde{t}) + (s,t)\,d(\widetilde{s},\widetilde{t})\\
&= \big[(\nabla_V s,t)+(s,\nabla_V t)\big](\widetilde{s},\widetilde{t}) + (s,t)\big[(\nabla_{\widetilde{V}}\widetilde{s},\widetilde{t})+(\widetilde{s},\nabla_{\widetilde{V}}\widetilde{t})\big]\\
&= (\nabla_V s,t)(\widetilde{s},\widetilde{t}) + (s,\nabla_V t)(\widetilde{s},\widetilde{t}) + (s,t)(\nabla_{\widetilde{V}}\widetilde{s},\widetilde{t}) + (s,t)(\widetilde{s},\nabla_{\widetilde{V}}\widetilde{t})\\
&= (\nabla_V s\otimes\widetilde{s}, t\otimes\widetilde{t}) + (s\otimes\widetilde{s},\nabla_V t\otimes\widetilde{t}) + (s\otimes\nabla_{\widetilde{V}}\widetilde{s}, t\otimes\widetilde{t}) + (s\otimes\widetilde{s}, t\otimes\nabla_{\widetilde{V}}\widetilde{t})\\
&= ((\nabla_V\otimes 1 + 1\otimes\nabla_{\widetilde{V}})(s\otimes\widetilde{s}), t\otimes\widetilde{t}) + (s\otimes\widetilde{s}, (\nabla_V\otimes 1 + 1\otimes\nabla_{\widetilde{V}})(t\otimes\widetilde{t})),
\end{aligned}$$

as required. Now, $\nabla^{0,1} = \nabla_V^{0,1}\otimes 1 + 1\otimes\nabla_{\widetilde{V}}^{0,1}$ and the latter two are $\overline{\partial}$. It follows that

$$\nabla^{0,1} = (\overline{\partial}\otimes 1 + 1\otimes\overline{\partial}).$$

If $(e_1,\ldots,e_m)$ is a holomorphic frame and $s=\sum_{i=1}^m s_ie_i$, and similarly, $\widetilde{s}=\sum_{i=1}^m s_i\widetilde{e}_i$ for $(\widetilde{e}_1,\ldots,\widetilde{e}_n)$, then as

$$\overline{\partial}(s_i\widetilde{s}_j) = (\overline{\partial}s_i)s_j + s_i(\overline{\partial}s_j) = (\overline{\partial}\otimes 1 + 1\otimes\overline{\partial})(s_i\otimes\widetilde{s}_j).$$

Therefore, ∇ is uniholo and so, it is $\nabla_{V\otimes\widetilde{V}}$.

(3) For the dual bundle, say $(e_1,\ldots,e_n)$ is a local unitary frame. Let $e_1^D,\ldots,e_n^D$ be the dual frame and decree it shall be unitary. We get an hermitian form on V^D and this is independent of the choice of the unitary frame. The bundle V^D is clearly holomorphic, so V^D is hermitian. Check (DX),

$$(s,\nabla_{V^D}t^D) = d(s,t^D) - (\nabla_V s, t^D)$$

define the uniholo connection in V^D. $\square$

Now, say V is holomorphic and ∇ is a holomorphic connection. So, $\nabla^{0,1}=\overline{\partial}$, but we know $\overline{\partial}^2 = 0$. Therefore, $\nabla^{0,2} = (\nabla^{0,1})^2 = 0$. We deduce that

$$0 = \nabla^{0,2}e_i = \sum_{j=1}^n \Theta_{ij}^{0,2}e_j$$

and by linear independence, $\Theta^{0,2}=(0)$. If we change frame, $\widetilde{\Theta}^{0,2} = B\Theta^{0,2}B^{-1}$, so $\Theta^{0,2}=0$ in any base. If V is hermitian and ∇ is uniholo then we know Θ is skew hermitian in a unitary frame, i.e.,

$$\Theta = -\Theta^* = -\overline{\Theta}^{\top}.$$

Consequently, $-(\Theta^{2,0})^{\top} = -\overline{\Theta^{0,2}}^{\top}$, which means that $\Theta^{2,0}=(0)$, too! Therefore, Θ is a $(1,1)$-matrix. Now, $\overline{\omega}=-\omega$ if ω is a $(1,1)$-form, and since Θ is skew hermitian, we get

Proposition 2.30 *For a holomorphic bundle and holomorphic connection, ∇, the curvature matrix, Θ, of ∇ in any frame has $\Theta^{0,2}=(0)$. If V is an hermitian bundle and if $\nabla=\nabla_V$ is the uniholo connection, then in a unitary frame, Θ is an hermitian matrix of $(1,1)$-forms.*

Now, look at V^D, for V hermitian. Say θ is the connection matrix for ∇_V in a unitary frame and θ^D the connection matrix for ∇_{V^D} in the dual frame. We have

$$\nabla_V(e_i) = \sum_{j=1}^n \theta_{ij}e_j \quad\text{and}\quad \nabla_{V^D}(e_k^D) = \sum_{r=1}^n \theta_{kr}^D e_r^D.$$

Now, $(e_i, e_k^D) = \delta_{ik}$, which implies $d(e_i, e_k^D) = 0$,, for all i, k. It follows that

$$\begin{aligned} 0 &= d(e_i, e_k^D) = (\nabla_V e_i, e_k^D) + (e_i, \nabla_{V^D} e_k^D) \\ &= \sum_j \theta_{ij}(e_j, e_k^D) + \sum_r \theta_{kr}^D(e_i, e_r^D) \\ &= \theta_{ik} + \theta_{ki}^D. \end{aligned}$$

Therefore, $\theta_{ki}^D = -\theta_{ik}$, i.e., $\theta^D = -\theta^\top$.

Proposition 2.31 *If V is a hermitian bundle, V^D, its dual, and ∇_V and ∇_{V^D} their uniholo connections, in a unitary frame (and coframe) for V (and V^D), the connection matrices satisfy $\theta^D = -\theta^\top$.*

Back to the Sobolev theorem and Rellich lemma. Instead of $\partial/\partial X_i$ locally on a real manifold, X, use n independent vector fields, $v_1, \ldots, v_n$. If f is a C^∞-function on X, locally on some open, U, for all $x \in U$, we have

$$v_i(x) = \sum_{j=1}^n a_j^{(i)} \frac{\partial}{\partial X_j},$$

for some C^∞ functions, $a_j^{(i)}$, and so,

$$v_i(x)(f) = \sum_{j=1}^n a_j^{(i)} \frac{\partial f}{\partial X_j}.$$

Note that in general, $v_i v_k(f) \neq v_k v_i(f)$. Indeed, we have

$$\begin{aligned} v_i v_k(f) &= v_i \left(\sum_{j=1}^n a_j^{(k)} \frac{\partial f}{\partial X_j} \right) \\ &= \sum_{r=1}^n a_r^{(i)} \frac{\partial}{\partial X_r} \left(\sum_{j=1}^n a_j^{(k)} \frac{\partial f}{\partial X_j} \right) \\ &= \sum_{r,j=1}^n a_r^{(i)} a_j^{(k)} \frac{\partial^2 f}{\partial X_r \partial X_j} + \sum_{r,j=1}^n a_r^{(i)} \frac{\partial a_j^{(k)}}{\partial X_r} \frac{\partial f}{\partial X_j}. \end{aligned}$$

Interchanging k and i, we have

$$v_k v_i(f) = \sum_{r,j=1}^n a_r^{(k)} a_j^{(i)} \frac{\partial^2 f}{\partial X_r \partial X_j} + \sum_{r,j=1}^n a_r^{(k)} \frac{\partial a_j^{(i)}}{\partial X_r} \frac{\partial f}{\partial X_j}.$$

We conclude that $[v_i, v_j](f) = v_i v_k(f) - v_k v_i(f) =$ a sum involving only the $\frac{\partial}{\partial X_j}$'s, i.e., it is a first-order differential operator $= O_1(\partial, \overline{\partial})$.

If V is a vector bundle on X and ∇ is a connection, $v_1, \ldots, v_n$ are n independent vector fields and $v_1^D, \ldots, v_n^D$ are their duals, then in a local frame, $e_1, \ldots, e_t$, for V over U, for any section, $s \in \Gamma(U, V)$, if

$$s = \sum_{j=1} s_j e_j,$$

then, by definition,

$$\nabla(s) = \sum_{i,j} (\nabla_{v_i}(s))_j \, v_i^D \otimes e_j$$

defines the covariant derivative of s. Write ∇_i for ∇_{v_i}. But, we have

$$\begin{aligned}\nabla(s) &= \nabla\Big(\sum_{j=1}^t s_j e_j\Big)\\ &= \sum_{j=1}^t ds_j \otimes e_j + \sum_{j=1}^t s_j \nabla e_j\\ &= \sum_{j=1}^t ds_j \otimes e_j + \sum_{j,r=1}^t s_j\theta_{jr}\otimes e_r,\end{aligned}$$

where the θ_{jr} are 1-forms involving the v_i^D. Furthermore,

$$ds_j = \sum_{k=1}^n \frac{\partial s_j}{\partial X_k}\, dX_k = \sum_{k,\mu=1}^n \frac{\partial s_j}{\partial X_k}\, b_k^{(\mu)} v_\mu^D = \sum_{\mu=1}^n v_\mu(s_j)\, v_\mu^D$$

and

$$ds_j \otimes e_j = \sum_k v_k(s_j)\, v_k^D \otimes e_j,$$

so, we get

$$\nabla(s) = \sum_{j,l} (\nabla_j(s))_l\, v_j^D \otimes e_l = \sum_{j=1}^t\sum_{l=1}^n v_l(s_j)\, v_l^D \otimes e_j + \sum_{j,r=1}^t s_j\theta_{jr}\otimes e_r.$$

This shows that $(\nabla_i(s))_j$ and $v_i(s_j)$ differ at most by $O_0(\partial, \overline{\partial})$ and $(\nabla_i(s))_j$ involves the v_i operating on $s_j +$ a term using the connection matrix.

We can repeat this process with $\Gamma(U, V\otimes T_X^D)$ and get $\nabla_i(\nabla_j)$ and $\nabla_j(\nabla_i)$ and from before, we find that

$$\nabla_i(\nabla_j) - \nabla_j(\nabla_i) = O_1(\partial, \overline{\partial}).$$

This can be extended to multi-indices. For $\alpha = (\alpha_1, \ldots, \alpha_n)$, define

$$\nabla^\alpha f = \nabla_{\alpha_1}(\nabla_{\alpha_2}(\cdots(\nabla_{\alpha_n} f)\cdots)).$$

The above term is well define and is independent of the order taken, up to $O_{<|\alpha|}(\partial, \overline{\partial})$.

Say V has a(n hermitian) metric and ∇ is a unitary connection and assume X is compact. For any $f \in \mathcal{C}^\infty(X, V)$, we have $\|\nabla^\alpha f\|_{L^2}$, so we can define

$$\|f\|_s^2 = \sum_{|\alpha|\leq s} \|\nabla^\alpha f\|_{L^2}^2 = \sum_{|\alpha|\leq s}\int_X |\nabla^\alpha f(x)|^2 \Phi(x),$$

where $\Phi(x)$ is the volume form on X. This is the Sobolev s-norm on $\mathcal{C}^\infty(X, V)$. Locally, where V is trivial, the above computations show that this norm is equivalent to the Sobolev s-norm defined before using the D^α's. When we complete $\mathcal{C}^\infty(X, V)$ w.r.t. the (global) s-Sobolev norm we get $H(X, V)_s$, a Hilbert space. Then, cover our compact space, X, by a finite number of opens where V is trivial, take a partition of unity subordinate to our cover, $\{U_\alpha\}$, use the sup of the finitely many constants relating global s-norm restricted to U_α and local s-norm on U_α, and we get

Lemma 2.32 *(Global Sobolev Embedding Lemma) If X is a compact (real or complex) manifold and $m \in \mathbb{Z}_{\geq 0}$, V is a vector bundle with metric and unitary connection, ∇, then*

$$H(X, V)_{\sigma(m)} \hookrightarrow \mathcal{C}^m(X, V).$$

In particular, $\bigcap_s H(X, V)_s = \mathcal{C}^\infty(X, V)$.

We also get

Lemma 2.33 *(Global Rellich Lemma) For X compact, V a vector bundle with metric and unitary connection, if $s > r$, then the embedding*

$$H(X, V)_s \hookrightarrow H(X, V)_r$$

is a compact operator.

Now, we are going to apply the above theorem to T_X^D and $\bigwedge^{p,q}(X)$. Consider the spaces $\bigwedge^{p,q}(X) = \Gamma(X, \bigwedge^{p,q} T_X^D)$ and suppose X is metrized (and compact), i.e., T_X (hence T_X^D) are bundles with a metric. If we have a local unitary frame for T_X over U, say $e_1, \ldots, e_n$, and dual coframe $e_1^D, \ldots, e_n^D$, for T_X^D, then we know $\bigwedge^{p,q}(X)$ is a pre-Hilbert space. Recall that if $|I| = p$ and $|J| = q$, then

$$\{e_I^D \wedge \overline{e}_J^D\}_{I,J}$$

is a basis for $\bigwedge^{p,q}(U)$ and decree that the $e_I^D \wedge \overline{e}_J^D$ are mutally orthogonal with size

$$\|e_I^D \wedge \overline{e}_J^D\|^2 = 2^{p+q}.$$

We found that for $\xi, \eta \in \bigwedge^{p,q}(X)$,

$$(\xi, \eta) = \int_X (\xi_x, \eta_x)\Phi(x),$$

where $\Phi(x)$ is the volume form. If $\omega \in \bigwedge^{p,q}(X)$, we can form (as usual), $\nabla^\alpha\omega$, for ∇ a unitary connection on T_X^D, and thus, we have $\|\nabla^\alpha\omega\|_{L^2}^2$, where

$$\|\nabla^\alpha\omega\|_{L^2}^2 = \int_X (\nabla^\alpha\omega, \nabla^\alpha\omega)_x\Phi(x).$$

Then, $\bigwedge^{p,q}(X)$ has the s-Sobolev norm:

$$\|\omega\|_s^2 = \sum_{|\alpha|\leq s} \|\nabla^\alpha\omega\|_{L^2}^2 = \sum_{|\alpha|\leq s} \int_X (\nabla^\alpha\omega, \nabla^\alpha\omega)_x\Phi(x),$$

where $\omega \in \bigwedge^{p,q}(X)$. Complete $\bigwedge^{p,q}(X)$ w.r.t. the s-Sobolev norm and get the Sobolev space, $H(X)_s^{p,q}$.

We even have another norm on (p, q)-forms in the complex, compact, case, the *Dirichlet norm*. Say $\xi, \eta \in \bigwedge^{p,q}(X)$, and set

$$(\xi, \eta)_D = (\xi, \eta)_{L^2} + (\overline{\partial}\xi, \overline{\partial}\eta)_{L^2} + (\overline{\partial}^*\xi, \overline{\partial}^*\eta)_{L^2}.$$

(Here, $\overline{\partial}^*$ is the formal adjoint of $\overline{\partial}$.) The Dirichlet norm is given by

$$\|\xi\|_D^2 = (\xi, \xi)_D.$$

We can motivate the definition of the Dirichlet norm as follows: Observe that

$$(\overline{\partial}\xi, \overline{\partial}\eta) = (\xi, \overline{\partial}^*\overline{\partial}\eta) \quad \text{and} \quad (\overline{\partial}^*\xi, \overline{\partial}^*\eta) = (\xi, \overline{\partial}\,\overline{\partial}^*\eta),$$

from which we conclude that

$$(\xi, \eta)_D = (\xi, \eta)_{L^2} + (\xi, \square\eta)_{L^2}.$$

Therefore,

$$(\xi, \eta)_D = (\xi, (I + \square)\eta)_{L^2}. \tag{$\dagger$}$$

The connection between the Dirichlet norm and the Sobolev 1-norm on $\bigwedge^{p,q}(X)$ is

Theorem 2.34 *(Gårding Inequality) If X is a compact, complex and Hermitian metrized, with its uniholo connection, then for all $\omega \in \bigwedge^{p,q}(X)$, we have*

$$\|\omega\|_1 \leq C\|\omega\|_D,$$

where $C > 0$ is independent of ω. In fact, $\| \, \|_1$ and $\| \, \|_D$ are equivalent norms on $\bigwedge^{p,q}(X)$.

Using the Gårding inequality we can prove

Lemma 2.35 *(Weyl's Regularity Lemma (1940)) Say $\xi \in H(X)^{p,q}_s$ and $\eta \in H(X)^{p,q}_0$ (actually, $\eta \in H(X)^{p,q}_{-t}$ will do for $t > 0$), so that η is a weak solution of $\Box\eta = \xi$ (i.e., for all $\zeta \in \bigwedge^{p,q}(X)$, we have $(\xi, \zeta) = (\eta, \Box\zeta)$.) Then, η is actually in $H(X)^{p,q}_{s+2}$ (resp. $\eta \in H(X)^{p,q}_{s-t+2}$).*

Applications of the Weyl Lemma:

Say $\eta \in H(X)^{p,q}_0$ is a (p, q)-eigenform (eigenvalue, $\lambda \in \mathbb{C}$) for $\Box$:

$$\Box\eta = \lambda\eta.$$

Now, $\eta \in H(X)^{p,q}_0$ implies $\lambda\eta \in H(X)^{p,q}_0$. By Weyl's Regularity Lemma, $\eta \in H(X)^{p,q}_2$. By repeating this process, we see that $\eta \in H(X)^{p,q}_\infty$. Therefore

Corollary 2.36 *Every L^2 weak (p, q)-form for $\Box$ is actually $\mathcal{C}^\infty$ and is an honest (p, q)-eigenform.*

In particular, in the case $\lambda = 0$, which means $\Box\eta = 0$, we get

Corollary 2.37 *Every L^2 weak harmonic (p, q)-form, η, is automatically $\mathcal{C}^\infty$ and is a (p, q)-harmonic form ($\eta \in \mathcal{H}^{p,q}(X)$) in the standard sense.*

Now, $\Box$ is self-adjoint (on $\bigwedge^{p,q}(X)$), which implies that all eigenvalues are *real*. Let $\omega \in \bigwedge^{p,q}(X)$ be an eigenform, with λ the corresponding eigenvalue. Then, we have

$$\begin{aligned}
(\omega, \Box\omega) &= \lambda(\omega, \omega) \\
&= (\omega, \overline{\partial}^*\overline{\partial}\omega) + (\omega, \overline{\partial}\,\overline{\partial}^*\omega) \\
&= (\overline{\partial}\omega, \overline{\partial}\omega) + (\overline{\partial}^*\omega, \overline{\partial}^*\omega) \geq 0.
\end{aligned}$$

Corollary 2.38 *All the eigenvalues of $\Box$ are non-negative.*

Corollary 2.39 *The operator $I + \Box$ has zero kernel.*

Proof. If $(I + \Box)(\omega) = 0$, then $\Box\omega = -\omega$. Yet, the eigenvalues of $\Box$ are non-negative, so $\omega = 0$. $\Box$

We now have all the necessary analytic tools to prove Hodge's theorem. Recall the statement of Hodge's Theorem:

Theorem 2.17 (W.V.D. Hodge, 1941) *Let X be a complex manifold and assume that X is compact. Then, for all $p, q \geq 0$,*

(1) The space $H^{p,q}_{\overline{\partial}} \cong \mathcal{H}^{p,q}(X)$ is finite-dimensional.

(2) There exists a projection, $\mathrm{H}\colon \bigwedge^{p,q}(X) \to \mathcal{H}^{p,q}(X)$ (= $\mathrm{Ker}\,\Box$ on $\bigwedge^{p,q}(X)$), so that we have the **orthogonal decomposition** *(Hodge decomposition)*

$$\bigwedge^{p,q}(X) = \mathcal{H}^{p,q}(X) \coprod^{\perp} \overline{\partial} \bigwedge^{p,q-1}(X) \coprod^{\perp} \overline{\partial}^* \bigwedge^{p,q+1}(X).$$

(3) There exists a **parametrix** *(= pseudo-inverse), G, for $\Box$, and it is is uniquely determined by*

(a) $\mathrm{id} = \mathcal{H} + \Box G = \mathcal{H} + G\Box$, *and*

(b) $G\overline{\partial} = \overline{\partial} G$, $G\overline{\partial}^* = \overline{\partial}^* G$ *and* $G \restriction \mathcal{H}^{p,q}(X) = 0$.

Proof. All the statements of the theorem about orthogonal decomposition have been shown to be consequences of the finite dimensionality of the harmonic space $\mathcal{H}^{p,q}(X)$ and the fact that

(a) $\mathrm{id} = \mathcal{H} + \Box G = \mathcal{H} + G\Box$, and

(b) $G\overline{\partial} = \overline{\partial} G$, $G\overline{\partial}^* = \overline{\partial}^* G$ and $G \restriction \mathcal{H}^{p,q}(X) = 0$.

Break up the rest of the proof into three steps.

Step I. (Main analytic step). I claim:

For every $\varphi \in H(X)_0^{p,q} = L^2$-$(p,q)$-forms, there is a unique $\psi \in H(X)_1^{p,q}$, so that, for all $\eta \in \bigwedge^{p,q}(X)$,

$$(\psi, (I + \Box)(\eta))_1 = (\varphi, \eta)_0$$

and the map $\varphi \mapsto S(\varphi) = \psi$ is a bounded linear transformation $H(X)_0^{p,q} \longrightarrow H(X)_1^{p,q}$.

Consider $(I + \Box) \bigwedge^{p,q}(X) \subseteq \bigwedge^{p,q}(X)$. As $I + \Box$ is a monomorphism, there is only one η giving $(I + \Box)(\eta)$. Look at the conjugate linear functional, l, on $(I + \Box) \bigwedge^{p,q}(X)$ given by

$$l((I + \Box)(\eta)) = (\varphi, \eta)_0.$$

We have

$$|l((I + \Box)(\eta))| = |(\varphi, \eta)_0| \leq \|\varphi\|_0 \|\eta\|_0 .$$

(By Cauchy-Schwarz.) Thus,

$$|l((I + \Box)(\eta))| \leq \|\varphi\|_0 \|\eta\|_0 \leq \|\varphi\|_0 \|I + \Box)(\eta)\|_0$$

and by Gårding inequality, the D-norm is equivalent to the Sobolev 1-norm. Therefore, l is indeed a bounded linear functional on $(I + \Box) \bigwedge^{p,q}(X) \hookrightarrow H(X)_1^{p,q}$. By the Hahn-Banach Theorem, l extends to a linear functional, with the same bound, on all of $H(X)_1^{p,q}$. But, $H(X)_1^{p,q}$ is a Hilbert space, so, by Riesz Theorem, there is some $\psi \in H(X)_1^{p,q}$ so that

$$l((I + \Box)(\eta)) = (\psi, (I + \Box)(\eta))_1.$$

Check that ψ is unique (DX). Then, we have

$$(\psi, (I + \Box)(\eta))_1 = (\varphi, \eta)_0$$

for every $\eta \in \bigwedge^{p,q}(X)$. Write $\psi = S(\varphi)$; then, formally, S is self-adjoint as $I + \Box$ is itself self-adjoint. For boundedness of S, compute

$$\|S\varphi\|_1^2 \leq C \|S\varphi\|_D^2 \leq C(S\varphi, (I + \Box)S\varphi)_0,$$

by Gårding inequality, but $\Box S = 0$, so

$$C(S\varphi, (I + \Box)S\varphi)_0 \leq C(S\varphi, \varphi)_0 \leq \|S\varphi\|_0 \|\varphi\|_0 .$$

(By Cauchy-Schwarz.) Now, S inverts $I + \Box$ and therefore is an integral operator on L^2-(p,q)-forms. Therefore, there is some $K > 0$ so that

$$\|S\varphi\|_0 \leq K \|\varphi\|_0 ,$$

and we conclude that
$$\|S\varphi\|_1^2 \leq CK\,\|\varphi\|_0\,\|\varphi\|_0 = CK\,\|\varphi\|_0^2,$$
which shows that S is indeed a bounded operator from $H(X)_0^{p,q}$ to $H(X)_1^{p,q}$.

Step II. (Application of spectral theory).

Consider
$$H(X)_0^{p,q} \xrightarrow{S} H(X)_1^{p,q} \hookrightarrow H(X)_0^{p,q},$$
where the last map is a compact embedding. Apply Rellich's lemma and find that our operator $H(X)_0^{p,q} \longrightarrow H(X)_0^{p,q}$ is a compact self-adjoint endomorphism. In this case, the spectral theorem says that $H(X)_0^{p,q}$ splits into a countable orthogonal coproduct of eigenspaces for S, each of finite dimension, write $S(\lambda_m)$ for the λ_m-eigenspace of the operator S (i.e., $S(\lambda_m) = \{\varphi \mid S(\varphi) = \lambda_m\varphi\}$),
$$H(X)_0^{p,q} = \coprod_{m\geq 0} S(\lambda_m).$$

Now, $S\varphi = 0$ implies $0 = (S\varphi, (I+\Box)(\eta)) = (\varphi,\eta)_0$, for all $\eta \in \bigwedge^{p,q}(X)$, so we deduce $\varphi = 0$, by denseness of $\bigwedge^{p,q}(X) \subseteq H(X)_0^{p,q}$. Therefore, $\lambda_m \neq 0$, for all m. Look at $\varphi \in S(\lambda_m)$, so $S(\varphi) = \lambda_m\varphi$. Then,
$$(I+\Box)S\varphi = \varphi = (I+\Box)(\lambda_m\varphi) = \lambda_m\varphi + \lambda_m(\Box\varphi),$$
from which we get
$$\left(\frac{(1-\lambda_m)}{\lambda_m}\right)\varphi = \Box\varphi.$$
As we can reverse the argument, we conclude that there is an isomorphism between $S(\lambda_m)$ and $\Box\left(\frac{(1-\lambda_m)}{\lambda_m}\right)$. Set $\mu_m = \frac{(1-\lambda_m)}{\lambda_m}$, so the μ_m are the eigenvalues of $\Box$, and we know that they are real and positive. Arrange the μ_m in ascending order
$$\mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots.$$
As $\mu_m = \frac{(1-\lambda_m)}{\lambda_m}$ we have $\lambda_m = \frac{1}{(1+\mu_m)}$ and we see that $\mu_m \uparrow \infty$ and $\lambda_m \downarrow 0$ and $\Box(\mu_m) = S(\lambda_m)$. We conclude

(a)
$$H(X)_0^{p,q} = \coprod_{m\geq 0} \Box(\mu_m) = \mathcal{H}^{p,q}(X) \overset{\perp}{\amalg} \overset{\perp}{\coprod_{m\geq 1}} \Box(\mu_m),$$
and each is a finite dimensional space, so $\mathcal{H}^{p,q}(X)$ is finite dimensional. Further

(b) Each subspace, $\Box(\mu_m)$, consists of C^∞ (p,q)-forms (by Weyl's lemma).

Step III. (The Green's operator).

On $(\mathcal{H}^{p,q})^\perp$, we have
$$\Box\varphi = \Box\Big(\sum_{n=1}^{\infty}\varphi_n\Big) = \sum_{n=1}^{\infty}\Box\varphi_n = \sum_{n=1}^{\infty}\mu_n\varphi_n,$$
so
$$\|\Box\varphi\|_0^2 = \sum_{n=1}^{\infty}\mu_n^2\,\|\varphi_n\|_0^2 \geq \mu_1^2\sum_{n=1}^{\infty}\|\varphi_n\|_0^2 = \mu_1^2\,\|\varphi\|_0^2.$$
Therefore,
$$\|\Box\varphi\|_0^2 \geq \mu_1^2\,\|\varphi\|_0$$

on $\mathcal{H}^{p,q}(X)$, which implies that if we construct a parametrix, G, it is automatically bounded above, i.e., continuous.

Set $G \equiv 0$ on $\mathcal{H}^{p,q}(X)$ (by definition) and on $\Box(\mu_m)$, set

$$G(\varphi) = \frac{1}{\mu_m}\,\varphi.$$

Then, this G is compact (as $H(X)_0^{p,q}$ is an orthogonal coproduct of its eigenspaces and they are finite dimensional, its bound is $\leq \frac{1}{\mu_1}$),

$$\Box G = G\Box,$$

as this holds on each piece. Consider $\varphi \in H(X)_0^{p,q}$ and look at $\varphi - \varkappa(\varphi)$. We have

$$G\Box(\varphi - \varkappa(\varphi)) = \Box G(\varphi - \varkappa(\varphi)) = \varphi - \varkappa(\varphi).$$

As $\Box G(\varphi) = \Box(G\varphi) = \Box G(\varphi - \varkappa(\varphi)) = \varphi - \varkappa(\varphi)$, we get

$$\varphi = \varkappa(\varphi) + \Box G\varphi$$

i.e., $I = \varkappa + \Box G$. $\Box$

We get a raft of corollaries.

Theorem 2.40 *(Hodge–Dolbeault Isomorphism and Finiteness Theorem) If X is a complex and compact manifold, then the spaces $H^q(X, \Omega_X^p)$ are finite dimensional for all $p, q \geq 0$ and we have the isomorphisms*

$$\mathcal{H}^{p,q}(X) \cong H_{\overline{\partial}}^{p,q}(X) \cong H^q(X, \Omega_X^p) \cong \check{H}^q(X, \Omega_X^p).$$

(Recall that $\Box\varphi = 0$, i.e., $\varphi \in \mathcal{H}^{p,q}(X)$, iff $\overline{\partial}\varphi = \overline{\partial}^*\varphi = 0$ iff φ is $\overline{\partial}$-closed and $\|\varphi\|$ is an absolute minimum in its $\overline{\partial}$-cohomology class.)

Corollary 2.41 *(Riemann's Theorem on Meromorphic Functions) Say X is a compact Riemann surface (1-dimensional complex, compact, manifold). Given $\zeta_1, \ldots, \zeta_t$, distinct points in X and integers $a_1, \ldots, a_t \geq 0$, if $\sum_{j=1}^t a_j \geq g+1$, where $g = \dim H^1(X, \mathcal{O}_X)$, then there exists a non-constant meromorphic function, f, on X having poles only at $\zeta_1, \ldots, \zeta_t$ and for all i, $1 \leq i \leq t$, the order of the pole of f at ζ_i is at most a_i.*

Proof. Note that $\dim H^1(X, \mathcal{O}_X)$ is finite by Hodge-Dolbeault (case $p = 0$, $q = 1$). At each ζ_i, pick a small open neighborhood, U_i, and an analytic isomorphism, $\varphi_i \colon U_i \to \Delta(0,1)$, where $\Delta(0,1) = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\varphi_i(\zeta_i) = 0$. Choose the U_i small enouh so that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Let $V = X - \{\zeta_1, \ldots, \zeta_t\}$. Then, $\{U_1, \ldots, U_t, V\}$ is an open cover of X. We know

$$H^1(\{U_i, V \longrightarrow X\}, \mathcal{O}_X) \hookrightarrow \check{H}^1(X, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X).$$

Therefore, $\dim H^1(\{U_i, V \longrightarrow X\}, \mathcal{O}_X) \leq g$. Consider the 1-cocycles

$$\psi_k^{(i)} = \left(\frac{1}{\varphi_i}\right)^k \quad \text{on} \quad U_i \cap V,$$

where $1 \leq i \leq t$; $1 \leq k \leq a_i$. Consider k; then, we have $a_1 + \cdots + a_t$ cocycles, which implies that there are at least $g+1$ cocycles. But, the dimension of $H^1(\{U_i, V \longrightarrow X\}, \mathcal{O}_X)$ is at most g, so these cocycles yield linearly dependent cohomology classes and we deduce that there exist some $c_{ik} \in \mathbb{C}$ so that the sum

$$F = \sum_{\substack{1 \leq i \leq t \\ 1 \leq k \leq a_i}} c_{ik}\psi_k^{(i)}$$

is a coboundary. Therefore, $F = g_i - g_V$ on $U_i \cap V$, where $g_i \in \Gamma(U_i, \mathcal{O}_X)$ and $g_V \in \Gamma(V, \mathcal{O}_X)$, for all i. Now, set

$$f = \begin{cases} \sum_{i,k=1}^{t,a_i} c_{ik}\psi_k^{(i)} - g_l & \text{on } U_l \cap V \\ -g_V & \text{on } V. \end{cases}$$

Observe that f is meromorphic and the local definitions agree on the overlaps, $U_l \cap V$, by choice, so f is indeed globally defined. The poles of $\psi_k^{(i)}$ are only ζ_i with order $\leq a_i$, which concludes the proof. $\Box$

What is going on?

Say X is a complex manifold and let $\mathcal{M}_X$ be the sheaf of germs of meromorphic functions. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{P}_X \longrightarrow 0, \tag{$*$}$$

where $\mathcal{P}_X$ is the sheaf cokernel of the sheaf map $\mathcal{O}_X \longrightarrow \mathcal{M}_X$. Pick an open, U, and look at $\Gamma(U, \mathcal{P}_X)$. By definition, $\sigma \in \Gamma(U, \mathcal{P}_X)$ iff there is an open cover, $\{U_\alpha \longrightarrow U\}$, so that $\sigma_\alpha = \sigma \upharpoonright U_\alpha$. Lift each σ_α to a meromorphic function, $f_\alpha \in \Gamma(U_\alpha, \mathcal{M}_X)$. On overlaps, $f_\alpha - f_\beta$ goes to zero on passing to $\Gamma(U_\alpha \cap U_\beta, \mathcal{P}_X)$. It follows that $f_\alpha - f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$. Therefore, $\Gamma(U, \mathcal{P}_X)$ is the set of pairs

$$\{\langle \{U_\alpha \longrightarrow U\}, f_\alpha \in \Gamma(U, \mathcal{M}_X)\rangle \mid f_\alpha - f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)\}.$$

This set is called *Cousin data of type* 1 *for the open* U.

Cousin Type 1 *Problem*: Given Cousin data of type 1 on U, does there exist a meromorphic function, f, where $f \in \Gamma(U, \mathcal{M}_X)$, so that $f \upharpoonright U_\alpha = f_\alpha$?

Write down the cohomology sequence for $(*)$ over U:

$$0 \longrightarrow \Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U, \mathcal{M}_X) \longrightarrow \Gamma(U, \mathcal{P}_X) \xrightarrow{\delta} H^1(U, \mathcal{O}_X).$$

Consequentely, Cousin 1 is solvable iff $H^1(U, \mathcal{O}_X) = (0)$.

Example 1. Take $U = \mathbb{C}^1$, then the Mittag-Leffler theorem holds iff $H^1(\mathbb{C}^1, \mathcal{O}_{\mathbb{C}^1}) = (0)$.

Example 2. $U = \mathbb{C}^n$. Again, OK.

Corollary 2.42 *(Case $t = 1$) Say X is a compact Riemann surface and $\zeta \in X$. Write $g = \dim H^1(X, \mathcal{O}_X)$. Then, there exists a non-constant meromorphic function, f, on X having a pole only at ζ and the order of the pole is at most $g + 1$.*

Consider a compact, complex, manifold, X, with a metric, so we have $*$ and look at $*\Box$ (on (p, q)-forms):

$$\begin{aligned}
*\Box &= *(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) \\
&= *\overline{\partial}(-*\overline{\partial}*) + *(-*\overline{\partial}*)\overline{\partial} \\
&= -(*\overline{\partial}*)\overline{\partial}* + (-1)^{p+q+1}\overline{\partial}*\overline{\partial} \\
&= \overline{\partial}^*\overline{\partial}* + \overline{\partial}*\overline{\partial}(-1)^{p+q+1} \\
&= \overline{\partial}^*\overline{\partial}* + \overline{\partial}(-*\overline{\partial}*)* \\
&= \Box *.
\end{aligned}$$

Say $\xi \in \mathcal{H}^{p,q}(X)$, then $\Box\xi = 0$ implies $*\Box(\xi) = 0$, and from the above, $\Box(*\xi) = 0$. Therefore, $*\xi \in \mathcal{H}^{n-p,n-q}(X)$ and we get an isomorphism

$$*\colon \mathcal{H}^{p,q}(X) \to \mathcal{H}^{n-p,n-q}(X).$$

This isomorphism depends on the metric.

Is there a canonical choice? Answer: No.

Is there a duality? Answer: Yes.

Take $p = q = 0$, then we have
$$*\colon \mathcal{H}^{0,0}(X) \to \mathcal{H}^{n,n}(X).$$
But, X is compact and connected, so $\mathcal{H}^{0,0}(X) = \mathbb{C}$. As $*1 = \Phi$ (the volume form) in $\mathcal{H}^{n,n}(X)$, we deduce that
$$\mathcal{H}^{n,n}(X) = \mathbb{C} \cdot \Phi.$$
Now, we have the pairing
$$\Omega_X^p \otimes \Omega_X^{p'} \longrightarrow \Omega_X^{p+p'}, \quad \text{where} \quad \xi \otimes \eta \mapsto \xi \wedge \eta.$$
The above induces the cup-product on cohomology:
$$H^\bullet(X, \Omega_X^p) \otimes H^\bullet(X, \Omega_X^{p'}) \longrightarrow H^{\bullet+\bullet}(X, \Omega_X^{p+p'}).$$
By Dolbeault, $H_{\overline{\partial}}^{p,q}(X) \cong H^q(X, \Omega_X^p)$. But, we have the pairing on the groups in the Dolbeault complex:
$$\bigwedge^{p,q}(X) \otimes \bigwedge^{p',q'}(X) \longrightarrow \bigwedge^{p+p',q+q'}(X), \quad \text{where} \quad \xi \otimes \eta \mapsto \xi \wedge \eta.$$
Moreover, $\overline{\partial}(\xi \wedge \eta) = \overline{\partial}\xi \wedge \eta + (-1)^{\deg \xi}\, \xi \wedge \overline{\partial}\eta$, an this implies (DX, elementary homological algebra) that we get the pairing
$$H_{\overline{\partial}}^{p,q}(X) \otimes H_{\overline{\partial}}^{p',q'}(X) \longrightarrow H_{\overline{\partial}}^{p+p',q+q'}(X)$$
and the diagram
$$\begin{array}{ccc}
H_{\overline{\partial}}^{p,q}(X) \otimes H_{\overline{\partial}}^{p',q'}(X) & \longrightarrow & H_{\overline{\partial}}^{p+p',q+q'}(X) \\
\downarrow \qquad\quad \downarrow & & \downarrow \\
H^q(X, \Omega_X^p) \otimes H^{q'}(X, \Omega_X^{p'}) & \longrightarrow & H^{q+q'}(X, \Omega_X^{p+p'})
\end{array}$$
commutes up to sign. This will give us the theorem

Theorem 2.43 *(Serre Duality–First Case) Let X be a compact, complex manifold of complex dimension n, then*

(1) There exists a canonical isomorphism (trace map)
$$H^n(X, \Omega_X^n) \overset{\mathrm{tr}}{\cong} \mathbb{C},$$
and

(2) The cup-product pairings
$$H^q(X, \Omega_X^p) \otimes H^{n-q}(X, \Omega_X^{n-p}) \longrightarrow H^n(X, \Omega_X^n) \xrightarrow{\mathrm{tr}} \mathbb{C}$$
are perfect dualities of finite dimensional vector spaces.

Proof. Define $\mathrm{tr}\colon H^n(X, \Omega_X^n) \longrightarrow \mathbb{C}$ as follows: Take $\zeta \in H^n(X, \Omega_X^n)$ and represent it by a C^∞ (n,n)-form, $\eta \in \bigwedge^{n,n}(X)$. Define
$$\mathrm{tr}(\zeta) = \int_X \eta.$$

Another lift is $\eta + \overline{\partial}\xi$, where $\xi \in \bigwedge^{n,n-1}(X)$. We have

$$\int_X (\eta + \overline{\partial}\xi) = \int_X \eta + \int_X \overline{\partial}\xi.$$

But, $d\xi = \partial\xi + \overline{\partial}\xi$ and $\partial\xi = 0$ (since $\xi \in \bigwedge^{n,n-1}(X)$), so we deduce $d\xi = \overline{\partial}\xi$. Consequently,

$$\int_X \overline{\partial}\xi = \int_X d\xi = \int_{\partial X} \xi = 0,$$

since X has no boundary (by Stokes). Therefore, the trace map exists canonically. Pick a metric on X, then Hodge implies

$$H^{n,n}_{\overline{\partial}}(X) \cong \mathcal{H}^{n,n}(X) = \mathbb{C} \cdot \Phi,$$

and we conclude that

$$\mathrm{tr}(\Phi) = \int_X \Phi = \mathrm{Vol}(X) > 0.$$

Therefore, $\mathrm{tr}(\Phi) \neq 0$ and tr is a nonzero linear map between two 1-dimensional spaces, so it must be an isomorphism.

(2) We already have our pairing:

$$\xi \otimes \eta \mapsto \int_X \xi \wedge \eta.$$

Put a metric on X, then Hodge says our pairing is the same pairing but considered on

$$\mathcal{H}^{p,q}(X) \otimes \mathcal{H}^{n-p,n-q}(X) \longrightarrow \mathcal{H}^{n,n}(X).$$

Now, given ξ, take $*\xi$, then

$$\xi \otimes *\xi \mapsto \int_X \xi \wedge *\xi = \|\xi\|_{L^2} > 0$$

(on (p,q)-forms). Since $\xi \neq 0$ implies that there is some η so that $\int_X \xi \wedge \eta \neq 0$, our pairing is nondegenerate. $\square$

Corollary 2.44 *If X is the Riemann sphere, $S^2 = \mathbb{P}^1_{\mathbb{C}}$, then the genus of X is 0.*

Proof. We must prove $H^1(X, \mathcal{O}_X) = (0)$. By Serre duality, this means $H^0(X, \Omega^1_X) = (0)$, i.e., the sphere carries no nonzero global holomorphic 1-forms. Cover $\mathbb{P}^1_{\mathbb{C}}$ by its two patches, parameters z and w (opens, U, V and on $U \cap V$, $w = 1/z$). If ω is a global holomorphic 1-form, then

$$f \restriction U = f(z)dz, \quad f \text{ entire on } \mathbb{C}$$

and

$$f \restriction V = g(w)dw, \quad g \text{ entire on } \mathbb{C}.$$

On $U \cap V$, as $w = 1/z$, we have $dw = -(1/z^2)dz$, so

$$f(z)dz = g\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)dz,$$

i.e.,

$$f(z) = g\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right) \quad \text{on } U \cap V.$$

If we let z go to ∞ (i.e., $1/z \longrightarrow 0$), the right hand side goes to 0, as g is entire. Therefore, f is bounded and entire and similarly, g is bounded and entire. By Liouville, both f and g are constant. But,

$$f = g\left(-\frac{1}{z^2}\right) \quad \text{on } U \cap V$$

only if $f = g = 0$. $\square$

Corollary 2.45 *On $S^2 = \mathbb{P}^1_{\mathbb{C}}$, for each ζ, there exists a non-constant meromorphic function, f, having exactly a pole of order 1 at ζ and no other pole. The function f may be replaced by $\alpha f + \beta$, for $\alpha, \beta \in \mathbb{C}$ and no other replacements are possible.*

Proof. By Riemann, as $g = 0$, there is a meromorphic function, f, with a pole of order at most 1 at ζ and no other poles. Consequently, this pole must be of order 1 at ζ. Say g is another function with the same property. Near ζ, we have

$$f(z) = \frac{a}{z-\zeta} + \text{holo. function} \quad \text{and} \quad g(z) = \frac{b}{z-\zeta} + \text{holo. function},$$

with $a, b \neq 0$. Then, $g - \frac{b}{a} f$ has no pole at ζ and no other poles, which means that it must be constant. Therefore, $g = \frac{b}{a} f + \beta$, as claimed. $\square$

Corollary 2.46 *If X is a compact, complex manifold, then we have*

(1) $h^{n,n} = 1$, where $n = \dim_{\mathbb{C}} X$.

(2) $h^{p,q} < \infty$, for all $p, q \geq 0$.

(3) $h^{n-p,n-q} = h^{p,q}$,

where $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p_X)$ (the qp-th Hodge number of X).

The same kind of argument as we've used shows the pairing

$$H^r_{\mathrm{DR}}(X, \mathbb{C}) \otimes H^{n-r}_{\mathrm{DR}}(X, \mathbb{C}) \longrightarrow \mathbb{C}, \quad \text{where} \quad s \otimes t \mapsto \int_X s \wedge t \tag{$\dagger$}$$

is (for X compact) non-degenerate. Simply take, given s, the form $t = *s$, where $*$ is computed *mutatis mutandis* for d just as for $\overline{\partial}$. We'll show (just below) these de Rham groups are finite dimensional vector spaces over $\mathbb{C}$. We deduce

Theorem 2.47 *If X is a compact, complex manifold and $n = \dim_{\mathbb{C}} X$, then the pairing ($\dagger$) is an exact duality of finite dimensional vector spaces.*

We know from topology the Künneth formula for cohomology of a product

$$H^l(X \prod Y) \cong \coprod_{r+s=l} H^r(X) \otimes H^s(Y).$$

This actually holds for forms in the compact, complex case. Pick, X, Y, compact, complex, then we have maps

$$H^q(X \text{ or } Y, \Omega^p_X \text{ or } \Omega^p_Y) \longrightarrow H^q(X \prod Y, \Omega^p_{X \prod Y}),$$

by pr_1^*, or pr_2^*. This gives a map

$$\coprod_{\substack{p+p'=a \\ q+q'=b}} H^q(X, \Omega^p_X) \otimes H^{q'}(Y, \Omega^{p'}_Y) \longrightarrow H^b(X \prod Y, \Omega^a_{X \prod Y}). \tag{$*$}$$

Theorem 2.48 *(Künneth for forms (or $\overline{\partial}$-cohomology)) If X, Y are compact, complex manifolds, then $(*)$ is an isomorphism (for all p, p', q, q', a, b).*

Proof. Pick Hermitian metrics for X and Y and give $X\coprod Y$ the product metric. Then, Hodge's theorem reduces $(*)$ to the statement:

$$\coprod_{\substack{p+p'=a\\ q+q'=b}} \mathcal{H}^{p,q}(X)\otimes\mathcal{H}^{p',q'}(Y)\longrightarrow \mathcal{H}^{a,b}(X\coprod Y).$$

Step 1. (Enough forms). Notation: z's and w's for coordinates in X, resp. Y. If ξ is a (p,q)-form on X and η is a (p',q')-form on Y, then

$$\bigwedge^{p,q}(X)\otimes\bigwedge^{p',q'}(Y)\longrightarrow \bigwedge^{p+p',q+q'}(X\coprod Y)\quad \textit{via}\quad \xi\otimes\eta\mapsto \xi(z'\mathrm{s})\wedge\eta(w'\mathrm{s}).$$

Call the forms in $\bigwedge^{a,b}(X\coprod Y)$ of the right hand side *decomposable* or *pure* forms. I claim the decomposable forms are L^2-dense. Namely, choose $\zeta\in\bigwedge^{a,b}(X\coprod Y)$ and say

$$\int_{X\coprod Y}\zeta(z,w)\wedge *(\xi(z)\wedge\eta(w))=0,\quad \text{for all } \xi,\eta.$$

if $\zeta(z_0,w_0)\neq 0$, by multiplication by $e^{i\theta}$, for some θ, we get $\Re(\zeta(z_0,w_0))>0$. So, there is some neighborhood of (z_0,w_0) where $\Re(\zeta)>0$ and we can even assume that this neighborhood is $U\coprod V$, with $z_0\in U$ and $w_0\in V$. Pick forms, ξ,η with compact support in U, resp. V, and arrange (use another $e^{i\varphi}$) so that

$$\Re(\zeta(z_0,w_0)\wedge *(\xi(z_0)\wedge\eta(w_0)))>0$$

and cut by a bump function so that the form $\zeta\wedge *(\xi\wedge\eta)=0$ outside $U'\coprod V'$, with $U'\subseteq U$, $V'\subseteq V$ and $\Re(\text{form})>0$ on $U'\coprod V'$. Then,

$$0=\int_{X\coprod Y}\zeta\wedge *(\xi\wedge\eta)=\int_{U'\coprod V'}\zeta\wedge *(\xi\wedge\eta)>0,$$

a contradiction.

Step 2. ($\square_{X\coprod Y}$). We know that we have the uniholo connections on X, Y and $X\coprod Y$, so that

$$\overline{\partial}_{X\coprod Y}=\overline{\partial}_X+\overline{\partial}_Y. \tag{††}$$

A unitary coframe for $X\coprod Y$ (locally) has the form

$$(\varphi_1(z),\ldots,\varphi_l(z),\psi_1(w),\ldots,\psi_m(w)),$$

where the φ_i's and ψ_j's are unitary coframes on the factors. The Hodge $*$ on $X\coprod Y$ is computed separatly from $*$ on X and $*$ on Y. Since $\overline{\partial}^*_X=-*\overline{\partial}_X*$ and $\overline{\partial}^*_Y=-*\overline{\partial}_Y*$, we get

(a) $\overline{\partial}^*_{X\coprod Y}=\overline{\partial}^*_X+\overline{\partial}^*_Y$;

(b) $\overline{\partial}^*_X\overline{\partial}_Y+\overline{\partial}_Y\overline{\partial}^*_X=0$;

(c) $\overline{\partial}^*_Y\overline{\partial}_X+\overline{\partial}_X\overline{\partial}^*_Y=0$.

The last two hold because $\overline{\partial}_X\overline{\partial}_Y+\overline{\partial}_Y\overline{\partial}_X=0$. Then (††) and (a), (b), (c) imply

$$\square_{X\coprod Y}=\overline{\partial}^*_{X\coprod Y}\overline{\partial}_{X\coprod Y}+\overline{\partial}_{X\coprod Y}\overline{\partial}^*_{X\coprod Y}=\square_X+\square_Y,$$

i.e.,

$$\square_{X\coprod Y}(\xi\otimes\eta)=(\square_X\xi)\otimes\eta+\xi\otimes(\square_Y\eta).$$

So, if $\Box_X(\xi) = \lambda\xi$ and $\Box_Y(\eta) = \mu\eta$, we deduce

$$\Box_{X\prod Y}(\xi\otimes\eta) = \lambda(\xi\otimes\eta) + \mu(\xi\otimes\eta) = (\lambda+\mu)(\xi\otimes\eta).$$

As decomposable forms are dense, the equation $\Box_{X\prod Y} = \Box_X + \Box_Y$ (on decomposables) determines $\Box_{X\prod Y}$. The decomposable eigenforms for $\Box_{X\prod Y}$ being dense by Step 1 and the density of the separate eigenforms for X and Y show that

(i) The eigenvalues of $\Box_{X\prod Y}$ are exactly $\lambda_i + \mu_j$, $0 \le i, j$.

(ii) The decomposable eigenforms are an L^2-basis on $X\prod Y$.

Step 3. (Harmonicity). Take ζ, an harmonic $X\prod Y$-form, say $\zeta = \xi\otimes\eta$. We have

$$0 = \Box_{X\prod Y}(\zeta) = (\lambda+\mu)(\xi\otimes\eta) = (\lambda+\mu)\zeta.$$

But, we know that $\lambda, \mu \ge 0$ and $\lambda+\mu = 0$ implies $\lambda = \mu = 0$. Therefore, ξ and η are also harmonic. $\Box$

Proposition 2.49 *If X, Y are compact, complex manifolds, then*

$$h(X\prod Y)^{a,b} = \sum_{\substack{p+p'=a\\ q+q'=b}} h(X)^{p,q}h(Y)^{p',q'}.$$

Now, we have a double complex, for a complex manifold, X, (not necessarily compact)

$$\bigwedge^{p,q}(X) = \Gamma(X, \bigwedge^{p,q} T_X^D),$$

with (partial) differentials $\partial, \overline{\partial}$ and total differential $d = \partial + \overline{\partial}$. Make the associated total complex, $K^\bullet$, where

$$K^l = \coprod_{p+q=l} \bigwedge^{p,q}(X).$$

It has differential d. We know that the cohomology of $K^\bullet$ is exactly the de Rham cohomology of X (with coefficients in $\mathbb{C}$):

$$H^r(K^\bullet) = H^r_{\mathrm{DR}}(X, \mathbb{C}).$$

A double complex always comes with a filtration (actually two)

$$K^l \supseteq F^pK^l = \coprod_{\substack{p'\ge p\\ p'+q=l}} \bigwedge^{p',q}(X).$$

(Locally, a form is in F^pK^l iff it has degre l and at least p of the $dz_1, \ldots, dz_n$). Check, $F^pK^\bullet \subseteq K^\bullet$ is a subcomplex (under d). So, we get maps

$$H^r(F^pK^\bullet) \longrightarrow H^r(K^\bullet) = H^r_{\mathrm{DR}}(X, \mathbb{C}).$$

By definition, the image of $H^r(F^pK^\bullet)$ is $F^pH^r_{\mathrm{DR}}(X, \mathbb{C})$ and this gives a decreasing filtration of $H^r_{\mathrm{DR}}(X, \mathbb{C})$, for each $r \ge 0$. This is the *Hodge filtration* on (de Rham) cohomology.

The double complex gives a spectral sequence from our Hodge filtration: The ending is

$$E_\infty^{p,q} = \mathrm{gr}_{p,q} H^\bullet_{\mathrm{DR}}(X, \mathbb{C}) = F^pH^{p+q}_{\mathrm{DR}}(X, \mathbb{C})/F^{p+1}H^{p+q}_{\mathrm{DR}}(X, \mathbb{C}).$$

It begins at the cohomology $H^\bullet(\mathrm{gr}(K^\bullet))$ taken with respect to $\overline{\partial}$. Recall

$$\mathrm{gr}_s(K^l) = \coprod_{\substack{p\geq s\\ p+q=l}} \bigwedge^{p,q}(X) / \coprod_{\substack{p\geq s+1\\ p+q=l}} \bigwedge^{p,q}(X) = \bigwedge^{s,q}(X),$$

with $s+q=l$. Therefore, $\mathrm{gr}_s(K^\bullet) = \bigwedge^{s,\bullet-s}(X)$ and we take the cohomology w.r.t. $\overline{\partial}$. Therefore, we have

$$E_1^{p,q} = H_{\overline{\partial}}^{p,q}(X),$$

the Dolbeault cohomology. We deduce the

Theorem 2.50 *(Fröhlicher) There exists for a complex manifold, X, a spectral sequence (the Hodge to de Rham, S.S., notation HDRSS)*

$$H_{\overline{\partial}}^{p,q}(X) = E_1^{p,q} \underset{p}{\Longrightarrow} H_{\mathrm{DR}}^{p+q}(X,\mathbb{C}).$$

Remarks:

(1) Assume that $E_1^{p,q} = H_{\overline{\partial}}^{p,q}(X)$ is finite-dimensional for all $p,q\geq 0$, which holds if X is compact. We know that $E_2^{p,q}$ comes from $E_1^{p,q}$ by using ∂ on $E_1^{p,q}$:

$$E_{r+1}^{p,q} = Z_r^{p,q}/B_r^{p,q}; \quad \text{where} \quad B_r^{p,q} \subseteq Z_r^{p,q} \subseteq E_r^{p,q}.$$

Therefore, $\dim_{\mathbb{C}} E_{r+1}^{p,q} \leq \dim_{\mathbb{C}} E_r^{p,q}$, for all $r\geq 1$ and all $p,q\geq 0$.

(2) $\dim_{\mathbb{C}} E_{r+1}^{p,q} = \dim_{\mathbb{C}} E_r^{p,q}$, for all $p,q\geq 0$, iff $d_r = 0$.

(3) We have $\dim_{\mathbb{C}} E_\infty^{p,q} \leq h^{p,q}$, for all p,q.

Let us look at the spaces $E_\infty^{p,q}$, where $p+q=l$. We have

$$F^l H_{\mathrm{DR}}^l(X,\mathbb{C}) = E_\infty^{l,0} \quad \text{and} \quad F^{l-1}H_{\mathrm{DR}}^l(X,\mathbb{C})/F^l H_{\mathrm{DR}}^l(X,\mathbb{C}) = E_\infty^{l-1,0}, \cdots.$$

We know $b_l = \dim H_{\mathrm{DR}}^l(X,\mathbb{C}) = l$-th Betti number of X and so,

$$b_l = \sum_{j=0}^{l} \dim E_\infty^{j,l-j} \leq \sum_{p+q=l} h^{p,q}.$$

So, we get: HDRSS degenerates at $r=1$ iff for every l, with $0\leq l\leq 2\dim_{\mathbb{C}} X$, $b_l = \sum_{p+q=l} h^{p,q}$ and then, (non-canonically, perhaps)

$$H_{\mathrm{DR}}^l(X,\mathbb{C}) \overset{\sim}{\longleftarrow} \coprod_{p+q=l} H_{\overline{\partial}}^{p,q}(X) \overset{\sim}{\longleftarrow} \coprod_{p+q=l} H^q(X,\Omega_X^p).$$

We know for a complex, the Euler characteristic, if defined, is equal to the Euler characteristic using the cohomology. As each E_r is the cohomology of the previous E_{r-1}, we get

$$\chi(X,\mathbb{C}) = \sum_{l=0}^{2\dim_{\mathbb{C}} X} (-1)^l b_l = \sum_{p,q} (-1)^{p+q} h^{p,q}.$$

Further, the inequality $b_l \leq \sum_{p+q=l} h^{p,q}$ implies in the compact case that all the de Rham cohomology groups are finite dimensional. We summarize all this as:

Theorem 2.51 *(Fröhlicher) There exists a spectral sequence, the HDRSS,*

$$H^{p,q}_{\overline{\partial}}(X) = E_1^{p,q} \underset{p}{\Longrightarrow} H^{p+q}_{\mathrm{DR}}(X, \mathbb{C}).$$

When X is compact we find that

(a) $\dim E_\infty^{p,q} \leq h^{p,q}$, *for all $p, q \geq 0$.*

(b) $b_l \leq \sum_{p+q=l} h^{p,q}$ *(Fröhlicher inequality)*

(c) $\chi(X, \mathbb{C}) = \sum_{l=0}^{2\dim_{\mathbb{C}} X} (-1)^l b_l = \sum_{p,q} (-1)^{p+q} h^{p,q}$ *(Fröhlicher relation)*

In the compact case, a n.a.s.c. that the HDRSS degenerates at $r = 1$ is that the inequalities in (b) are equalities for all l $(0 \leq l \leq 2\dim_{\mathbb{C}} X)$ and in this case, we have the (perhaps non-canonical) Hodge decomposition

$$H^l_{\mathrm{DR}}(X, \mathbb{C}) \xleftarrow{\sim} \coprod_{p+q=l} H^q(X, \Omega^p_X).$$

Remarks and Applications.

(A) Let X be a Riemann surface. Here, $\dim_{\mathbb{C}} = 1$, so we only have $h^{0,0} = 1$, $h^{0,1}, h^{1,0}$ and $h^{1,1} = 1$. We know if X is compact, $b_0 = 1$, $b_2 \geq 1$ (there is a volume form), but $b_2 \leq h^{1,1}$, by Fröhlicher, so $b_2 = 1$. By Serre duality, $h^{0,1} = h^{1,0} = \dim H^1(X, \mathcal{O}_X) = g$, the geometric genus of X. By Fröhlicher (c), we have

$$b_0 - b_1 + b_2 = h^{0,0} - (h^{0,1} + h^{1,0}) + h^{1,1},$$

i.e.,

$$2 - b_1 = 2 - 2g,$$

So, we conclude $b_1 = 2g$. Topologically, X is a sphere with m handles, therefore we get:

Corollary 2.52 *For a compact Riemann surface, X, the three numbers*

(a) $g = \dim H^1(X, \mathcal{O}_X) =$ *geometric genus;*

(b) $\dim H^0(X, \Omega^1_X) =$ *analytic genus (number of linearly independent holomorphic 1-forms (Riemann), and*

(c) The topological genus = number of handles describing X

are the same and we have the Hodge decomposition

$$H^l_{\mathrm{DR}}(X, \mathbb{C}) \xleftarrow{\sim} \coprod_{p+q=l} H^q(X, \Omega^p_X).$$

(B) $X =$ a complex, compact surface, X: $\dim_{\mathbb{C}} X = 2$. We have $b_0 = h^{0,0} = 1$ and $b_4 \leq h^{2,2} = 1$; but, $b_4 \geq 1$ (there is a volume form), so $b_4 = h^{2,2} = 1$ (this is also true in dimension n). We have

$$\begin{aligned} b_1 &\leq h^{1,0} + h^{0,1} \\ b_2 &\leq h^{2,0} + h^{1,1} + h^{0,2} \\ b_3 &\leq h^{2,1} + h^{1,2}. \end{aligned}$$

Poincaré duality says $b_1 = b_3$ and Serre duality says $h^{1,0} + h^{1,0} = h^{2,1} + h^{1,2}$. Again, Serre duality says $h^{2,0} = h^{0,2}$ and further

$$\begin{aligned} \chi(X, \mathbb{C}) &= b_0 - b_1 + b_2 - b_3 + b_4 \\ &= 2 - 2b_1 + b_2 \\ &= 2 - 2(h^{1,0} + h^{0,1}) + 2h^{2,0} + h^{1,1}. \end{aligned}$$

Hence, we have equivalent conditions for complex surfaces:

(i) $b_1 = h^{1,0} + h^{0,1}$;

(ii) $b_2 = 2h^{2,0} + h^{1,1}$;

(iii) $b_3 = h^{2,1} + h^{1,2}$;

(iv) HDRSS degenerates at $r = 1$ and Hodge decomposition for de Rham cohomology.

The following nomenclature is customary:

1. $p_g = \dim H^2(X, \mathcal{O}_X) = h^{0,2} =$ *geometric genus* of X;
2. $q = \dim H^1(X, \mathcal{O}_X) = h^{0,1}$, the *irregularity* of X;
3. $p_g - q = p_a =$ *arirthmetic genus* $= \chi(X, \mathcal{O}_X) - 1$.

(C) Some compact, complex surfaces. Look at $\lambda \in \mathbb{R}$, with $0 < \lambda < 1$ and $\mathbb{C}^2 - \{0\}$. Let $\Gamma \cong \mathbb{Z}$, i.e., $\Gamma = \{\lambda^n \mid n \in \mathbb{Z}\}$. We make Γ act on $\mathbb{C}^2 - \{0\}$ and we get $\Gamma \backslash (\mathbb{C}^2 - \{0\}) = X_\lambda$, a complex surface. Now $\mathbb{C}^2 - \{0\} \cong \mathbb{R}^+ \coprod S^3$, and λ operates on the $\mathbb{R}^+$-factor. Therefore,

$$X_\lambda \xrightarrow{\sim} S^1 \coprod S^3,$$

a compact suface. We can compute cohomology by Künneth:

$$\begin{array}{rcl}
H^0_{\mathrm{DR}}(X, \mathbb{C}) & \cong & \mathbb{C}; \\
H^4_{\mathrm{DR}}(X, \mathbb{C}) & \cong & \mathbb{C}; \\
H^1_{\mathrm{DR}}(X, \mathbb{C}) & \cong & \displaystyle\coprod_{p+q=1} H^p_{\mathrm{DR}}(S^1, \mathbb{C}) \otimes H^q_{\mathrm{DR}}(S^3, \mathbb{C}) \cong \mathbb{C}; \\
H^3_{\mathrm{DR}}(X, \mathbb{C}) & \cong & \mathbb{C}; \quad \text{(Poincaré duality)} \\
H^2_{\mathrm{DR}}(X, \mathbb{C}) & \cong & \displaystyle\coprod_{p+q=2} H^p_{\mathrm{DR}}(S^1, \mathbb{C}) \otimes H^q_{\mathrm{DR}}(S^3, \mathbb{C}) = (0).
\end{array}$$

We know $1 = b_1 \leq h^{0,1} + h^{1,0}$ (at least one ≥ 1), $0 \leq 2h^{0,0} + h^{1,1}$. Now, $\chi(X, \mathbb{C}) = 0$. Therefore,

$$2 - 2(h^{0,1} + h^{1,0}) + 2h^{2,0} + h^{1,1} = 0.$$

We can generalize X_λ as follows: Take complex λ_1, λ_2, with $0 \leq |\lambda_1| \leq |\lambda_2| < 1$ and make the abelian group Γ

$$z_1 \mapsto \lambda_1^m z_1, \qquad z_2 \mapsto \lambda_2^n z_2.$$

The group Γ acts on $\mathbb{C}^2 - \{0\}$ and we get the complex surface $X_{\lambda_1,\lambda_2} = \Gamma \backslash (\mathbb{C}^2 - \{0\})$.

Slight variation, choose $\lambda \in \mathbb{C}$, with $0 \leq |\lambda| < 1$; $k \in \mathbb{N}$ and act on $\mathbb{C}^2 - \{0\}$ by

$$z_1 \mapsto \lambda z_1, \qquad z_2 \mapsto \lambda z_2 + z_1^k.$$

We get a surface $X_{\lambda,k}$. These are the *Hopf surfaces*, X_{λ_1,λ_2} and $X_{\lambda,k}$.

(D) If X_t is a family of compact, complex n-manifolds, all are diffeomorphic and metrically equivalent, they may have different holomorphic (= complex) structures e.g., the Legendre family of elliptic curves

$$Y^2 = X(X-1)(X-t)).$$

All the de Rahm cohomology is the same, but the Hodge filtrations change, giving variations of the Hodge structure.

We can make from d and the metric, d^* and similarly, we can make from ∂ and the metric, ∂^*, and we can make from $\overline{\partial}$ and the metric, $\overline{\partial}^*$. So, we get $\square_d$, $\square_\partial$ and $\square_{\overline{\partial}} = \square$. In general, there are no relations among the three.

Even for the same manifold and the same $\overline{\partial}$, the wedge of two harmonic forms may be (in general is) *not* harmonic. Now, say $Y \hookrightarrow X$ (with X, Y compact), where Y is a complex submanifold, X metrized and Y has the induced structure, *still* a harmonic form on X when restricted to Y need *not* be harmonic. Such bad behavior does not happen with Kähler manifolds, the object of study of the next section.

2.5 Hodge III, The Kähler Case

Assume X is a hermitian manifold (= metrized and holomorphic). Both T_X and T_X^D are metrized bundles on X. We have the unique uniholo connection, ∇ on T_X and ∇^D on T_X^D:

$$\nabla^D\colon \Gamma(X, \mathcal{C}^\infty(X)) \longrightarrow \Gamma(X, T_X^{D1,0} \otimes T_X^D) \cong \Gamma(X, T_X^{D1,0} \otimes T_X^{D1,0}) \coprod \Gamma(X, T_X^{D1,0} \otimes T_X^{D0,1}).$$

Compare ∇^D with d:

$$d\colon \Gamma(X, T_X^{D1,0}) \longrightarrow \Gamma(X, T_X^{D2,0}) \coprod \Gamma(X, T_X^{D1,0} \otimes T_X^{D0,1}).$$

As $\nabla^{D\,0,1} = \overline{\partial}$, we get the same image on the second factor.

Write $ds^2 = \sum_{i,j} h_{ij}\, dz_i \otimes d\overline{z}_j = \sum_k \varphi_k \otimes \overline{\varphi}_k$, for the metric.

Claim. Everywhere locally, there exists a unique (given by the coordinates, z_j) matrix of 1-forms, ψ_{ij}, so that

$$\overline{\partial}\varphi_i = \sum_j \psi_{ij} \wedge \varphi_j$$

and the ψ_{ij} are computable in terms of the Gram-Schmidt matrix taking the dz_j's to the φ_k's.

Since the φ_j form a basis, existence and uniqueness is clear. Let α be the Gram-Schmidt matrix given by

$$\varphi_i = \sum_j \alpha_{ij} dz_j.$$

We get

$$\overline{\partial}\varphi_i = \sum_j \overline{\partial}\alpha_{ij} \wedge dz_j.$$

But,

$$dz_j = \sum_k \alpha_{jk}^{-1} \varphi_k,$$

so we get

$$\overline{\partial}\varphi_i = \sum_{j,k} \overline{\partial}\alpha_{ij} \wedge \alpha_{jk}^{-1}\varphi_k = \sum_k \Big(\sum_j \overline{\partial}\alpha_{ij}\, \alpha_{jk}^{-1}\Big) \wedge \varphi_k.$$

Therefore,

$$\psi = \overline{\partial}\alpha \cdot \alpha^{-1}.$$

Make a skew-hermitian matrix from ψ, call it Ψ, *via*:

$$\Psi^{0,1} = \psi; \quad \Psi^{1,0} = -\overline{\psi^\top} \quad \text{and} \quad \Psi = \Psi^{0,1} + \Psi^{1,0}.$$

Clearly, $\Psi = -\overline{\psi^\top} + \psi$ and it follows that

$$\Psi^\top = (-\overline{\psi^\top} + \psi)^\top = -\overline{\psi} + \psi^\top = -(\overline{\psi} - \psi^\top) = -\overline{(\psi - \overline{\psi^\top})} = -\overline{\Psi}.$$

This show that Ψ is indeed skew-hermitian. We also have

$$\begin{aligned} d\varphi_i &= \partial\varphi_i + \overline{\partial}\varphi_i = \partial\varphi_i + \sum_j \Psi_{ij}^{0,1} \wedge \varphi_j \\ &= \partial\varphi_i - \sum_j \Psi_{ij}^{1,0} \wedge \varphi_j + \sum_j \Psi_{ij} \wedge \varphi_j \\ &= \tau_i + \sum_j \Psi_{ij} \wedge \varphi_j. \end{aligned}$$

where $\tau_i = \partial\varphi_i - \sum_j \Psi_{ij}^{1,0} \wedge \varphi_j$ is a $(2,0)$-form.

The vector $\tau = (\tau_1, \ldots, \tau_n)$ is the *torsion* of the metric and its compoenents are the *torsion components*. Hence, the torsion of the metric vanishes iff

$$\partial\varphi_i = \sum_j \Psi_{ij}^{1,0} \wedge \varphi_j$$

and we always have

$$\overline{\partial}\varphi_i = \sum_j \Psi_{ij}^{0,1} \wedge \varphi_j.$$

Let θ (resp. θ^D) be the connection matrix for ∇ (resp. ∇^D) in a unitary frame and its dual, φ_i. We know

(1) $\theta^D = -\overline{(\theta^D)^\top}$;

(2) $\theta = -(\theta^D)^\top$;

(3) $\Psi = -\overline{\Psi}^\top$ (by construction);

Now, $\nabla^{D\,0,1} = \overline{\partial}^{0,1}$ and, locally in the coframe, the left hand side is wedge with $\theta^{D\,0,1}$ and the right hand side is wedge with $\Psi^{0,1}$ and both are skew-hermitian. Therefore, $\theta^D = \Psi$, i.e., $\theta = -\Psi^\top$. Hence we proved:

Proposition 2.53 *If X is a hermitian manifold with metric ds^2 and if in local coordinates $z_1, \ldots, z_n$, we have $ds^2 = \sum_{i,j} h_{ij}\, dz_i \otimes d\overline{z}_j$ and $\varphi_1, \ldots, \varphi_n$ form a unitary coframe for ds^2 with $\varphi = \sum_j \alpha_{ij} dz_j$ (where α = Gram-Schmidt matrix), then for the uniholo connections ∇ and ∇^D on T_X and T_X^D with connection matrices θ and θ^D in the unitary frame and coframe, we have:*

(1) The matrix $\Psi = -\overline{\psi^\top} + \psi$, where $\psi = \overline{\partial}\alpha \cdot \alpha^{-1}$ is skew-hermitian and

(2) $\overline{\partial}\varphi_i = \sum_j \Psi_{ij}^{0,1} \wedge \varphi_j$ while $\partial\varphi_i = \sum_j \Psi_{ij}^{1,0} \wedge \varphi_j + \tau_i$ and τ is a $(2,0)$-form;

(3) $d\varphi_i = \sum_j \Psi_{ij} \wedge \varphi_j + \tau_i$;

(4) $\theta = -\Psi^\top$ and $\theta^D = \Psi$;

(5) $h = \alpha^\top \cdot \overline{\alpha}$.

Moreover, $\tau = 0$ iff $d\varphi_i = \sum_j \Psi_{ij}^{1,0} \wedge \varphi_j$.

Proof. Only statement (5) remains to be proved. We know

$$\begin{aligned}
ds^2 &= \sum_{i,j} h_{ij}\, dz_i \otimes d\overline{z}_j = \sum_k \varphi_k \otimes \overline{\varphi}_k \\
&= \Big(\sum_l \alpha_{kl} dz_j\Big) \otimes \Big(\sum_m \overline{\alpha}_{km} d\overline{z}_m\Big) \\
&= \sum_{k,l} \alpha_{kl}\, \overline{\alpha}_{km}\, dz_j \otimes d\overline{z}_m \\
&= \sum_{k,l} \alpha^{\top}_{lk}\, \overline{\alpha}_{km}\, dz_j \otimes d\overline{z}_m \\
&= (\alpha^{\top} \cdot \overline{\alpha})_{lm}\, dz_j \otimes d\overline{z}_m.
\end{aligned}$$

Therefore, $h = \alpha^{\top} \cdot \overline{\alpha}$. $\square$

Example. Let X be a Riemann surface, not necessarily compact. There is a single local coordinate, z, and $\varphi = \alpha dz$, where we may assume $\alpha > 0$ after multiplication by some suitable complex number of the form $e^{i\beta}$. Thus,

$$ds^2 = \varphi \otimes \overline{\varphi} = \alpha^2\, dz \otimes d\overline{z},$$

and $h = \alpha^2$. We also have

$$\psi = \overline{\partial}\alpha \cdot \alpha^{-1} = \overline{\partial}(\log \alpha).$$

It follows that $\overline{\psi^{\top}} = \partial(\log \alpha)$, so

$$\Psi = (\overline{\partial} - \partial)(\log \alpha) \quad \text{and} \quad \theta = (\partial - \overline{\partial})(\log \alpha).$$

Since the curvature form, Θ, is given by

$$\Theta = d\theta - \theta \wedge \theta$$

and

$$\theta = \frac{\partial \log \alpha}{\partial z}\, dz - \frac{\partial \log \alpha}{\partial \overline{z}}\, d\overline{z},$$

we deduce that $\theta \wedge \theta = 0$. Therefore, $\Theta = d\theta$, i.e., $\Theta = (\partial + \overline{\partial})(\partial - \overline{\partial})(\log \alpha)$ and we get

$$\Theta = -2\partial\overline{\partial}(\log \alpha) = -2\frac{\partial^2 \log \alpha}{\partial z \partial \overline{z}}\, dz \wedge d\overline{z}.$$

But, $dz \wedge d\overline{z} = -2i dx \wedge dy$, so

$$\Theta = 4i\frac{\partial^2 \log \alpha}{\partial z \partial \overline{z}}\, dx \wedge dy$$

and

$$\Delta(\log \alpha) = 4\frac{\partial^2 \log \alpha}{\partial z \partial \overline{z}},$$

so we obtain

$$\Theta = i\Delta(\log \alpha)\, dx \wedge dy.$$

Recall that the $(1, 1)$-form, ω, associated with ds^2 is given by

$$\omega = \frac{i}{2}\, \varphi \wedge \overline{\varphi} = \frac{i}{2}\, \alpha^2\, dz \wedge d\overline{z} = \alpha^2\, dx \wedge dy,$$

so

$$i\Theta = -\Delta(\log \alpha)\, dx \wedge dy = K\omega,$$

where

$$K = -\frac{\Delta(\log \alpha)}{\alpha^2},$$

the Gaussian curvature. As $\dim_{\mathbb{C}} X = 1$, we get $\tau = 0$, for any Riemann surface.

Pick any hermitian metric, ds^2, on the complex manifold, X, and let

$$\omega = \frac{i}{2}\sum_k \varphi_k \wedge \overline{\varphi}_k = \frac{i}{2}\sum_{i,j} h_{ij}\, dz_i \wedge d\overline{z}_j$$

(in a unitary coframe, φ_i) be the corresponding $(1,1)$-form (determining ds^2).

Definition 2.11 The metric $ds^2 = \sum_{i,j} h_{ij}\, dz_i \wedge d\overline{z}_j$ is *Kähler* iff ω is a closed form (i.e, $d\omega = 0$). The complex hermitian manifold, X, is a *Kähler manifold* iff it possesses at least one Kähler metric.

Examples. Not every (even compact) complex manifold is Kähler, there are topological restrictions in the Kähler case. Say X is compact, Kähler and look at ω and ω^k, with $0 \leq k \leq n$. As each ω^k is closed we get a de Rham cohomology class in $H^{2k}_{\mathrm{DR}}(X, \mathbb{C})$. Now, $\frac{1}{n!}\omega^n$, the volume form, is given by

$$\frac{1}{n!}\omega^n = \bigwedge_{1\leq j\leq n} \frac{i}{2}\, dz_j \wedge d\overline{z}_j\ \det(h_{ij}) = \det(h_{ij}) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n > 0.$$

So,

$$\int_X \frac{1}{n!}\omega^n = \mathrm{Vol}(X) \in H^{2n}_{\mathrm{DR}}(X, \mathbb{C}) \cong \mathbb{C}, \quad \text{with} \quad \mathrm{Vol}(X) > 0.$$

It follows that $\int_X \omega^k \wedge \omega^{n-k} \neq 0$, so ω^k defines a nonzero class in $H^{2k}_{\mathrm{DR}}(X, \mathbb{C})$. We get

Proposition 2.54 *If X is a compact, complex, Kähler manifold, then for every k, $0 \leq k \leq n$, we have $H^{2k}_{\mathrm{DR}}(X, \mathbb{C}) \neq (0)$.*

The Hodge surface, X_λ, is compact, yet it is not Kähler, because $H^2_{\mathrm{DR}}(X, \mathbb{C}) = H^2_{\mathrm{DR}}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = (0)$.

Remarks:

(1) Every Riemann surface is Kähler. For, ω is a 2-form and $\dim_{\mathbb{R}} X = 2$, so $d\omega = 0$ (it is a 3-form).

(2) $\mathbb{P}^n_{\mathbb{C}}$ is Kähler, for every $n \geq 1$. We know that the Fubini-Study metric has an ω given by

$$\omega = \frac{i}{2\pi}\partial\overline{\partial} \log \|F\|^2,$$

where F is a holomorphic section: $U \subseteq \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{C}^{n+1} - \{0\}$. We have

$$d\omega = (\partial + \overline{\partial})(\partial\overline{\partial}(\ )) = \overline{\partial}\partial\overline{\partial}(\ ) = -\partial\overline{\partial}\,\overline{\partial}(\ ) = 0.$$

(3) If X is Kähler and Y is a complex submanifold of X, then Y is Kähler in the induced metric. Because the $(1,1)$-form of the induced metric is the pullback of the $(1,1)$-form of the parent metric, Y is Kähler and (2) & (3) imply (4):

(4) Every (compact) complex manifold embeddable in $\mathbb{P}^n_{\mathbb{C}}$, in particular, each projective algebraic variety, is Kähler.

(5) If X is Kähler and Y is Kähler then so is $X \prod Y$ in the product metric.

Theorem 2.55 *If X is a complex manifold and ds^2 is a metric on it with associated $(1,1)$-form ω, then the following are equivalent:*

(1) $d\omega = 0$, i.e., ds^2 is Kähler.

(2) Locally everywhere, there exists a C^∞-function, α, so that $\omega = \partial\overline{\partial}(\alpha)$. The function α is called a Kähler potential.

(3) The torsion of the metric, τ, vanishes.

(4) The metric ds^2 is tangent everywhere up to order 2 to the local Euclidean metric. That is, near z_0,

$$\sum_{i,j} h_{ij}\, dz_i \otimes d\overline{z}_j = \sum_{i,j} (\delta_{ij} + g_{ij})\, dz_i \otimes d\overline{z}_j,$$

where g_{ij} vanishes up to (not including) order 2 terms in the Taylor series at z_0.

Before giving the proof, observe: ds^2 is Kähler iff $\partial\omega = 0$ iff $\overline{\partial}\omega = 0$. Indeed, we know ds^2 is Kähler iff $d\omega = 0$ iff $(\partial + \overline{\partial})\omega = 0$, i.e., $\partial\omega + \overline{\partial}\omega = 0$. As the first is a $(2,1)$-form and the second is a $(1,2)$-form, $d\omega = 0$ iff $\partial\omega$ and $\overline{\partial}\omega$ both vanish. But, ω is real, so

$$\overline{\partial\omega} = \overline{\partial}\overline{\omega} = \overline{\partial}\,\omega,$$

i.e., $\partial\omega = 0$ iff $\overline{\partial}\omega = 0$.

Now, recall that

$$\omega = \frac{i}{2}\sum_{i,j} h_{ij}\, dz_i \wedge d\overline{z}_j,$$

so we get

$$\partial\omega = \frac{i}{2}\sum_{i,j,k} \frac{\partial h_{ij}}{\partial z_k}\, dz_k \wedge dz_i \wedge d\overline{z}_j.$$

Therefore, $d\omega = 0$ iff $\partial\omega = 0$ iff

$$\frac{\partial h_{ij}}{\partial z_k} = \frac{\partial h_{kj}}{\partial z_i}, \quad \text{for all } i, j, k. \tag{$\dagger$}$$

Proof of Theorem 2.55. (1) $\Longleftrightarrow$ (2). The metric ds^2 is Kähler iff $d\omega = 0$. Apply $\partial\overline{\partial}$-Poincaré, to get $d\omega = 0$ iff $\omega = \partial\overline{\partial}(\alpha)$, locally everywhere.

(1) $\Longleftrightarrow$ (3). Write

$$\omega = \frac{i}{2}\sum_j \varphi_j \wedge \overline{\varphi}_j,$$

in a unitary coframe, φ_j. We know

$$d\varphi_j = \sum_k \Psi_{jk} \wedge \varphi_k + \tau_j.$$

But then,

$$d\omega = \frac{i}{2}\Big(\sum_j d\varphi_j \wedge \overline{\varphi}_j - \sum_l \varphi_l \wedge d\overline{\varphi}_l\Big),$$

so we deduce

$$\frac{2}{i}d\omega = \sum_{j,k} \Psi_{jk} \wedge \varphi_k \wedge \overline{\varphi}_j + \sum_j \tau_j \wedge \overline{\varphi}_j - \sum_{l,k} \varphi_l \wedge \overline{\Psi_{lk}} \wedge \overline{\varphi}_k - \sum_l \varphi_l \wedge \overline{\tau}_l.$$

Now, $\overline{\Psi_{lk}} = -\Psi_{kl}$, so we deduce

$$-\sum_{l,k} \varphi_l \wedge \overline{\Psi_{lk}} \wedge \overline{\varphi}_k = \sum_{l,k} \varphi_l \wedge \Psi_{kl} \wedge \overline{\varphi}_k = -\sum_{l,k} \Psi_{kl} \wedge \varphi_l \wedge \overline{\varphi}_k,$$

from which we get

$$\frac{2}{i} d\omega = \sum_j \tau_j \wedge \overline{\varphi}_j - \sum_j \varphi_j \wedge \overline{\tau}_j.$$

By type, $d\omega = 0$ iff both

$$\sum_j \tau_j \wedge \overline{\varphi}_j = 0 \quad \text{and} \quad \sum_j \varphi_j \wedge \overline{\tau}_j = 0.$$

As the φ_j are everywhere locally linearly independent, we get $\tau_j = 0$ (and $\overline{\tau}_j = 0$) for all j, i.e., $\tau = 0$.

(4) $\Longrightarrow$ (1). Say

$$ds^2 = \sum_{i,j} (\delta_{ij} + g_{ij}) dz_i \otimes d\overline{z}_j,$$

where $g_{ij} = 0$ at z_0 up to and including first derivatives. Then,

$$d\omega = \frac{i}{2} \sum_{i,j} dg_{ij} \wedge dz_i \wedge d\overline{z}_j$$

and $d\omega(z_0) =$ right hand side at z_0, which vanishes, so (1) holds.

(1) $\Longrightarrow$ (4). We have

$$\omega = \frac{i}{2} \sum_{i,j} h_{ij}\, dz_i \wedge d\overline{z}_j$$

and we can always pick local coordinates so that $h_{ij}(z_0) = \delta_{ij}$ (by Gram-Schmidt at z_0). Find a change of coordinates and by Taylor and our condition, we are reduced to seeking a change of the form

$$z_j = w_j + \frac{1}{2} \sum_{r,s} c_{jrs} w_r w_s, \tag{$*$}$$

and we may assume $c_{jrs} = c_{jsr}$. Write (Taylor for h_{ij})

$$h_{ij} = \delta_{ij} + \sum_k (a_{ijk} z_k + b_{ijk} \overline{z}_k) + O(2). \tag{$**$}$$

Since h_{ij} is hermitian, $\overline{h_{ij}} = h_{ji}$, so

$$\overline{a}_{ijk} \overline{z}_k + \overline{b}_{ijk} z_k = a_{jik} z_k + b_{jik} \overline{z}_k,$$

and we conclude that

$$b_{ijk} = \overline{a}_{jik},$$

i.e., the a's determine the b's.

Since (1) holds, i.e., $d\omega = 0$, we have

$$\frac{i}{2} \sum_{i,j} dh_{ij} \wedge dz_i \wedge d\overline{z}_j = 0,$$

that is,

$$\frac{i}{2} \Big(\sum_{i,j,k} a_{ijk}\, dz_k \wedge dz_i \wedge d\overline{z}_j + \text{similar terms with } b\text{'s} + O(2) \Big) = 0.$$

We conclude that
$$a_{ijk} = a_{kji}.$$
We now use $(*)$ and $(**)$ in ω:
$$\frac{i}{2}\omega = \sum_{i,j,k}(\delta_{ij} + a_{ijk}z_k + b_{ijk}\overline{z}_k)\Big(dw_i + \frac{1}{2}\sum_{r,s} c_{irs}(w_s dw_r + w_r dw_s)\Big) \wedge \Big(d\overline{w}_j + \frac{1}{2}\sum_{m,n}\overline{c}_{jmn}(\overline{w}_n d\overline{w}_m + \overline{w}_m d\overline{w}_n\Big) + O(2).$$
Thus, we get
$$\frac{i}{2}\omega = \sum_{i,j,k}(\delta_{ij} + a_{ijk}z_k + b_{ijk}\overline{z}_k)\Big(dw_i \wedge d\overline{w}_j + dw_i \wedge \sum_{m,n}\overline{c}_{jmn}\overline{w}_m d\overline{w}_n - d\overline{w}_j \wedge \sum_{r,s} c_{irs}w_r dw_s\Big) + O(2)$$
i.e.,
$$\begin{aligned}\frac{i}{2}\omega \quad = \quad & \sum_{i,j,k}\delta_{ij}dw_i \wedge d\overline{w}_j + \sum_{i,j,k} a_{ijk}w_k dw_i \wedge d\overline{w}_j + \sum_{j,m,n}\overline{c}_{jmn}\overline{w}_m dw_j \wedge d\overline{w}_n \\ & + \sum_{j,r,s} c_{jrs}w_r dw_s \wedge d\overline{w}_j + \sum_{i,j,k} b_{ijk}\overline{w}_k dw_i \wedge d\overline{w}_j + O(2).\end{aligned}$$
It suffices to take
$$c_{jki} = -a_{ijk}$$
and then, since $b_{ijk} = \overline{a}_{jik}$, the other two terms also cancel out. So, (4) is achieved. $\square$

The main use of the above is in the corollary below:

Corollary 2.56 *If X is a complex manifold and ds^2 is an hermitian metric on X, then the metric is Kähler iff for all z_0, there is an open, U, with $z_0 \in U$ and we can choose a unitary coframe, $\varphi_1, \ldots, \varphi_n$, so that $d\varphi_j(z_0) = 0$, for $j = 1, \ldots, n$.*

Remark: As
$$\omega = \frac{i}{2}\sum_j \varphi_j \wedge \overline{\varphi}_j,$$
we have
$$d\omega = \frac{i}{2}\Big(\sum_j d\varphi_j \wedge \overline{\varphi}_j - \sum_j \varphi_j \wedge d\overline{\varphi}_j;\Big)$$
consequently, $d\omega(z_0) = 0$ iff $d\varphi_j(z_0) = 0$, for $j = 1, \ldots, n$ (by linear independence of the coframe and decomposition into types).

Say we know (what we're about to prove): If X is compact and Kähler, then HDRSS degenerates at E_1. Then, $E_1^{p,q} = E_\infty^{p,q} = (p,q)$th graded piece of the Hodge filtration of $H_{\mathrm{DR}}^{p+q}(X, \mathbb{C})$. We know that $E_1^{p,q} = E_\infty^{p,q}$ implies that
$$E_\infty^{p,q} = H_{\overline{\partial}}^{p,q}(X).$$
Also, $F^p H_{\mathrm{DR}}^{p+q}/F^{p+1}H_{\mathrm{DR}}^{p+q} = E_\infty^{p,q}$. This implies this is an inclusion
$$E_\infty^{p,0} \hookrightarrow H_{\mathrm{DR}}^{p+q}(X, \mathbb{C}).$$
In the Kähler case, we get
$$H^0(X, \Omega_X^p) = E_1^{p,0} = E_\infty^{p,0} \hookrightarrow H_{\mathrm{DR}}^{p+q}(X, \mathbb{C}).$$

Corollary 2.57 *If X is compact, Kähler, then every global holomorphic p-form is d-closed and* **never** *exact (unless $= 0$).*

Interesting (DX): Prove this directly from the Kähler condition.

To prove degeneration of the HSRSS, we prove the Hodge identities. First introduce (à la Hodge)

$$d^c = \frac{i}{4\pi}(\overline{\partial} - \partial).$$

As $d = \partial + \overline{\partial}$, we get

$$dd^c = (\partial + \overline{\partial})\frac{i}{4\pi}(\overline{\partial} - \partial) = \frac{i}{4\pi}(\partial\overline{\partial} + \partial\overline{\partial}) = \frac{i}{2\pi}\partial\overline{\partial},$$

so

$$\omega_{FS} = dd^c(\log \|F\|^2)$$

(where ω_{FS} is the $(1,1)$-form associated with the Fubini-Study metric) and

$$d^c d = \frac{i}{4\pi}(\overline{\partial} - \partial)(\partial + \overline{\partial}) = \frac{i}{4\pi}(\overline{\partial}\partial + \overline{\partial}\partial) = \frac{i}{2\pi}\overline{\partial}\partial = -dd^c.$$

Therefore,

$$d^c d = -dd^c.$$

Both d and d^c are *real* operators (i.e., they are equal to their conjugate).

Now, for *any metric*, ds^2, introduce (à la Lefschetz) the operators L and Λ:

$$L\colon \bigwedge^{p,q}(X) \longrightarrow \bigwedge^{p+1,q+1}(X), \quad L(\xi) = \omega \wedge \xi,$$

where ω is the $(1,1)$-form associated with ds^2 and

$$\Lambda\colon \bigwedge^{p,q}(X) \longrightarrow \bigwedge^{p-1,q-1}(X), \quad \text{with} \quad \Lambda = L^*, \quad \text{the adjoint of } L.$$

The main necessary fact is this:

Proposition 2.58 *(Basic Hodge Identities) If X is Kähler, then*

(1) $[\Lambda, d] = -4\pi(d^c)^*$;

(2) $[L, d^*] = 4\pi d^c$;

(3) $[\Lambda, \overline{\partial}] = -i\partial^*$;

(4) $[\Lambda, \partial] = i\overline{\partial}^*$;

and (1)–(4) are mutually equivalent.

Proof. First, we prove the equivalence of (1)–(4). We have

$$[\Lambda, d] = [\Lambda, \partial + \overline{\partial}] = [\Lambda, \partial] + [\Lambda, \overline{\partial}]$$

and

$$(d^c)^* = \frac{i}{4\pi}(\overline{\partial} - \partial)^* = -\frac{i}{4\pi}(\overline{\partial}^* - \partial^*).$$

Consequently, (1) iff $[\Lambda, \partial] + [\Lambda, \overline{\partial}] = i(\overline{\partial}^* - \partial^*)$. By types, (1) iff both (3) and (4).

(3) $\Longleftrightarrow$ (4). We have $L\xi = \xi \wedge \omega$ and ω is real, so L is real and $\Lambda = L^*$ is real as well. It follows that

$$\overline{[\Lambda, \overline{\partial}]} = [\overline{\Lambda}, \partial] = [\Lambda, \partial].$$

Consequently, (3) iff $[\Lambda, \overline{\partial}] = -i\partial^*$ iff $\overline{[\Lambda, \overline{\partial}]} = i\overline{\partial}^*$ iff $[\Lambda, \partial] = i\overline{\partial}^*$, i.e., (4).

We conclude that (1)–(3) are all equivalent. As

$$[\Lambda, d]^* = (\Lambda d - d\Lambda)^* = d^*\Lambda^* - \Lambda^* d^* = -[L, d^*],$$

we see that (1) and (2) are equivalent. Therefore, (1)–(4) are all equivalent.

Next, we prove (4) for $\mathbb{C}^n$ and the standard ω, i.e.,

$$\omega = \frac{i}{2}\sum_j dz_j \wedge d\overline{z}_j,$$

where $z_1, \ldots, z_n$ are global coordinates on $\mathbb{C}^n$. Since every form on $\mathbb{C}^n$ is uniformly approximable (on compact sets) up to any preassigned number of derivatives by forms with compact support, we may assume all forms to be delt with below to have compact support. Break up all into components on $\bigwedge_0^{p,q}(X)$ and define operators, e_k, $\overline{e}_k$, f_k, $\overline{f}_k$ as follows:

$$\begin{aligned}
e_k(dz_I \wedge d\overline{z}_J) &= dz_k \wedge dz_I \wedge d\overline{z}_J\\
\overline{e}_k(dz_I \wedge d\overline{z}_J) &= d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J\\
f_k &= e_k^*\\
\overline{f}_k &= \overline{e}_k^*.
\end{aligned}$$

Observe, e_k, $\overline{e}_k$, f_k, $\overline{f}_k$ are C^∞-linear. I claim

(A) $f_k e_k + e_k f_k = 2$, for all k;

(B) $f_k e_j + e_j f_k = 0$, for all $j \neq k$;

(C) $\overline{f}_j e_k + e_k \overline{f}_j = 0$, for all $j \neq k$.

Observe that trivially (by definition),

$$e_j e_k + e_k e_j = 0 \quad \text{and} \quad f_j f_k + f_k f_j = 0, \quad \text{for all } j, k$$

and similarly for $\overline{e}_k$, $\overline{f}_k$. To prove (A)–(C), by C^∞-linearity, it is enough to check them on a basis $dz_I \wedge d\overline{z}_J$. First, let us compute $f_k(dz_I \wedge d\overline{z}_J)$. Say $k \notin I$, then

$$(f_k(dz_I \wedge d\overline{z}_J), dz_R \wedge d\overline{z}_S) = (dz_I \wedge d\overline{z}_J, e_k(dz_R \wedge d\overline{z}_S)) = (dz_I \wedge d\overline{z}_J, dz_k \wedge dz_R \wedge d\overline{z}_S) = 0,$$

by our definition of the inner product. Therefore, as R and S are arbitrary,

$$f_k(dz_I \wedge d\overline{z}_J) = 0 \quad \text{if } k \notin I.$$

Similarly,

$$\overline{f}_k(dz_I \wedge d\overline{z}_J) = 0 \quad \text{if } k \notin J.$$

The case $k \in I$ is taken care of as follows. First, assume $dz_I = dz_k \wedge dz_{I'}$, then

$$(f_k(dz_k \wedge dz_{I'} \wedge d\overline{z}_J), dz_R \wedge d\overline{z}_S) = (dz_k \wedge dz_{I'} \wedge d\overline{z}_J, dz_k \wedge dz_R \wedge d\overline{z}_S) = 2(dz_{I'} \wedge d\overline{z}_J, dz_R \wedge d\overline{z}_S)$$

so we get

$$(f_k(dz_k \wedge dz_{I'} \wedge d\overline{z}_J), dz_R \wedge d\overline{z}_S) = \begin{cases} 0 & \text{if } I' \neq R \text{ or } J \neq S \\ 2(dz_{I'} \wedge d\overline{z}_J, dz_{I'} \wedge d\overline{z}_J) & \text{if } I' = R \text{ and } J = S. \end{cases}$$

It follows that

$$f_k(dz_k \wedge dz_{I'} \wedge d\overline{z}_J) = 2dz_{I'} \wedge d\overline{z}_J \quad \text{if } k \in I$$

and so,

$$f_k e_k(dz_I \wedge d\overline{z}_J) = f_k(dz_k \wedge dz_I \wedge d\overline{z}_J) = \begin{cases} 2dz_I \wedge d\overline{z}_J & \text{if } k \notin I \\ 0 & \text{if } k \in I. \end{cases}$$

We also have

$$e_k f_k(dz_I \wedge d\overline{z}_J) = \begin{cases} (-1)^b e_k f_k(dz_k \wedge dz_{I'} \wedge d\overline{z}_J) & \text{if } k \in I \\ 0 & \text{if } k \notin I. \end{cases}$$

But,

$$\begin{aligned} e_k f_k(dz_I \wedge d\overline{z}_J) &= (-1)^b e_k f_k(dz_k \wedge dz_{I'} \wedge d\overline{z}_J) \\ &= 2(-1)^b e_k(dz_{I'} \wedge d\overline{z}_J) \\ &= 2(-1)^b dz_k \wedge dz_{I'} \wedge d\overline{z}_J \\ &= 2dz_I \wedge d\overline{z}_J, \end{aligned}$$

if $k \in I$, and we conclude that $f_k e_k + e_k f_k = 2$, for all k.

(B) Take $j \neq k$. If $k \notin I$, then

$$f_k e_j(dz_I \wedge d\overline{z}_J) = f_k(dz_j \wedge dz_I \wedge d\overline{z}_J) = 0.$$

If $k \in I$, then

$$\begin{aligned} f_k e_j(dz_I \wedge d\overline{z}_J) &= (-1)^b f_k(dz_j \wedge dz_k \wedge dz_{I'} \wedge d\overline{z}_J) \\ &= (-1)^{b+1} f_k(dz_k \wedge dz_j \wedge dz_{I'} \wedge d\overline{z}_J) \\ &= 2(-1)^{b+1} dz_j \wedge dz_{I'} \wedge d\overline{z}_J. \end{aligned}$$

We also have

$$e_j f_k(dz_I \wedge d\overline{z}_J) = \begin{cases} 0 & \text{if } k \notin I \\ 2(-1)^b dz_j \wedge dz_{I'} \wedge d\overline{z}_J & \text{if } k \in I, \end{cases}$$

and we conclude that $f_k e_j + e_j f_k = 0$.

(C) The proof is similar.

Now, we have

$$\begin{aligned} L(\xi) &= \xi \wedge \omega = \frac{i}{2} \sum_j \xi \wedge dz_j \wedge d\overline{z}_j \\ &= \frac{i}{2} \sum_j dz_j \wedge d\overline{z}_j \wedge \xi \\ &= \frac{i}{2} \sum_j (e_j \overline{e}_j)(\xi), \end{aligned}$$

that is $L = \frac{i}{2} \sum_j e_j \overline{e}_j$, so we get $\Lambda = -\frac{i}{2} \sum_j \overline{f_j} f_j = \frac{i}{2} \sum_j f_j \overline{f_j}$.

If $\xi = \sum_{I,J} \varphi_{IJ} dz_I \wedge d\overline{z}_J$, set

$$\partial_k(\xi) = \sum_{I,J} \frac{\partial \varphi_{IJ}}{\partial z_k} dz_I \wedge d\overline{z}_J,$$

and

$$\overline{\partial}_k(\xi) = \sum_{I,J} \frac{\partial \varphi_{IJ}}{\partial \overline{z}_k}\, dz_I \wedge d\overline{z}_J.$$

This is now the part where we need compact support. Namely, I claim:

(D) $\partial_k^* = -\overline{\partial}_k$ and $\overline{\partial}_k^* = -\partial_k$.

Using integration by parts, we have

$$\begin{aligned}
(\partial_k^*(\xi), g dz_R \wedge d\overline{z}_S) &= (\xi, \partial_k(g dz_R \wedge d\overline{z}_S)) \\
&= (\varphi_{RS} dz_R \wedge d\overline{z}_S, \frac{\partial g}{\partial z_k}\, dz_R \wedge d\overline{z}_S) \\
&= 2^{|R|+|S|} \int_{\mathbb{C}^n} \varphi_{RS} \overline{\frac{\partial g}{\partial z_k}} \\
&= 2^{|R|+|S|} \int_{\mathbb{C}^n} \varphi_{RS} \frac{\partial \overline{g}}{\partial \overline{z}_k} \\
&= -2^{|R|+|S|} \int_{\mathbb{C}^n} \frac{\partial \varphi_{RS}}{\partial \overline{z}_k}\, \overline{g} \\
&= \Big(-\frac{\partial \varphi_{RS}}{\partial \overline{z}_k}\, dz_R \wedge d\overline{z}_S, g dz_R \wedge d\overline{z}_S\Big) \\
&= (-\overline{\partial}_k(\xi), g dz_R \wedge d\overline{z}_S).
\end{aligned}$$

The second identity follows by taking complex conjugates.

What are ∂ and $\overline{\partial}$ in these terms? We have

$$\partial(\xi) = \partial\Big(\sum_{I,J} \varphi_{IJ} dz_I \wedge d\overline{z}_J\Big) = \sum_{I,J} \frac{\partial \varphi_{RS}}{\partial z_k}\, dz_k \wedge dz_I \wedge d\overline{z}_J = \sum_k \partial_k e_k(\xi) = \sum_k e_k \partial_k(\xi).$$

Therefore,

$$\partial = \sum_k \partial_k e_k = \sum_k e_k \partial_k.$$

From this, we deduce that

$$\overline{\partial} = \sum_k \overline{\partial}_k \overline{e}_k = \sum_k \overline{e}_k \overline{\partial}_k,$$

and

$$\partial^* = \sum_k \partial_k^* e_k^* = -\sum_k \overline{\partial}_k f_k = -\sum_k f_k \overline{\partial}_k$$

as well as

$$\overline{\partial}^* = -\sum_k \partial_k \overline{f}_k = -\sum_k \overline{f}_k \partial_k.$$

Finally, we have

$$\begin{aligned}
[\Lambda, \partial] &= \Lambda\partial - \partial\Lambda \\
&= \frac{i}{2}\Big(\sum_{j,k} f_j\overline{f}_j\partial_k e_k - \sum_{j,k}\partial_k e_k f_j\overline{f}_j\Big) \\
&= \frac{i}{2}\Big(\sum_{j,k} \partial_k f_j\overline{f}_j e_k - \sum_{j,k}\partial_k e_k f_j\overline{f}_j\Big) \\
&= -\frac{i}{2}\Big(\sum_{j,k} \partial_k\overline{f}_j f_j e_k + \sum_{j,k}\partial_k \overline{f}_j e_k f_j\Big) \\
&= -\frac{i}{2}\Big(\sum_{j\neq k} \partial_k\overline{f}_j (f_j e_k + e_k f_j) + \sum_{j=k}\partial_j \overline{f}_j (f_j e_j + e_j f_j)\Big) \\
&= -i\sum_j \partial_j\overline{f}_j = i\overline{\partial}^*,
\end{aligned}$$

by (A) and (B), as claimed. This proves the identity for $\mathbb{C}^n$.

in the general case (Kähler case, not necessarily compact), we have to show

$$[\Lambda, \partial](z_0) = i\overline{\partial}^*(z_0), \quad \text{for every } z_0 \in X.$$

At z_0, pick a local unitary coframe, $\varphi_1, \ldots, \varphi_n$, so that, near z_0,

$$\omega = \frac{i}{2}\sum_j \varphi_j \wedge \overline{\varphi}_j,$$

and, as X is Kähler, a previous Corollary shows that we can choose $\varphi_1, \ldots, \varphi_n$ so that $d\varphi_j(z_0) = d\overline{\varphi}_j(z_0) = 0$. As before, we set

$$L = \frac{i}{2}\sum_k e_j\overline{e}_j \quad \text{and} \quad \Lambda = \frac{i}{2}\sum_k f_j\overline{f}_j.$$

(Here, $e_j\xi = \varphi_j \wedge \xi$). We make the same computations as before (A)–(D) and we get the same results on commutativity, except for extra terms involving the differentials of the φ_j's. However, at z_0, these differentials vanish and we get (A)–(D) as the error term, $d\varphi_j(z_0)$ is 0 for all j. Therefore, (1)–(4) are correct at z_0. But, z_0 is arbitrary, so the theorem is proved. $\square$

Corollary 2.59 *If X is Kähler (not necessarily compact) then $\square_d = dd^* + d^*d$ (d-Laplacian) commutes with L and Λ.*

Proof. Write $\square$ for $\square_d$ and observe that

$$[\square, \Lambda]^* = [\Lambda^*, \square^*] = [L, \square],$$

so $[\square, \Lambda] = 0$ iff $[L, \square] = 0$ and it suffices to prove $[\Lambda, \square] = 0$. Let us check that $[L, d] = 0$. Pick any ξ, then

$$\begin{aligned}
[L, d](\xi) &= (Ld - dL)(\xi) \\
&= L(d\xi) - d(L\xi) \\
&= d\xi \wedge \omega - d(\xi \wedge \omega) \\
&= d\xi \wedge \omega - d\xi \wedge \omega \pm \xi \wedge d\omega = 0,
\end{aligned}$$

since $d\omega = 0$ and so, $[L, d] = 0$. by adjointness, $[\Lambda, d^*] = 0$. Then, we have

$$[\Lambda, \square] = \Lambda\square - \square\Lambda = \Lambda dd^* + \Lambda d^* d - dd^*\Lambda - d^*d\Lambda.$$

Use $[\Lambda, d^*] = 0$ in the middle terms to get

$$\begin{aligned}
[\Lambda, \Box] &= \Lambda d d^* - d\Lambda d^* + d^*\Lambda d - d^* d\Lambda \\
&= [\Lambda, d]d^* + d^*[\Lambda, d] \qquad \text{(by Basic Hodge)} \\
&= -4\pi d^{c*} d^* - 4\pi d^* d^{c*} \\
&= -4\pi(d^{c*} d^* + d^* d^{c*}).
\end{aligned}$$

But, $dd^c = -d^c d$; by applying $*$, we have $d^{c*} d^* + d^* d^{c*} = 0$, and the proof is complete. $\Box$

Corollary 2.60 *If X is Kähler (not necessarily compact) then*

$$\Box_d = 2\Box_\partial = 2\Box_{\overline{\partial}}.$$

Proof. It will be enough to show $\Box_\partial = \Box_{\overline{\partial}}$. To see this compute $\Box_d = dd^* + d^*d$. We have

$$\begin{aligned}
\Box_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\
&= \partial\partial^* + \overline{\partial}\,\partial^* + \partial\overline{\partial}^* + \overline{\partial}\,\overline{\partial}^* + \partial^*\partial + \partial^*\overline{\partial} + \overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} \\
&= \Box_\partial + \Box_{\overline{\partial}} + \overline{\partial}\,\partial^* + \partial^*\overline{\partial} + \partial\overline{\partial}^* + \overline{\partial}^*\partial.
\end{aligned}$$

I claim that $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. If so, its conjugate is also zero, i.e., $\overline{\partial}\,\partial^* + \partial^*\overline{\partial} = 0$, and we deduce that

$$\Box_d = \Box_\partial + \Box_{\overline{\partial}}.$$

By Basic Hodge (4), we have $[\Lambda, \partial] = i\overline{\partial}^*$, i.e., $-i[\Lambda, \partial] = \overline{\partial}^*$. Then,

$$\partial\overline{\partial}^* + \overline{\partial}^*\partial = -i\big(\partial[\Lambda, \partial] + [\Lambda, \partial]\partial\big) = -i\big(\partial\Lambda\partial - \partial\partial\Lambda + \Lambda\partial\partial - \partial\Lambda\partial\big) = 0.$$

Therefore, $\Box_d = \Box_\partial + \Box_{\overline{\partial}}$. If we prove that $\Box_\partial = \Box_{\overline{\partial}}$, we are done. We have

$$\begin{aligned}
\Box_{\overline{\partial}} &= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \\
&= -i\big(\overline{\partial}[\Lambda, \partial] + [\Lambda, \partial]\overline{\partial}\big) \qquad \text{(by Basic Hodge)} \\
&= -i\big(\overline{\partial}\,\Lambda\partial - \overline{\partial}\,\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial}\big).
\end{aligned}$$

Now, by Basic Hodge (3), $[\Lambda, \overline{\partial}] = -i\partial^*$ (and recall $\overline{\partial}\,\partial = -\partial\overline{\partial}$), so

$$\begin{aligned}
\Box_{\overline{\partial}} &= -i\big(\overline{\partial}\,\Lambda\partial - \Lambda\overline{\partial}\,\partial + \partial\overline{\partial}\,\Lambda - \partial\Lambda\overline{\partial}\big) \\
&= -i\big(-[\Lambda, \overline{\partial}]\partial - \partial[\Lambda, \overline{\partial}]\big) \\
&= -i\big(i\partial^*\partial + i\partial\partial^*\big) \\
&= \partial^*\partial + \partial\partial^* = \Box_\partial,
\end{aligned}$$

and this concludes the proof of the corollary. $\Box$

Corollary 2.61 *If X is Kähler (not necessarily compact), then any and all of $\Box_d$, $\Box_\partial$, $\Box_{\overline{\partial}}$ commute with all of $*$, ∂, ∂^*, $\overline{\partial}$, $\overline{\partial}^*$, L and Λ.*

Proof. (DX) $\Box$

Corollary 2.62 *If X is Kähler (not necessarily compact), then the de Rham Laplacian, $\Box_d$, preserves bidegree. That is, the diagram*

$$\begin{tikzcd}
\bigwedge(X) \arrow[r, "\Box_d"] \arrow[d, "pr^{p,q}"'] & \bigwedge(X) \arrow[d, "pr^{p,q}"] \\
\bigwedge^{p,q}(X) \arrow[r, "\Box_d"'] & \bigwedge^{p,q}(X)
\end{tikzcd}$$

commutes.

Proof. We have $\Box_d = 2\Box_{\overline{\partial}}$ and $\Box_{\overline{\partial}}$ preserves bidegree. $\Box$

Theorem 2.63 *(Hodge Decomposition, Kähler Case) Assume X is a complex, compact Kähler manifold. Then, there is a canonical isomorphism*

$$\coprod_{p+q=r} H^q(X, \Omega_X^p) \xrightarrow{\sim} H_{\mathrm{DR}}^r(X, \mathbb{C}).$$

Moreover, we have

$$H^q(X, \Omega_X^p) = \overline{H^p(X, \Omega_X^q)},$$

so $h^{p,q} = h^{q,p}$.

Proof. Pick a Kähler metric, d, on X and make $\Box = \Box_d$ (using the metric) and set

$$\mathcal{H}_d^{p,q} = \{\xi \in \bigwedge^{p,q}(X) \mid \Box(\xi) = 0\}$$

and

$$\mathcal{H}_d^r = \{\xi \in \bigwedge^r(X) \mid \Box(\xi) = 0\}.$$

Both of these groups depend on the metric. Pick $\xi \in \bigwedge^r(X)$ and write

$$\xi = \sum_j \xi_j, \quad \text{where} \quad \xi_j \in \bigwedge^{j,r-j}(X).$$

Then, $\Box(\xi) = \sum_j \Box(\xi_j)$. But, $\Box(\xi_j) \in \bigwedge^{j,r-j}(X)$, by Corollary 2.62. Therefore, $\Box(\xi) = \sum_j \Box(\xi_j)$ is the type decomposition of $\Box(\xi)$. Hence, $\Box(\xi) = 0$ iff $\Box(\xi_j) = 0$, for every j. Hence,

$$\mathcal{H}_d^r = \coprod_{p+q=r} \mathcal{H}_d^{p,q}.$$

But, $\Box = 2\Box_{\overline{\partial}}$ (Corollary 2.60), so $\mathcal{H}_d^{p,q} = \mathcal{H}_{\overline{\partial}}^{p,q}$. By the $\overline{\partial}$-Hodge Theorem (X is compact), we know

$$\mathcal{H}_{\overline{\partial}}^{p,q} \cong H_{\overline{\partial}}^{p,q} \cong H^q(X, \Omega_X^p).$$

In the same way, by Hodge's Theorem for d (on the compact, X), we get

$$\mathcal{H}_d^r \cong H_{\mathrm{DR}}^r(X, \mathbb{C})$$

and for *some* isomorphism,

$$\coprod_{p+q=r} H^q(X, \Omega_X^p) \xrightarrow{\sim} H_{\mathrm{DR}}^r(X, \mathbb{C}).$$

Let

$$K^{p,q} = Z_d^{p,q}/d(\bigwedge^{\bullet}(X) \cap Z_d^{p,q}).$$

Note, $K^{p,q}$ = all cohomological (de Rham) classes having a (p,q)-representative. The set $K^{p,q}$ depends only on the complex structure (hence, the topology—independent of any metric). When X is Kähler and we choose a metric, then

$$\mathcal{H}^{p,q} \subseteq K^{p,q}$$

and we know $H^q(X, \Omega_X^p) \cong \mathcal{H}^{p,q}$. Pick $\xi \in K^{p,q}$ and write ξ, again, for a $Z^{p,q}$-representative. The d-Hodge Theorem yields

$$\xi = \varkappa(\xi) + d(d^* G(\xi)) + d^*(dG(\xi)).$$

But, $\square$ commutes with d, so G commutes with d, and as $d\xi = 0$, we get

$$\xi = \mathcal{H}(\xi) + d(d^* G(\xi)).$$

But, G preserves (p,q)-type, so ξ as element of $K^{p,q}$ is $\mathcal{H}(\xi)$. As $\mathcal{H}(\xi) \in \mathcal{H}^{p,q}$, we see that $K^{p,q} \subseteq \mathcal{H}^{p,q}$, from which we conclude that

$$K^{p,q} = \mathcal{H}^{p,q}.$$

For the conjugation, pick $\xi \in H^q(X, \Omega_X^p)$ and represent it by $\mathcal{H}(\xi) \in \mathcal{H}_d^{p,q}$ (the unique harmonic representative). Take $\eta = \mathcal{H}(\xi)$ and observe that as $\square_d$ is real, $\overline{\square(\eta)} = \square(\overline{\eta})$. Therefore, $\overline{\eta}$ is harmonic (i.e., ξ is harmonic iff $\overline{\xi}$ is harmonic). But, $\xi \in \bigwedge^{p,q}(X)$ iff $\overline{\xi} \in \bigwedge^{q,p}(X)$, and our map $\xi \mapsto \overline{\xi}$ is a sesquilinear isomorphism

$$H^q(X, \Omega_X^p) \longrightarrow \overline{H^p(X, \Omega_X^q)},$$

which concludes the proof. $\square$

Corollary 2.64 *If X is a compact, Kähler manifold, in particular if X is a nonsingular complex projective variety, then the odd-degree Betti numbers of X are even integers.*

Proof. We have

$$H_{\mathrm{DR}}^{2r+1}(X, \mathbb{C}) \cong \coprod_{p+q} H^q(X, \Omega_X^p) \cong \coprod_{\substack{0 \le p \le r \\ p+r=2r+1}} H^q(X, \Omega_X^p) \amalg \coprod_{\substack{r+1 \le p \le 2r+1 \\ p+r=2r+1}} H^q(X, \Omega_X^p)$$

and each term has an isomorphic term in the other sum, so

$$b_{2r+1} = 2 \sum_{\substack{0 \le p \le r \\ p+r=2r+1}} h^{p,q}$$

which is even. $\square$

Code all the informatin in the *Hodge diamond* (shown in Figure 2.2). At height r, you insert $h^{p,q}$ at position (p,q) where $p+q=r$.

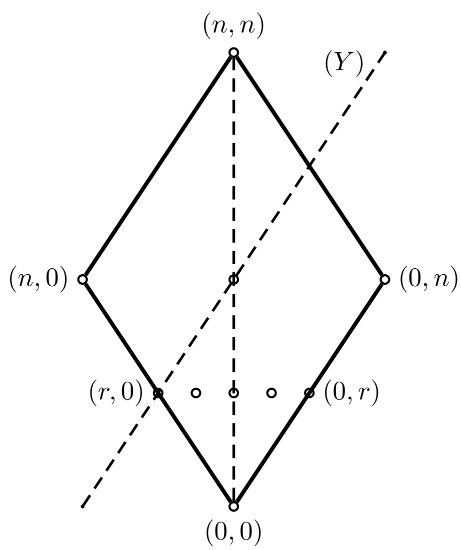

Figure 2.2: The Hodge Diamond

(a) The sum of the row numbers at height r is b_r (the rth Betti number).

(b) $h^{p,q} = h^{q,p}$, which means that the diamond is the same on either side of the vertical (as a mirror).

(c) $*$ (Serre duality) says there is symmetry about the central point.

(d) If X is Calabi-Yau and Y is its mirror, then the Hodge diamond of Y is that for X reflected (as a mirror) in the line (Y).

Corollary 2.65 *If $X = \mathbb{P}^n_{\mathbb{C}}$, then $h^{p,q} = 0$ if $p \neq q$ and $h^{p,p} = 1$, if $0 \leq p \leq n$.*

Proof. Remember that $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\mathrm{pt}\}$, $\mathbb{P}^2_{\mathbb{C}} = \mathbb{C}^2 \cup \mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^2 \cup \mathbb{C}^1 \cup \mathbb{C}^0$ and generally,

$$\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^1 \cup \mathbb{C}^0.$$

By elementary topology,

$$H_r(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \begin{cases} (0) & \text{if } r \text{ is odd or } r \geq 2n \\ \mathbb{Z} & \text{if } r \leq 2n \text{ is even.} \end{cases}$$

Then,

$$H^r(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \begin{cases} (0) & \text{if } r \text{ is odd or } r \geq 2n \\ \mathbb{Z} & \text{if } r \leq 2n \text{ is even} \end{cases}$$

and this implies

$$H^r(\mathbb{P}^n_{\mathbb{C}}, \mathbb{C}) = \begin{cases} (0) & \text{if } r \text{ is odd or } r \geq 2n \\ \mathbb{C} & \text{if } r \leq 2n \text{ is even.} \end{cases}$$

We deduce

$$b_{2r+1}(\mathbb{P}^n_{\mathbb{C}}) = 0, \quad b_{2r}(\mathbb{P}^n_{\mathbb{C}}) = 1, \quad 0 \leq r \leq n.$$

But, $b_r = \sum_{p+q=r} h^{p,q}$, so $h^{p,q} = 0$ if $p+q$ is odd. We also have

$$b_{2r} = \sum_{p+q=2r} h^{p,q} = h^{r,r} + \sum_{0 \leq p \leq r-1} h^{p,2r-p} + \sum_{r+1 \leq p \leq 2r} h^{p,2r-p} = h^{r,r} + 2 \sum_{0 \leq p \leq r-1} h^{p,2r-p}.$$

But, $b_{2r} = 1$, which implies $h^{r,r} = 1$ and $h^{p,2r-p} = 0$, for $0 \leq p \leq r-1$. $\square$

Corollary 2.66 *On a Kähler manifold, for any Kähler metric, a global holomorphic p-form is always harmonic.*

Proof. For any $\xi \in H^0(X, \Omega^p_X)$, by Dolbeault, $H^0(X, \Omega^p_X) \hookrightarrow \bigwedge^{p,0}(X)$ and in fact, ξ is given by $\overline{\partial}\xi = 0$. Now, $\overline{\partial}^* \equiv 0$ on $\bigwedge^{0,0}$. Therefore, $\overline{\partial}\xi = \overline{\partial}^*\xi = 0$, which means that ξ is harmonic. $\square$

Say $Y \subseteq X \subseteq \mathbb{P}^N_{\mathbb{C}}$, with X, Y some analytic (= algebraic) smooth varieties. Say Y is codimension t in X and $\dim_{\mathbb{C}} X = n$. So, $\dim_{\mathbb{C}} Y = n - t$. The inclusion, $i\colon Y \hookrightarrow X$, yields the map, $i^*\colon (T^{1,0}_X)^D \to (T^{1,0}_Y)^D$, i.e., $i^*\colon \Omega^1_X \to \Omega^1_Y$. Therefore, we get a map

$$\bigwedge^{n-t} \Omega^1_X = \Omega^{n-t}_X \longrightarrow \Omega^{n-t}_Y = \bigwedge^{n-t} \Omega^1_Y,$$

so we get a map

$$H^\bullet(X, \Omega^{n-t}_X) \longrightarrow H^\bullet(X, \Omega^{n-t}_Y) = H^\bullet(Y, \Omega^{n-t}_Y).$$

By Serre duality, we get

$$H^{n-t}(X, \Omega^{n-t}_X) \longrightarrow H^{n-t}(Y, \Omega^{n-t}_Y) \overset{\mathrm{tr}}{\cong} \mathbb{C}.$$

Therefore, Y gives an element, $l_Y \in H^{n-t}(X, \Omega^{n-t}_X)^D$. By Serre duality, the latter group is

$$H^t(X, \Omega^t_X) = H^{t,t}_{\overline{\partial}}(X).$$

Hence, each Y of codimension t in X gives a cohomology class, $l_Y \in H^{t,t}_{\overline{\partial}}(X)$, called its *cohomology class*.

Alternatively: Since $\dim_{\mathbb{R}} Y = 2(n-t)$, the variety Y is a $2(n-t)$-chain in X and in fact, it is a cycle. Thus, it gives an element of $H_{2(n-t)}(X)$. We get a linear form on $H^{2(n-t)}(X)$ and, by Poincaré duality, $H^{2(n-t)}(X)^D \cong H^{2t}(X)$. So, Y gives $\lambda_Y \in H^{2t}(X)$. In fact, $\lambda_Y = l_Y$ in $H^{t,t}_{\overline{\partial}}(X)$.

Hodge Conjecture: If $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$, then $H^{2t}(X, \mathbb{Q}) \cap H^{t,t}_{\overline{\partial}}(X)$ is generated by the cohomology classes l_Y as Y ranges over codimension t smooth subvarieties of X.

2.6 Hodge IV: Lefschetz Decomposition & the Hard Lefschetz Theorem

Proposition 2.67 *(Basic Fact) Say X is a Kähler manifold, then on $\bigwedge^{p,q}(X)$, the operator $[L, \Lambda]$ is just multiplication by $p+q-n$, where $n = \dim_{\mathbb{C}} X$. Therefore, $[L, \Lambda]$ on $\bigwedge^{\bullet}(X)$ is a diagonal operator, its eigenspaces are the $\bigwedge^{r}(X)$ and the eigenvalue on this eigenspace is $r-n$.*

Proof. Both L and Λ are algebraic operators so don't involve either ∂ or $\overline{\partial}$. By the Kähler principle, we may assume $X = \mathbb{C}^n$ with the standard Kähler metric and prove it there. Revert to the component decomposition:

$$L = \frac{i}{2} \sum_j e_j \overline{e}_j, \quad \Lambda = \frac{i}{2} \sum_j \overline{f}_j f_j,$$

so

$$[L, \Lambda] = L\Lambda - \Lambda L = \frac{1}{4}\Big(\sum_{j,k} e_j \overline{e}_j \overline{f}_k f_k - \overline{f}_k f_k e_j \overline{e}_j\Big).$$

Recall our commutation relations:

(A) $f_k e_k + e_k f_k = 2$, for all k;

(B) $f_k e_j + e_j f_k = 0$, for all $j \neq k$;

(C) $\overline{f}_j e_k + e_k \overline{f}_j = 0$, for all $j \neq k$.

For $j \neq k$, we have

$$\begin{aligned} e_j \overline{e}_j \overline{f}_k f_k - \overline{f}_k f_k e_j \overline{e}_j &= e_j \overline{e}_j \overline{f}_k f_k - e_j \overline{f}_k f_k \overline{e}_j \\ &= e_j \overline{e}_j \overline{f}_k f_k - e_j \overline{e}_j \overline{f}_k f_k = 0. \end{aligned}$$

Consequently,

$$[L, \Lambda] = \frac{1}{4}\Big(\sum_j e_j \overline{e}_j \overline{f}_j f_j - \overline{f}_j f_j e_j \overline{e}_j\Big).$$

As $f_je_j = 2 - e_jf_j$, we get

$$\begin{aligned}
[L, \Lambda] &= \frac{1}{4}\Big(\sum_j e_j\overline{e}_j\overline{f}_jf_j - \overline{f}_je_jf_j\overline{e}_j - 2\overline{f}_j\overline{e}_j\Big) \\
&= \frac{1}{4}\Big(\sum_j e_j\overline{e}_j\overline{f}_jf_j - \overline{f}_je_j\overline{e}_jf_j - 2\overline{f}_j\overline{e}_j\Big) \\
&= \frac{1}{4}\Big(\sum_j e_j\overline{e}_j\overline{f}_jf_j - e_j\overline{f}_j\overline{e}_jf_j - 2\overline{f}_j\overline{e}_j\Big) \\
&= \frac{1}{4}\Big(\sum_j e_j\overline{e}_j\overline{f}_jf_j - e_j\overline{e}_j\overline{f}_jf_j + 2e_jf_j - 2\overline{f}_j\overline{e}_j\Big) \\
&= \sum_j\Big(1 - \frac{1}{2}(f_je_j + \overline{f}_j\overline{e}_j)\Big) \\
&= n - \frac{1}{2}\sum_j(f_je_j + \overline{f}_j\overline{e}_j).
\end{aligned}$$

Now, take $\xi \in \bigwedge^{p,q}(X)$ and compute $[L, \Lambda](\xi)$. We may assume $\xi = dz_I \wedge d\overline{z}_J$, with $|I| = p$ and $|J| = q$. As

$$f_je_j(dz_I \wedge d\overline{z}_J) = \begin{cases} 0 & \text{if } j \in I \\ 2dz_I \wedge d\overline{z}_J & \text{if } j \notin I \end{cases}$$

and

$$\overline{f}_j\overline{e}_j(dz_I \wedge d\overline{z}_J) = \begin{cases} 0 & \text{if } j \in J \\ 2dz_I \wedge d\overline{z}_J & \text{if } j \notin J \end{cases}$$

we get

$$\sum_j(f_je_j + \overline{f}_j\overline{e}_j)(dz_I \wedge d\overline{z}_J) = \sum_{j\notin I} 2dz_I \wedge d\overline{z}_J + \sum_{j\notin J} 2dz_I \wedge d\overline{z}_J = (2(n-p) + 2(n-q))dz_I \wedge d\overline{z}_J.$$

We deduce that

$$-\frac{1}{2}\sum_j(f_je_j + \overline{f}_j\overline{e}_j) = p + q - 2n \quad \text{on } \textstyle\bigwedge^{p,q}(X),$$

and so,

$$[L, \Lambda] = n + p + q - 2n = p + q - n \quad \text{on } \textstyle\bigwedge^{p,q}(X),$$

finishing the argument. $\square$

Set

$$\mathfrak{H} = [\Lambda, L], \quad \mathfrak{X} = [\mathfrak{H}, L], \quad \mathfrak{Y} = [\mathfrak{H}, \Lambda].$$

For any $\xi \in \bigwedge^r(X)$, observe that $\mathfrak{H}$ on $\bigwedge^r(X)$ is just $(n-r)I$ (by Lefschetz). Futhermore, as

$$\mathfrak{X}(\xi) = (\mathfrak{H}L - \mathfrak{H}L)(\xi))$$

we have $L(\xi) \in \bigwedge^{r+2}(X)$, so

$$\mathfrak{X}(\xi) = (n - (r+2))L(\xi) - (n-r)L(\xi) = -2L(\xi).$$

Therefore,

$$[\mathfrak{H}, L] = -2L.$$

Taking adjoints, we get

$$[\Lambda, \mathfrak{H}^*] = -2\Lambda$$

and as $\mathfrak{H}^* = \mathfrak{H}$, we deduce

$$[\mathfrak{H}, \Lambda] = 2\Lambda.$$

In summary, we have the identities

$$\begin{aligned} [\mathfrak{H}, L] &= -2L \\ [\mathfrak{H}, \Lambda] &= 2\Lambda \\ [\Lambda, L] &= \mathfrak{H}. \end{aligned}$$

This means that we should look at the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (of 2×2 complex matrices with zero trace). Its generators are

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We check that

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

So, send $h \mapsto \mathfrak{H}$, $x \mapsto \Lambda$ and $y \mapsto L$, then we get a representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\bigwedge^\bullet(X)$, i.e., a Lie algebra map

$$\mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathrm{End}_{\mathcal{C}^\infty}(\bigwedge^\bullet(X)).$$

Now, $\Box_d$ commutes with Λ and L, so our representation gives a representation on harmonic forms. By Hodge, we get a representation

$$\mathfrak{sl}(2, \mathbb{C}) \longrightarrow H^\bullet(X).$$

(Recall, if X is compact, then $H^\bullet(X)$ is a finite-dimentional vector space over $\mathbb{C}$.)

Remark: On $\mathrm{End}_{\mathcal{C}^\infty}(\bigwedge^\bullet(X))$, we could use the "graded commutator"

$$[A, B] = AB - (-1)^{ab} BA,$$

where $a = \deg A$ and $b = \deg B$. But, L, Λ, $[L, \Lambda]$ have even degree, so everything is the same.

Say we have a representation of $\mathfrak{sl}(2, \mathbb{C})$ on V (some finite-dimensional $\mathbb{C}$-linear space), i.e., a Lie algebra map

$$\mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathrm{End}_{\mathbb{C}}(V).$$

Since $\mathbf{SL}(2, \mathbb{C})$ is connected and simply-connected, we get a map of Lie groups

$$\mathbf{SL}(2, \mathbb{C}) \longrightarrow \mathbf{GL}(2, \mathbb{C}).$$

This is a representation of $\mathbf{SL}(2, \mathbb{C})$. Conversely, given a representation, $\mathbf{SL}(2, \mathbb{C}) \longrightarrow \mathbf{GL}(2, \mathbb{C})$, the tangent map at the identity yields a Lie algebra representation

$$\mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathrm{End}_{\mathbb{C}}(V).$$

Therefore, there is a one-to-one correspondence between group representations of $\mathbf{SL}(2, \mathbb{C})$ and Lie algebra representations of $\mathfrak{sl}(2, \mathbb{C})$.

Say G is a compact Lie group and V is a finite-dimensional $\mathbb{C}$-space and put a hermitian metric, h, on V. Write $d\sigma$ for the Haar measure on G and define (after Weyl)

$$h_0(v, w) = \int_G h(\sigma v, \sigma w) d\sigma.$$

Check: h_0 is a left-invariant hermitian metric on V.

Say $W \subseteq V$ is a G-submodule (a subrepresentation of V), then we can form $W^\perp$ (w.r.t. h_0), it is a subrepresentation as h_0 is G-invariant. Therefore,

$$V = W \amalg W^\perp \qquad \text{in } G\text{-mod}.$$

Lemma 2.68 *(Weyl-Hurwitz) For any compact Lie group G, every finite-dimensional representation is a coproduct of G-irreducible representations.*

Now, $\mathbf{SU}(2,\mathbb{C})$ is compact and its complexification is $\mathbf{SL}(2,\mathbb{C})$. Therefore, Lemma 2.68 holds for $\mathbf{SL}(2,\mathbb{C})$ and thus, for $\mathfrak{sl}(2,\mathbb{C})$.

Corollary 2.69 *Every finite-dimensional complex representation of $\mathbf{SL}(2,\mathbb{C})$ is a finite coproduct of finite-dimensional irreducible representations of $\mathbf{SL}(2,\mathbb{C})$.*

Now, we study the finite-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules, say V. The crucial idea is to examine the eigenspaces of h on V. Let V_λ be the eigenspace where $h(v) = \lambda v$. If v is an eigenvector of h, then $x(v) \in V_{\lambda+2}$ and $y(v) \in V_{\lambda-2}$. Indeed,

$$h(x(v)) = [h,x](v) + xh(v) = 2x(v) + \lambda x(v) = (\lambda+2)x(v),$$

and similarly for $y(v)$. We get

$$hx^r(v) = (\lambda + 2r)x^r(v) \quad \text{and} \quad hy^r(v) = (\lambda - 2r)y^r(v).$$

But, V is finite-dimensional, so both x and y are nilpotent on the eigenvectors of h in V.

Definition 2.12 *(Lefschetz)* An element, v, of the finite-dimensional representation space, V, for $\mathfrak{sl}(2,\mathbb{C})$ is *primitive* iff it is an eigenvector for h and $x(v) = 0$.

As V is a finite-dimensional $\mathbb{C}$-space, a primitive element must exist. Indeed, h has at least some eigenvalue, λ, and if $v \in V_\lambda$, then $x^r(v) \in V_{\lambda+2r}$, for all r. Since $V_{\lambda+2r} \cap V_{\lambda+2s} = (0)$, for $r \neq s$ and V is finite-dimensional, there is a smallest r so that $x^r(v) \neq 0$ and $x^{r+1}(v) = 0$. The vector $x^r(v)$ is a primitive element.

Proposition 2.70 *Say V is a finite-dimensional irreducible representation space for $\mathfrak{sl}(2,\mathbb{C})$. Pick any primitive vector, v, in V. Then, the vectors*

$$v, y(v), y^2(v), \ldots, y^t(v),$$

where $y^{t+1}(v) = 0$, form a basis for V. Hence,

(1) $\dim_{\mathbb{C}} V = t+1 =$ *index of nilpotence of Y on V.*

(2) Any two primitive v's give the same index of nilpotence.

Proof. Consider

$$W = \mathrm{span}(v, y(v), y^2(v), \ldots, y^t(v)).$$

If we show that h, x, y take W to itself, irreducibility of V implies $W = V$. Clearly, $y(W) \subseteq W$. As

$$hy^r(v) = (\lambda - 2r)y^r(v), \quad \text{if} \quad h(v) = \lambda v,$$

we also have $h(W) \subseteq W$. For x, we prove by induction on l that $xy^l(v) \in W$. When $l = 0$, we get $x(v) = 0$, and the claim holds trivially. Assume the claim holds for $l-1$. We have

$$\begin{aligned} xy^l(v) &= xy\, y^{l-1}(v) \\ &= (h + yx)(y^{l-1}(v)) \\ &= (\lambda - 2(l-1))y^{l-1}(v) + y(xy^{l-1}(v)), \end{aligned}$$

and $xy^{l-1}(v) \in W$, by the induction hypothesis. So, both terms on the right hand side are in W and the induction step is done. Now, $v, y(v), y^2(v), \ldots, y^t(v)$ are eigenvectors with distinct eigenvalues, so they must be linearly independent. Therefore, they form a basis of V. The rest is obvious. $\square$

Call an eigenspace for h on any (finite-dimensional) representation space a *weight space* and the *weight* is just the eigenvalue. We get

Corollary 2.71 *Every irreducible finite-dimensional representation, V, of $\mathfrak{sl}(2,\mathbb{C})$ is a finite coproduct of one-dimensional weight spaces, V_λ,*

$$V = \coprod_{\lambda} V_\lambda.$$

The "highest weight space" consists of 0 *and all the primitive vectors (each a multiple of the other).*

Proposition 2.72 *Say V is a finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module, then every eigenvalue of V is an integer. If V is irreducible, these are*

$$-t, -t+2, \ldots, t-2, t,$$

where $\dim_{\mathbb{C}} V = t+1 =$ index of nilpotence of Y on V. Therefore, the irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules are in one-to-one correspondence with the non-negative integers, t, via

$$t \mapsto V(t) = \coprod_{0\leq 2j\leq t} V_{-t+2j} \amalg \coprod_{0\leq 2j\leq t} V_{t-2j}$$

with $\dim_{\mathbb{C}} V = t+1$.

Proof. As V is finite-dimensional, there is a primitive element, v, and let λ be its weight (eigenvalue). Look at $xy^l(v)$. I claim:

$$xy^l(v) = (l\lambda - l(l-1))y^{l-1}(v).$$

This is shown by induction on l. For $l=0$, this is trivial $(0=0)$. Assume the claim hols for l. We have

$$\begin{aligned} xy^{l+1}(v) &= xy(y^l(v)) \\ &= h(y^l(v)) + yx(y^l(v)) \\ &= (\lambda - 2l)y^l(v) + y(l\lambda - l(l-1))y^{l-1}(v) \\ &= (\lambda - 2l + l\lambda - l^2 + l)y^l(v) \\ &= ((l+1)\lambda - (l+1)l)y^l(v), \end{aligned}$$

proving the induction hypothesis. Now, we know that there is some $t \geq 0$ so that $y^t(v) \neq 0$ and $y^{t+1}(v) = 0$, so let $l = t+1$. We get

$$0 = xy^{t+1}(v) = ((t+1)\lambda - (t+1)t)y^l(v),$$

that is,

$$(t+1)\lambda - (t+1)t = 0,$$

which means that $\lambda = t$, an integer. Now, say V is irreducible and t is the maximum weight in V. If V has weight t, then $x(v)$ has weight $t+2$, a contradiction, unless $x(v) = 0$. Therefore, v is primitive. Now, Proposition 2.70 implies that V is as claimed. $\square$

A useful alternate description of $V(t)$ is: $V(t) = \mathrm{Sym}^t(\mathbb{C}^2)$, with the natural action. For, a basis of $\mathrm{Sym}^t(\mathbb{C}^2)$ is

$$\xi^0\eta^t, \xi^1\eta^{t-1}, \ldots, \xi^t\eta^0.$$

Also,

$$\begin{aligned} h(\xi^i\eta^j) &= (i-j)\xi^i\eta^j \\ x(\xi^i\eta^j) &= \xi^{i+1}\eta^{j-1} \\ y(\xi^i\eta^j) &= \xi^{i-1}\eta^{j+1}. \end{aligned}$$

Now, say we look at V_k (the k weight space for some $\mathfrak{sl}(2,\mathbb{C})$-module, V). Observe that

$$y^k \colon V_k \to V_{-k}, \quad x^k \colon V_{-k} \to V_k,$$

and each is an isomorphism.

Suppose V is some finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module (not necessarily irreducible). Define PV = the *primitive part* of V by

$$PV = \mathrm{Ker}\ x.$$

We get

$$V = PV \coprod yPV \coprod y^2 PV \coprod \cdots,$$

the *Lefschetz decomposition* of V. We also have

$$(\mathrm{Ker}\ x) \cap V_k = \mathrm{Ker}\,(y^{k+1}\colon V_k \longrightarrow V_{-k-2}).$$

We can apply the above to X, a compact Kähler manifold and

$$V = H^{\bullet}_{\mathrm{DR}}(X, \mathbb{C}) = \coprod_{0 \leq r \leq 2n} H^r(X),$$

where $n = \dim_{\mathbb{C}} X$. The maps

$$h \mapsto [\Lambda, L], \quad x \mapsto \Lambda, \quad y \mapsto L$$

give a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $H^{\bullet}_{\mathrm{DR}}(X, \mathbb{C})$. Now, $H^r(X)$ is a weight space and the weight is $n - r$, so

$$H^r(X) = V_{n-r}.$$

Then,

$$L^{n-r}\colon H^r(X) \xrightarrow{\sim} V_{r-n} = H^{2n-r}(X)$$

and if we let

$$P^{n-k}(X) = (\mathrm{Ker}\,\Lambda) \cap H^{n-k}(X) = \mathrm{Ker}\,(L^{k+1}\colon H^{n-k}(X) \longrightarrow H^{n+k+2}(X)),$$

then the Lefschetz decomposition says

$$H^r(X) = P^r(X) \coprod LP^{r-2}(X) \coprod \cdots \coprod L^{[\frac{r}{2}]} P^{r-[\frac{r}{2}]}(X).$$

As a consequence, we get

Theorem 2.73 *(Hard Lefschetz Theorem) If X is a compact, Kähler manifold, then*

(1) $L^k\colon H^{n-k}(X) \longrightarrow H^{n+k}(X)$ is an isomorphism, $0 \leq k \leq n$.

(2) The cohomology has the Lefschetz decomposition

$$H^r(X) = P^r(X) \coprod LP^{r-2}(X) \coprod \cdots \coprod L^k P^{r-2k}(X) \coprod \cdots,$$

where $P^{n-k}(X) = (\mathrm{Ker}\,\Lambda) \cap H^{n-k}(X) = \mathrm{Ker}\,(L^{k+1}\colon H^{n-k}(X) \longrightarrow H^{n+k+2}(X))$.

(3) The primitive cohomology commutes with the (p,q)-decomposition, that is: If $P^{p,q} = P^r \cap H^{p,q}$, then

$$P^r = \coprod_{p+q=n} P^{p,q},$$

and a cohomology class is primitive iff each (p,q)-piece is primitive.
(Recall, $P^r(X) = H^r(X) \cap \mathrm{Ker}\ \Lambda = H^r(X) \cap \mathrm{Ker}\ L^{n+1-r}$.)

Proof. We already proved (1) and (2). But, (3) is clear as L and Λ commute with bidegree ((1, 1) and (−1, −1)). $\square$

Interpretation à la Lefschetz

Say $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ is a closed complex submanifold of $\mathbb{P}^N_{\mathbb{C}}$ equipped with the Fubini-Study metric, ω and let $n = \dim_{\mathbb{C}} X$. Recall that ω is given locally by

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log \|F\|^2,$$

where $F\colon U \subseteq \mathbb{P}^N_{\mathbb{C}} \to \mathbb{C}^{n+1} - \{0\}$ is any holomorphic local lifting. We know that ω is a real $(1, 1)$-form and it is d-closed but *not* d-exact. Let $[\omega]$ be the cohomology class of ω in $H^2_{\mathrm{DR}}(X, \mathbb{R})$.

By the duality between homology and cohomology,

$$H^2_{\mathrm{DR}}(X, \mathbb{R})^D \cong H_2(X, \mathbb{R}) \cong H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

where the pairing is: Given $[\alpha] \in H^2_{\mathrm{DR}}(X, \mathbb{R})$ and $[\eta] \in H_2(X, \mathbb{R})$,

$$([\alpha], [\eta]) = \int_\eta \alpha \in \mathbb{R}.$$

We also know that $H^k_{\mathrm{DR}}(X, \mathbb{R})$ and $H^{2n-k}_{\mathrm{DR}}(X, \mathbb{R})$ are Poincaré dual and this is given by

$$([\alpha], [\beta]) = \int_X \alpha \wedge \beta.$$

By duality, we get the a pairing

$$H_k(X, \mathbb{R}) \otimes H_{2n-k}(X, \mathbb{R}) \longrightarrow \mathbb{R},$$

a nondegenerate pairing and, geometrically, this is the intersection pairing

$$([\alpha], [\beta]) \mapsto [\alpha \cap \beta].$$

Poincaré duality shows $[\omega]$ *is* a homology class in $H_{2N-2}(\mathbb{P}^N, \mathbb{R})$. But, a generator for the latter group is the class of H, where H is a hyperplane. Consequently, there is some $\lambda \in \mathbb{R}$ so that $[\omega] = \lambda[H]$. Take a complex line, l, then we have the pairing

$$([l], [H]) \in H_0(\mathbb{P}^N, \mathbb{R}),$$

namely (as above), this number is $\#([H \cap l]) = 1$. Therefore,

$$([\omega], [l]) = \lambda.$$

But, $([\omega], [l])$ is computable. We can take l to be the line

$$z_2 = z_3 = \cdots = z_n = 0.$$

So, l is given by $(z_0\colon z_1\colon 0 \cdots\colon 0)$ and l is covered by $U_0 \cap l$ and $U_1 \cap l$. Now $l = (U_0 \cap l) \cup \{\mathrm{pt}\}$, so

$$\int_l \omega = \int_{U_0 \cap l} \omega$$

and a lifting on $U_0 \cap l$ is just

$$F((1\colon z\colon 0\colon \cdots\colon 0)) = (1, z, 0, \cdots, 0).$$

Consequently, $\|F\|^2 = 1 + |z|^2 = 1 + z\overline{z}$. We get

$$\overline{\partial}\log \|F\|^2 = \overline{\partial}\log(1 + z\overline{z}) = \frac{z d\overline{z}}{1 + z\overline{z}},$$

and

$$\partial\overline{\partial}\log\|F\|^2 = \frac{(1+z\overline{z})dz\wedge d\overline{z} - \overline{z}dz\wedge zd\overline{z}}{(1+z\overline{z})^2} = \frac{dz\wedge d\overline{z}}{(1+z\overline{z})^2}.$$

As $dz = dx + idy$ and $d\overline{z} = dx - idy$, we get $dz\wedge d\overline{z} = -2idx\wedge dy$. Now, $l\cap U_0 = \mathbb{C}$, so

$$\int_{l\cap U_0}\omega = \frac{i}{2\pi}\int_{\mathbb{C}}\frac{-2idx\wedge dy}{1+x^2+y^2)^2}.$$

If we use polar coordinates, then the right hand side is just

$$\frac{i}{2\pi}(-i)\int_0^{2\pi}\int_0^{\infty}\frac{2rdrd\theta}{(1+r^2)^2} = \int_0^{\infty}\frac{2rdr}{(1+r^2)^2} = \int_1^{\infty}\frac{du}{u^2} = \left[-\frac{1}{u}\right]_1^{\infty} = 1.$$

Therefore, $\lambda = 1$, and $[\omega] = [H] \in H_{2N-2}(\mathbb{P}^N, \mathbb{Z})$. Of course $\omega = \omega \restriction X = [H\cap X]$. Therefore,

$$L^k\colon H^{n-k}(X,\mathbb{C}) \xrightarrow{\sim} H^{n+k}(X,\mathbb{C})$$

becomes in homology,

$$\bigcap \text{ with } \mathbb{P}^{N-k}\colon H_{n+k}(X,\mathbb{C}) \xrightarrow{\sim} H_{n-k}(X,\mathbb{C}).$$

This is the geometric interpretation of Hard Lefschetz.

How about primitive cohomology (or homology)?

By definition, the sequence

$$0 \longrightarrow P^{n-k}(X) \longrightarrow H^{n-k}(X) \xrightarrow{L^{k+1}} H^{n+k+2}(X) \quad \text{is exact.}$$

When we dualize, we get

$$\begin{tikzcd}
H_{n+k+2}(X) \arrow[rr, "\cap\,\mathbb{P}^{N-(k+1)}"] \arrow[dr, "\cap H"'] & & H_{n-k}(X) \arrow[r] & P_{n-k}(X) \arrow[r] & 0 \\
& H_{n+k}(X) \arrow[ur, "\cap\,\mathbb{P}^{N-k}"'] & & &
\end{tikzcd}$$

Therefore, a cycle of dimension $n-k$ is primitive iff it does not cut the "hyperplane at infinity", i.e., if it arise from $H_{n-k}(X - X\cap H)$ in the map

$$H_{n-k}(X - X\cap H) \longrightarrow H_{n-k}(X).$$

(These are the "finite cycles")

We now consider the "Hodge-Riemann bilinear relations". Given X, compact, Kähler, we have the Poincaré duality

$$H^{n-k}(X,\mathbb{R})\otimes H^{n+k}(X,\mathbb{R}) \longrightarrow H^{2n}(X,\mathbb{R}) \cong \mathbb{R},$$

where $\dim_{\mathbb{C}} X = n$, given by

$$([\alpha],[\beta]) \mapsto \int_X \alpha\wedge\beta.$$

By the hard Lefschetz Theorem, $\beta = L^k(\gamma)$, for some unique $\gamma \in H^{n-k}(X)$, so we can define a bilinear pairing on $H^{n-k}(X)$ *via*

$$Q_{n-k}(\alpha,\gamma) = \int_X \alpha\wedge\gamma\wedge\omega^k.$$

The following properties hold:

(1) If $n-k$ is even, then Q_{n-k} is symmetric. Indeed,

$$\begin{aligned} Q_{n-k}(\gamma,\alpha) &= \int_X \gamma\wedge\alpha\wedge\omega^k \\ &= (-1)^{\deg(\alpha)\deg(\gamma)}\int_X \alpha\wedge\gamma\wedge\omega^k \\ &= Q_{n-k}(\alpha,\gamma). \end{aligned}$$

(2) If $n-k$ is odd, then Q_{n-k} is alternating.

(3) Q_{n-k} is a real form. This is because

$$\begin{aligned} \overline{Q_{n-k}(\alpha,\beta)} &= \overline{\int_X \alpha\wedge\gamma\wedge\omega^k} \\ &= \int_X \overline{\alpha}\wedge\overline{\gamma}\wedge\omega^k \\ &= Q_{n-k}(\overline{\alpha},\overline{\gamma}). \end{aligned}$$

(Recall, ω is real.)

By Hodge,
$$H^{n-k}(X,\mathbb{C}) = \coprod_{p+q=n-k} H^{p,q}.$$

Claim. For all $\alpha\in H^{p,q}$ and all $\beta\in H^{p',q'}$, we have $Q(\alpha,\beta)=0$ unless $p=q'$ and $p'=q$.

We have $p+q=p'+q'=n-k$ and
$$Q(\alpha,\beta)=\int_X \alpha\wedge\beta\wedge\omega^k$$
with $\alpha\wedge\beta\wedge\omega^k\in H^{p+p'+k,q+q'+k}$. As the only $2n$ form on X is an (n,n)-form, $Q(\alpha,\beta)\neq 0$ implies $p+p'+k=n=q+q'+k$, i.e., $p+p'=q+q'$. Hence (together with $p+q=p'+q'$) we deduce

(4) $Q(\alpha,\beta)\neq 0$ implies $p=q'$ and $p'=q$.

(1)-(4) suggest we consider
$$W_{n-k}(\alpha,\beta)=i^{n-k}Q_{n-k}(\alpha,\overline{\beta}).$$

(5) W_{n-k} is an Hermitian form on $H^{n-k}(X,\mathbb{C})$.

Now, when $n-k$ is even, we have
$$W_{n-k}(\beta,\alpha)=i^{n-k}Q_{n-k}(\beta,\overline{\alpha})=i^{n-k}Q_{n-k}(\overline{\alpha},\beta),$$
and so,
$$\overline{W_{n-k}(\beta,\alpha)}=\overline{i^{n-k}Q_{n-k}(\overline{\alpha},\beta)}=(-1)^{n-k}i^{n-k}Q_{n-k}(\alpha,\overline{\beta})=i^{n-k}Q_{n-k}(\alpha,\overline{\beta})=W(\alpha,\beta),$$
as $n-k$ is even. A similar argument applies when $n-k$ is odd, and W_{n-k} is an Hermitian form. We will need the following lemma:

Lemma 2.74 *If X is a compact, Kähler manifold of dimension $\dim_{\mathbb{C}} X=n$ and $\eta\in\bigwedge^{p,q}(X)\subseteq\bigwedge^k(X)$ (with $p+q=k$) with η primitive, then*
$$\overline{*\eta}=(-1)^{\binom{k+1}{2}}i^{p-q}\frac{1}{(n-k)!}L^{n-k}\eta.$$

Proof. A usual computation either to be supplied or for Homework. Let us check that, at least, both sides are of the same type. Indeed, the left hand side belongs to $\overline{\bigwedge^{n-p,n-q}} X = \bigwedge^{n-q,n-p} X$. Moreover,

$$L^{n-k}\eta \in \bigwedge^{p+n-k,q+n-k} X.$$

If we put $p+q=k$, then $p-k=-q$ and $q-k=-p$, so we have $L^{n-k}\eta \in \bigwedge^{n-q,n-p} X$.
$\square$

Theorem 2.75 *(Hodge-Riemann Bilinear Relations) Let X be compact, Kähler, with $\dim_{\mathbb{C}} X = n$, and examine $H^{n-k}(X,\mathbb{C})$ $(0 \le k \le n)$.*

(1) The form W_{n-k} makes both the Lefschetz and Hodge decomposition orthogonal coproducts.

(2) On $P^{p,q} \subseteq H^{n-k}(X,\mathbb{C})$, the form

$$(-1)^{\binom{n-k}{2}} i^{p-q-(n-k)} W_{n-k}$$

is positive definite. That is,

$$(-1)^{\binom{n-k}{2}} i^{p-q} Q_{n-k}(\alpha,\overline{\alpha}) > 0$$

whenever $\alpha \in P^{n-k}(X)$ and $\alpha \neq 0$.

(3) When W_{n-k} is restricted to $L^l P^{n-k-2l}$, it becomes $(-1)^l W_{n-k-2l}$.

Proof. (1) The Hodge components are orthogonal as $W(\alpha,\beta) = Q(\alpha,\overline{\beta})$ and use (4) above.

Observe that $W_r(\xi,\eta) = cQ_r(\xi,\overline{\eta})$, so we can replace W_r by Q_r. Then, we get

$$Q_r(\xi,\eta) = Q_{r-2}(\xi',\eta'),$$

where $\xi = L\xi'$ and $\eta = L\eta'$. Now assume $\xi = L^m\xi'$, $\eta = L^t\eta'$, with ξ,η' primitive and $m \neq t$. We may assume $m < t$. We have $\xi' \in \bigwedge^{r-2m}(X)$ and $\eta' \in \bigwedge^{r-2t}(X)$ and as ξ' is primitive, $L^{n+1-r+2m}\xi' = 0$. Then,

$$Q(\xi,\eta) = Q(L^m\xi', L^t\eta') = \int_X \xi' \wedge \eta' \wedge \omega^{n-r+m+t}.$$

Now, as $m < t$, we have $2m < m+t$, so $2m+1 \le m+t$ and

$$\xi' \wedge \eta' \wedge \omega^{n-r+m+t} = L^{n-r+m+t}\xi' \wedge \eta'$$

and $n-r+m+t \ge n-r+2m+1$; As $L^{n+1-r+2m}\xi' = 0$, we also have $L^{n-r+m+t}\xi' \wedge \eta' = 0$.

(2) (Bilinear Relations). Pick $\xi \in P^{p,q}$ and compute: By Lemma 2.74 (for k),

$$\overline{*\xi} = (-1)^{\binom{k+1}{2}} i^{p-q} \frac{1}{(n-k)!} L^{n-k}\xi,$$

so we get

$$*\xi = (-1)^{\binom{k+1}{2}} (-1)^{p-q} i^{p-q} \frac{1}{(n-k)!} L^{n-k}\overline{\xi},$$

and

$$L^{n-k}\overline{\xi} = (-1)^{\binom{k+1}{2}} (-1)^{p-q} i^{q-p} (n-k)! * \xi.$$

If we replace k by $n-k$, as $\xi \in \bigwedge^{n-k}(X)$, we get

$$L^k\overline{\xi} = (-1)^{\binom{n-k+1}{2}} (-1)^{p-q} i^{q-p} k! * \xi.$$

But,

$$\binom{n-k+1}{2} = \binom{n-k}{2} + p + q$$

(since $n - k = p + q$), so

$$(-1)^{\binom{n-k+1}{2}}(-1)^{p-q} = (-1)^{\binom{n-k}{2}}(-1)^{2p} = (-1)^{\binom{n-k}{2}},$$

which means that

$$L^k\overline{\xi} = (-1)^{\binom{n-k}{2}} i^{q-p} k! * \xi.$$

As

$$W_{n-k}(\alpha, \beta) = i^{n-k} Q_{n-k}(\alpha, \overline{\beta}),$$

we have

$$\begin{aligned}
(-1)^{\binom{n-k}{2})} i^{p-q-(n-k)} W(\xi, \xi) &= (-1)^{\binom{n-k}{2})} i^{p-q} Q(\xi, \overline{\xi}) \\
&= (-1)^{\binom{n-k}{2})} i^{p-q} \int_X \xi \wedge \overline{\xi} \wedge \omega^k \\
&= (-1)^{\binom{n-k}{2})} i^{p-q} \int_X \xi \wedge L^k\overline{\xi} \\
&= (-1)^{\binom{n-k}{2})} i^{p-q} (-1)^{\binom{n-k}{2})} i^{q-p} k! \int_X \xi \wedge *\xi \\
&= k!\, \|\xi\|_{L^2}^2 > 0, \quad \text{as } \xi \neq 0.
\end{aligned}$$

(3) We have

$$\begin{aligned}
W_r(\xi, \eta) &= i^r Q_r(\xi, \overline{\eta}) \\
&= i^r Q_{r-2l}(\xi', \overline{\eta'}) \\
&= \frac{i^r}{i^{r-2l}} W_{r-2l}(\xi', \overline{\eta'}) \\
&= (-1)^l W_{r-2l}(\xi', \overline{\eta'}),
\end{aligned}$$

which proves (3). $\square$

Remarks:

(1) For all $p \geq 0$, we have $H^{p,0} = P^{p,0}$ and $H^{0,p} = P^{0,p}$. It is enough to prove it for one of the two equations. Take $\xi \in H^{p,0}$. Then, ξ is primitive iff $L^x\xi = 0$, where $x + p = n + 1$. We deduce $x = n + 1 - p$ and then,

$$L^x\xi = L^{n+1-p}\xi \in \bigwedge^{n+1,n+1-p}(X) = (0),$$

as $\dim_{\mathbb{C}} X = n$.

(2) Lefschetz says

$$H^{p,q} = \coprod_{0 \leq k \leq \left[\frac{p+q}{2}\right]} L^k P^{p-k,q-k},$$

a coproduct of lower primitive cohomologies. But, $H^{p-1,q-1}$ itself is the coproduct of its lowr primitives, which are strictluy lower primitives of $H^{p,q}$. Therefore, we conclude that

$$H^{p,q} = P^{p,q} \amalg H^{p-1,q-1}.$$

(with $p + q \leq n$). Therefore, we have

(a) $h^{p,q} = \dim P^{p,q} + h^{p-1,q-1}$ $(p+q \leq n)$

(b) $H^{p,q} \geq h^{p-1,q-1}$, for $p+q \leq n$.

We have our pairing

$$Q_{n-k}\colon H^{n-k} \otimes H^{n-k} \to H^{n-k} \cong \mathbb{C},$$

given by

$$Q_{n-k}(\xi, \eta) = \int_X \xi \wedge \eta \wedge \omega^k.$$

When $n-k$ is even, our pairing is symmetric and when $n-k$ is odd, it is alternating. The most important case is when $k=0$, in which case,

$$Q = Q_n\colon H^n \otimes H^n \to \mathbb{C}$$

is given by

$$Q(\xi, \eta) = \int_X \xi \wedge \eta,$$

the intersection pairing (in homology).

Corollary 2.76 *If X is compact, Kähler, the forms Q_r on $H^r(X, \mathbb{C})$ are always nondegenerate.*

Proof. We have

$$H^r = \coprod_{0 \leq k \leq \left[\frac{r}{2}\right]} L^k P^{r-2k},$$

a Q-orthogonal decomposition. We need only look at the cofactors. On the cofactors, Q is Q_{lower} and this is (up to a constant) positive or negative, so each Q_{lower} is nondegenerate. $\square$

For $n = 2r$ and $\dim_{\mathbb{C}} X = n = 2r$ (so, $\dim_{\mathbb{R}} X \equiv 0\,(4)$) our form Q on H^n is symmetric, nondegenerate and real. By Sylvester's inertia theorem, Q is known if we know its signature ($= \mathrm{sgn}(Q)$).

The index of X, denoted $I(X)$ is by definition the signature, $\mathrm{sgn}(Q)$, where Q is the intersection form on the middle cohomology, $H^n(X, \mathbb{C})$, when n is even. So, $I(X)$ makes sense if $\dim_{\mathbb{R}} X \equiv 0\,(4)$.

Theorem 2.77 *(Hodge Index Theorem) If X is an even (complex) dimensional, compact, Kähler manifold, say $\dim_{\mathbb{C}} X = n = 2r$, then*

$$I(X) = \sum_{p,q} (-1)^p h^{p,q} = \sum_{p+q \text{ even}} (-1)^p h^{p,q}.$$

Proof. From the Lefschetz decomposition for $H^n(X, \mathbb{C})$, we have

$$H^n(X, \mathbb{C}) = \coprod_{0 \leq k \leq \frac{n}{2}} L^k P^{n-2k}(X).$$

Since this is a Q (and also a W) orthogonal decomposition, we have

$$I(X) = \mathrm{sgn}(Q) = \sum_{0 \leq k \leq \frac{n}{2}} \mathrm{sgn}(Q) \restriction P^{n-2k} = \sum_{0 \leq k \leq \frac{n}{2}} \mathrm{sgn}(W) \restriction P^{n-2k}.$$

Again, the (p,q)-decomposition is orthogonal, so

$$I(X) = \sum_{0 \leq k \leq \frac{n}{2}} \sum_{p+q = n-2k} \mathrm{sgn}(W) \restriction P^{p,q}.$$

But, we know that $\dim P^{p,q} = h^{p,q} - h^{p-1,q-1}$, by Remark (2) and W on $P^{p,q}$ is definite, with sign

$$(-1)^{\binom{p+q}{2}} i^{p-q}. \tag{$*$}$$

As $i^{p-q} = (-1)^{\frac{p-q}{2}}$, we have

$$(-1)^{\binom{p+q}{2}} = (-1)^{\frac{(p+q)(p+q-1)}{2}} = (-1)^{\frac{p+q}{2}} (-1)^{\frac{p+q}{2}(p+q)},$$

so

$$(*) = (-1)^{\frac{p-q}{2}} (-1)^{\frac{p+q}{2}} (-1)^{\frac{p+q}{2}(p+q)} = (-1)^p,$$

as $p + q = n - 2k$ and n is even. Thus,

$$\begin{aligned}
I(X) &= \sum_{\substack{p+q \text{ even} \\ p+q \leq n}} (-1) \dim P^{p,q} \\
&= \sum_{\substack{p+q \text{ even} \\ p+q \leq n}} (-1)^p (h^{p,q} - h^{p-1,q-1}) \\
&= \sum_{p+q=n} (-1)^p h^{p,q} - \sum_{p+q=n} (-1)^p h^{p-1,q-1} + \sum_{p+q=n-2} (-1)^p h^{p,q} - \sum_{p+q=n-2} (-1)^p h^{p-1,q-1} + \cdots \\
&= \sum_{p+q=n} (-1)^p h^{p,q} + 2 \sum_{\substack{p+q \text{ even} \\ p+q<n}} (-1)^p h^{p,q}.
\end{aligned}$$

But, as n is even and by duality,

$$(-1)^{n-p} h^{n-p,n-q} = (-1)^p h^{n-p,n-q} = (-1)^p h^{p,q},$$

so the right hand side above is

$$\sum_{p+q=n} (-1)^p h^{p,q} + \sum_{\substack{p+q \text{ even} \\ p+q \neq n}} (-1)^p h^{p,q},$$

so we get

$$I(X) = \sum_{p+q \text{ even}} (-1)^p h^{p,q}.$$

Now, we show that

$$\sum_{p+q \text{ odd}} (-1)^p h^{p,q} = 0.$$

Since X is Kähler, we know that $h^{p,q} = h^{q,p}$. Therefore,

$$(-1)^p h^{p,q} = (-1)^p h^{q,p} = -(-1)^q h^{q,p},$$

since $-1 = (-1)^{p+q} = (-1)^p (-1)^q$, as $p + q$ is odd. But,

$$\sum_{p+q \text{ odd}} (-1)^p h^{p,q} = \sum_{p+q \text{ odd}} (-1)^q h^{q,p} = - \sum_{p+q \text{ odd}} (-1)^p h^{p,q}.$$

So, $\sum_{p+q \text{ odd}} (-1)^p h^{p,q} = 0$, as claimed. $\square$

Example: The case of a complex, Kähler surface. In this case, $n = \dim_{\mathbb{C}} X = 2$, so

$$I(X) = \sum_{p+q \text{ even}} (-1)^p h^{p,q} = h^{0,0} + h^{2,0} - h^{1,1} + h^{0,2} + h^{2,2},$$

i.e.,

$$I(X) = 2 + 2h^{0,2} - h^{1,1}.$$

We know that $h^{0,2} = \dim_{\mathbb{C}} H^2(X, \mathcal{O}_X) = p_g =$ *geometric genus*. So,

$$I(X) = 2 + 2p_g - h^{1,1}.$$

The number $q = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = \frac{1}{2}b_1$ is called the *irregularity* of X. By Lefschetz, we also have

$$h^{1,1} = p^{1,1} + 1,$$

where $p^{r,s} = \dim_{\mathbb{C}} P^{r,s}$.

Let's look at Q restricted to $H^{1,1}(X)$. Now,

$$H^{1,1} = P^{1,1} \coprod^{\perp} LP^{0,0} = P^{1,1} \coprod^{\perp} LH^{0,0}.$$

So, $\mathrm{sgn}(Q) \restriction H^{1,1} = \mathrm{sgn}(Q) \restriction P^{1,1} + \mathrm{sgn}(Q) \restriction H^{0,0}$. We know that on $P^{1,1}$, the form $(-1)^{\binom{2}{2}} i^{1-1} Q(\alpha, \overline{\beta})$ is positive. Thus, Q is negative on $P^{1,1}$. On $H^{0,0}$ (up by L in $H^{1,1}$), we have $Q > 0$. Therefore,

$$\mathrm{sgn}(Q) \restriction H^{1,1} = 1 - \dim_{\mathbb{C}} P^{1,1}$$

and the one positive eigenvector (i.e., corresponding to the positive eigenvalue) is $[\omega] = [H \cap X]$, where H is a hyperplane of $\mathbb{P}^N_{\mathbb{C}}$. Therefore, we get

Corollary 2.78 *(Hodge Index for Holomorphic Cycles on a Surface) If X is a Kähler, compact, surface then in $H^{1,1}$ we can choose a basis so that:*

(a) The first basis vector is a multiple of $[\omega] = [H \cap X]$.

(b) the matrics of Q on $H^{1,1}$ is $\mathrm{diag}(1, -1, -1, \cdots, -1)$.

Let's examine Q on $H^{1,0}$ and $H^{0,1}$, this is Riemann's case: We know $P^{1,0} = H^{1,0}$; $P^{0,1} = H^{0,1}$. Now, $Q \restriction H^{1,0} = (-1)^{\binom{1}{2}} i^{1-0} Q > 0$, which means that

$$iQ(\xi, \overline{\xi}) > 0 \quad \text{if } \xi \text{ is a } (1,0) \text{ form.}$$

We also have $Q \restriction H^{0,1} = (-1)^{\binom{1}{2}} i^{0-1} Q > 0$, which means that

$$-iQ(\xi, \overline{\xi}) > 0 \quad \text{if } \xi \text{ is a } (0,1) \text{ form.}$$

Say X is a Kähler, complex, compact manifold. We have the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0,$$

where $\exp(f) = e^{2\pi i f}$. If we apply cohomology, we get

$$\begin{array}{l} 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow \\ \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c} \\ \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow \cdots \end{array}$$

We get the exact sequence

$$0 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c} H^2(X,\mathbb{Z}).$$

The group $H^1(X,\mathcal{O}_X^*)$ is called the (analytic) *Picard group* of X; notation: $\mathrm{Pic}(X)$.

I claim: The group $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathcal{O}_X) \cong \mathbb{C}^q$, where $b_1(X) = 2q$.

To see this, look at the inclusions

$$\mathbb{Z} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathcal{O}_X$$

and examine first $H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathbb{R})$. We compute these groups by Čech cohomology and all takes place for finite covers and opens that are diffeomorphic to convex opens. It follows (DX) that $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathbb{R})$. Examine the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \bigwedge^0 \arrow[r, "d"] \arrow[d] & \bigwedge^1 \arrow[r, "d"] \arrow[d] & \bigwedge^2 \arrow[r, "d"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \bigwedge^0 \arrow[r, "d"] \arrow[d] & \bigwedge^1 \arrow[r, "d"] \arrow[d] & \bigwedge^2 \arrow[r, "d"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \bigwedge^{0,0} \arrow[r, "\overline{\partial}"] & \bigwedge^{0,1} \arrow[r, "\overline{\partial}"] & \bigwedge^{0,2} \arrow[r, "\overline{\partial}"] & \cdots
\end{tikzcd}$$

where the vertical maps between the first two rows are complexification and the maps between the second and the third row are projection on $(0,-)$. This implies that our maps come from de Rham and Dolbeault. But,

$$H^1(X,\mathbb{C}) = H^1(X,\mathbb{R})z \amalg H^1(X,\mathbb{R})\overline{z}$$

and $H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O}_X)$ is the map

$$H^1(X,\mathbb{R})z \amalg H^1(X,\mathbb{R})\overline{z} \longrightarrow H^1(X,\mathbb{R})\overline{z},$$

so, the composite map

$$H^1(X,\mathbb{R}) \hookrightarrow H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O}_X)$$

is an isomorphism over $\mathbb{R}$. Therefore, the claim is proved.

Since $\mathrm{rk}\, H^1(X,\mathbb{Z}) = 2q$ (where $q = \dim_{\mathbb{R}} H^1(X,\mathbb{R})$), we deduce that

$$H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \cong \mathbb{C}^q/\mathbb{Z}^{2q}.$$

Therefore, $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is a q-dimensional complex torus. This torus, denoted $\mathrm{Pic}^0(X)$, is called the *Picard manifold* of X. The image of c into $H^2(X,\mathbb{Z})$ is called the *Néron-Severi group* of X; it is denoted $\mathrm{NS}(X)$. Observe that $\mathrm{NS}(X) \hookrightarrow H^2(X,\mathbb{Z}) =$ a finitely generated abelian group, as X is compact. Consequently, $\mathrm{NS}(X)$ is a finitely generated abelian group. Moreover, the sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0 \quad \text{is exact.}$$

2.7 Extensions of Results to Vector Bundles

Say X is a complex manifold and E is a holomorphic vector bundle on X. Put hermitian metrics on X and E. We know $(\bigwedge^{p,q} T_X^D)_x$ gets an inner product and E_x has one too; so, $\left(\bigwedge^{p,q} T_X^D\right) \otimes E$ has an inner product. We have a pairing

$$\begin{tikzcd} E \otimes E^D \arrow[r] \arrow[dr] & \mathcal{E}nd_{\mathcal{O}_X}(E) \arrow[d, "\mathrm{tr}"] \\ & \mathcal{O}_X \end{tikzcd}$$

a nondegenerate bilinear form. We also have a nondegenerate pairing

$$\Big(\bigwedge^{p,q} T_X^D\Big) \otimes \Big(\bigwedge^{n-p,n-q} T_X^D\Big) \longrightarrow \bigwedge^{n,n} T_X^D \cong \mathbb{C} \qquad (*)$$

(The isomorphism $\bigwedge^{n,n} T_X^D \cong \mathbb{C}$ is given by the volume form.) Using the metric, we have an isomorphism

$$\left(\Big(\bigwedge^{p,q} T_X^D\Big) \otimes E\right)^D \cong \Big(\bigwedge^{p,q} T_X^D\Big) \otimes E.$$

From $(*)$, we have the pairing

$$\left(\Big(\bigwedge^{p,q} T_X^D\Big) \otimes E\right) \otimes \left(\Big(\bigwedge^{n-p,n-q} T_X^D\Big) \otimes E^D\right) \longrightarrow \Big(\bigwedge^{n,n} T_X^D\Big) \otimes \mathcal{O}_X \cong \mathcal{O}_X,$$

so, we have the isomorphism

$$\left(\Big(\bigwedge^{p,q} T_X^D\Big) \otimes E\right)^D \cong \Big(\bigwedge^{n-p,n-q} T_X^D\Big) \otimes E^D.$$

The composite isomorphism

$$\Big(\bigwedge^{p,q} T_X^D\Big) \otimes E \cong \left(\Big(\bigwedge^{p,q} T_X^D\Big) \otimes E\right)^D \cong \Big(\bigwedge^{n-p,n-q} T_X^D\Big) \otimes E^D,$$

is the Hodge $*$ in the case of a v.b.:

$$*_E \colon \Big(\bigwedge^{p,q} T_X^D\Big) \otimes E \longrightarrow \Big(\bigwedge^{n-p,n-q} T_X^D\Big) \otimes E^D.$$

If X is compact, as

$$(\xi,\eta)_x(\text{vol. form})_x = \xi_x \wedge *\eta_x,$$

by definition, we set

$$(\xi,\eta) = \int_X \xi \wedge *\eta = \int_X (\xi,\eta)_x(\text{vol. form})_x.$$

This gives $\Gamma_{\mathcal{C}^\infty}(X, (\bigwedge^{p,q} T_X^D) \otimes E)$ an inner product. let ∇_E be the uniholo connection on E. Then, $- *_E \circ \nabla_E \circ *_E$ is the formal adjoint of ∇_E, denoted ∇_E^*. We have as well $\nabla_E^{1,0}$, $\nabla_E^{0,1} = \overline{\partial}_E$ and $\nabla_E^{*\,1,0} = \overline{\partial}_E^*$. We define $\Box_E$; $\Box_E^{1,0}$; $\Box_E^{0,1}$; *via*:

$$\begin{aligned} \Box_E &= \nabla_E\nabla_E^* + \nabla_E^*\nabla_E \\ \Box_E^{1,0} &= \nabla_E^{1,0}\nabla_E^{*,\,1,0} + \nabla_E^{*\,1,0}\nabla_E^{1,0} \\ \Box_E^{0,1} &= \nabla_E^{0,1}\nabla_E^{*,\,0,1} + \nabla_E^{*\,0,1}\nabla_E^{0,1}. \end{aligned}$$

Theorem 2.79 *(Dolbeault's Theorem, V.B. Case) If X is a complex manifold and E is a holomorphic vector bundle, then*

$$H^q(X, \Omega_X^p \otimes E) \cong H^{p,q}_{\nabla_E^{0,1}}(X, E).$$

(Here, $\Omega_X^p = \Gamma(X, \bigwedge^{p,0} T_X^D)$.)

Theorem 2.80 *(Hodge's Theorem, compact manifold, V.B. Case) If X is a complex, compact manifold and E is a holomorphic vector bundle on X, both X and E having hermitian metrics, then*

$$\Gamma_{\mathcal{C}^\infty}\Big(X, (\bigwedge^{p,q} T_X^D) \otimes E\Big) = \mathcal{H}^{p,q} \overset{\perp}{\coprod} \mathrm{Im}\, \nabla_E^{0,1} \overset{\perp}{\coprod} \mathrm{Im}\, \nabla_E^{*,\,0,1}$$

where this is an orthogonal coproduct and

$$\mathcal{H}^{p,q} = \mathrm{Ker}\, \Box_E^{0,1}$$

and $\mathcal{H}^{p,q}$ is finite dimensional, for all p, q.

We also have

Theorem 2.81 *(Serre Duality, V.B. Case) If X is a complex, compact manifold of dimension $n = \dim_{\mathbb{C}} X$ and E is a holomorphic vector bundle on X, then the pairing*

$$(\xi, \eta) \mapsto \int_X \xi \wedge \eta$$

is a nondegenerate pairing of finite dimensional vector spaces

$$H^{p,q}(X, \mathcal{O}_X(E)) \otimes H^{n-p,n-q}(X, \mathcal{O}_X(E^D)) \longrightarrow \mathbb{C},$$

where $H^{p,q}(X, \mathcal{O}_X(E)) = H^q(X, \Omega_X^p \otimes E)$. That is, we have the isomorphism

$$H^{n-q}(X, \Omega_X^{n-q} \otimes E^D) \cong (H^q(X, \Omega_X^p \otimes E))^D.$$

When $p = 0$, set $\omega_X = \Omega_X^n$, then

$$H^{n-q}(X, \omega_X \otimes E^D) \cong (H^q(X, E))^D.$$

Theorem 2.82 *(Hodge's Theorem, Kähler and V.B. Case) If X is a complex, compact, Kähler manifold (of dimension $n = \dim_{\mathbb{C}} X$) and E is a holomorphic vector bundle on X with a flat connection (i.e., the curvature of the connection is identically zero), then there is a canonical isomorphism*

$$\coprod_{p+q=k} H^{p,q}(X, \mathcal{O}_X(E)) \cong H^k(X, \mathcal{O}_X(E))$$

and moreover, there are isomorphisms

(a)
$$H^{p,q}_{\mathrm{BC}}(X, \mathcal{O}_X(E)) \cong H^{p,q}(X, \mathcal{O}_X(E))$$

(where, $H^{p,q}_{\mathrm{BC}}(X, \mathcal{O}_X(E)) = (\mathrm{Ker}\, \nabla_E$ on $\bigwedge^{p,q}(X, E))/(\mathrm{Im}\, \nabla_E^{1,0} \nabla_E^{0,1}$ from $\bigwedge^{p-1,q-1}(X, E))$.)

(b)
$$\overline{H^{p,q}_{\mathrm{BC}}(X, \mathcal{O}_X(E))} \cong H^{q,p}(X, \mathcal{O}_X(E^D)).$$

The proof is the same as before.

A nice example of the Hodge Index theorem

Let X be a compact, Kähler surface (so, $\dim_{\mathbb{C}} X = 2$). Then, we already saw that

$$I(X) = 2 + 2h^{0,0} - h^{1,1}.$$

Let

$$\chi_{\mathrm{top}}(X) = \text{Euler-Poincaré}(X) = \sum_{i=0}^{2} (-1)^i b_i = \sum_{p,q=0}^{2} (-1)^{p+q} h^{p,q} = 2 - 4h^{0,1} - 2h^{0,2} + h^{1,1}.$$

Observe that

$$\begin{aligned}
I(X) + \chi_{\mathrm{top}}(X) &= 4 - 4h^{0,1} + 4h^{0,2} \\
&= = 4(\dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) + \dim H^2(X, \mathcal{O}_X)) \\
&= 4(1 - q + p_g) \\
&= 4(1 + p_a) = 4\chi(X, \mathcal{O}_X).
\end{aligned}$$

(Recall $p_a = p_g - q$.) Now, the Hirzebruch-Riemann-Roch Theorem (HRR) for X is equivalent with

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2),$$

where c_1, c_2 are the *Chern classes* of $T_X^{1,0}$, with $c_j \in H^{2j}(X, \mathbb{Z})$ and one shows that $\chi_{\mathrm{top}}(X) = c_2$. In fact, for any compact, complex manifold, X, of complex dimension n, the top Chern class of $T_X^{1,0}$, namely, c_d (the *Euler class*) is equal to $\chi_{\mathrm{top}}(X)$. Therefore,

$$I(X) + c_2 = \frac{1}{3}(c_1^2 + c_2),$$

iff HRR holds. Consequently,

$$I(X) = \frac{1}{3}(c_1^2 - 2c_2) \quad \text{iff HRR holds.}$$

This last statement is the Hirzebruch signature theorem for a complex, compact surface and the Hirzebruch signature theorem is equivalent to HRR.

Case of a Compact Riemann Surface

Compute c_1 = the highest Chern class of $T_X^{1,0}$. We have

$$\begin{aligned}
c_1 &= \chi_{\mathrm{top}}(X) \\
&= b_0 - b_1 + b_2 \\
&= 2 - b_1 \\
&= 2(1 - h^{0,1}) \\
&= 2((\dim H^0(X, \mathcal{O}_X) - (\dim H^1(X, \mathcal{O}_X)) \\
&= 2\chi(X, \mathcal{O}_X).
\end{aligned}$$

We get a form of the Riemann-Roch theorem:

$$\chi(X, \mathcal{O}_X) = \frac{1}{2}c_1.$$

Since $\chi(X, \mathcal{O}_X) = 1 - g$ (by definition, $g = \dim H^1(X, \mathcal{O}_X)$), we get

$$c_1 = 2 - 2g.$$

Chapter 3

The Hirzebruch-Riemann-Roch Theorem

3.1 Line Bundles, Vector Bundles, Divisors

From now on, X will be a complex, irreducible, algebraic variety (not necessarily smooth). We have

(I) X with the Zariski topology and $\mathcal{O}_X$ = germs of algebraic functions. We will write X or X_{Zar}.

(II) X with the complex topology and $\mathcal{O}_X$ = germs of algebraic functions. We will write $X_{\mathbb{C}}$ for this.

(III) X with the complex topology and $\mathcal{O}_X$ = germs of holomorphic functions. We will write X^{an} for this.

(IV) X with the complex topology and $\mathcal{O}_X$ = germs of $\mathcal{C}^{\infty}$-functions. We will write $X_{\mathcal{C}^{\infty}}$ or X_{smooth} in this case.

Vector bundles come in four types: Locally trivial in the Z-topology (I); Locally trivial in the $\mathbb{C}$-topology (II, III, IV).

Recall that a *rank r vector bundle over X* is a space, E, together with a surjective map, $p\colon E \to X$, so that the following properties hold:

(1) There is some open covering, $\{U_\alpha \longrightarrow X\}$, of X and isomorphisms

$$\varphi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \prod \mathbb{C}^r \qquad (\textit{local triviality})$$

We also denote $p^{-1}(U_\alpha)$ by $E \restriction U_\alpha$.

(2) For every α, the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "p"'] & & U_\alpha \prod \mathbb{C}^r \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

(3) Consider the diagram

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] & U_\alpha \prod \mathbb{C}^r \\
p^{-1}(U_\alpha \cap U_\beta) \arrow[u, hook] \arrow[r, "\varphi_\alpha"] \arrow[d, equal] & (U_\alpha \cap U_\beta) \prod \mathbb{C}^r \arrow[u, hook] \arrow[d, "g_\alpha^\beta"] \\
p^{-1}(U_\beta \cap U_\alpha) \arrow[r, "\varphi_\beta"] \arrow[d, hook] & (U_\beta \cap U_\alpha) \prod \mathbb{C}^r \arrow[d, hook] \\
p^{-1}(U_\beta) \arrow[r, "\varphi_\beta"] & U_\beta \prod \mathbb{C}^r
\end{tikzcd}$$

where $g_\alpha^\beta = \varphi_\beta \circ \varphi_\alpha^{-1} \restriction p^{-1}(U_\alpha \cap U_\beta)$. Then,

$$g_\alpha^\beta \restriction U_\alpha \cap U_\beta = \mathrm{id} \quad \text{and} \quad g_\alpha^\beta \restriction \mathbb{C}^r \in \mathrm{GL}_r(\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X))$$

and the functions g_α^β in the glueing give type II, III, IV.

On triple overlaps, we have

$$g_\beta^\gamma \circ g_\alpha^\beta = g_\alpha^\gamma \quad \text{and} \quad g_\beta^\alpha = (g_\alpha^\beta)^{-1}.$$

This means that the $\{g_\alpha^\beta\}$ form a 1-cocycle in $Z^1(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r)$. Here, we denote by $\mathbb{GL}_r(X)$, or simply $\mathbb{GL}_r$, the sheaf defined such that, for every open, $U \subseteq X$,

$$\Gamma(U, \mathbb{GL}_r(X)) = \mathrm{GL}_r(\Gamma(U, \mathcal{O}_X)),$$

the group of invertible linear maps of the free module $\Gamma(U, \mathcal{O}_X)^r \cong \Gamma(U, \mathcal{O}_X^r)$. When $r = 1$, we also denote the sheaf $\mathbb{GL}_1(X)$ by $\mathbb{G}_m$, or $\mathcal{O}_X^*$.

Say $\{\psi_\alpha\}$ is another trivialization. We may assume (by refining the covers) that $\{\varphi_\alpha\}$ and $\{\psi_\alpha\}$ use the same cover. Then, we have an isomorphism, $\sigma_\alpha \colon U_\alpha \prod \mathbb{C}^r \to U_\alpha \prod \mathbb{C}^r$:

$$\begin{tikzcd}
 & U_\alpha \prod \mathbb{C}^r \arrow[dd, "\sigma_\alpha"] \\
p^{-1}(U_\alpha) \arrow[ur, "\varphi_\alpha"] \arrow[dr, "\psi_\alpha"'] & \\
 & U_\alpha \prod \mathbb{C}^r
\end{tikzcd}$$

We see that $\{\sigma_\alpha\}$ is a 0-cochain in $C^0(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r)$. Let $\{h_\alpha^\beta\}$ be the glueing data from $\{\psi_\alpha\}$. Then, we have

$$\begin{aligned}
\varphi_\beta &= g_\alpha^\beta \circ \varphi_\alpha \\
\psi_\beta &= h_\alpha^\beta \circ \psi_\alpha \\
\psi_\alpha &= \sigma_\alpha \circ \varphi_\alpha.
\end{aligned}$$

From this, we deduce that $\sigma_\beta \circ \varphi_\beta = \psi_\beta = h_\alpha^\beta \circ \sigma_\alpha \circ \varphi_\alpha$, and then

$$\varphi_\beta = (\sigma_\beta^{-1} \circ h_\alpha^\beta \circ \sigma_\alpha) \circ \varphi_\alpha,$$

so

$$g_\alpha^\beta = \sigma_\beta^{-1} \circ h_\alpha^\beta \circ \sigma_\alpha.$$

This gives an equivalence relation, $\sim$, on $Z^1(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r)$. Set

$$H^1(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r) = Z^1/\sim.$$

This is a pointed set. If we pass to the right limit over covers by refinement and call the pointed set from the limit $\check{H}^1(X, \mathbb{GL}_r)$, we get

Theorem 3.1 *If X is an algebraic variety of one of the types $T =$ I, II, III, IV, then the set of isomorphism classes of rank r vector bundles, $\mathrm{Vect}_{T,r}(X)$, is in one-to-one correspondence with $\check{H}^1(X, \mathbb{GL}_r)$.*

Remarks:

(1) If F is some "object" and $\mathrm{Aut}(F) =$ is the group of automorphisms of F (in some catgeory), then an X-*torsor* for F is just an "object, E, over X", locally (on X) of the form $U \prod F$ and glued by the pairs (id, g), where $g \in \mathrm{Maps}(U \cap V, \mathrm{Aut}(F))$ on $U \cap V$. The theorem says: $\check{H}^1(X, \mathrm{Aut}(F))$ classifies the X-torsors for F.

Say $F = \mathbb{P}^r_{\mathbb{C}}$, we'll show that in the types I, II, III, $\mathrm{Aut}(F) = \mathbb{PGL}_r$, where

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{GL}_{r+1} \longrightarrow \mathbb{PGL}_r \longrightarrow 0 \quad \text{is exact.}$$

(2) Say $1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$ is an exact sequence of sheaves of (not necessarily commutative) groups. Check that

$$\begin{tikzcd}
1 \arrow[r] & G'(X) \arrow[r] & G(X) \arrow[r] & G''(X) \arrow[dll, "\delta_0"] \\
& \check{H}^1(X, G') \arrow[r] & \check{H}^1(X, G) \arrow[r] & \check{H}^1(X, G'')
\end{tikzcd}$$

is an exact sequence of pointed sets. To compute $\delta_0(\sigma)$ where $\sigma \in G''(X)$, proceed as follows: Cover X by suitable U_α and pick $s_\alpha \in G(U_\alpha)$ mapping to $\sigma \restriction U_\alpha$ in $G''(U_\alpha)$. Set

$$\delta_0(\sigma) = s_\alpha s_\beta^{-1} \quad \text{on } U_\alpha \cap U_\beta/\sim.$$

We find that $\delta_0(\sigma) \in \check{H}^1(X, G')$. When $G' \subseteq Z(G)$, we get the exact sequence

$$\begin{tikzcd}
1 \arrow[r] & G'(X) \arrow[r] & G(X) \arrow[r] & G''(X) \arrow[dll, "\delta_0"] \\
& \check{H}^1(X, G') \arrow[r] & \check{H}^1(X, G) \arrow[r] & \check{H}^1(X, G'') \arrow[dll, "\delta_1"] \\
& \check{H}^2(X, G') & &
\end{tikzcd}$$

(3) Apply the above to the sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{GL}_{r+1} \longrightarrow \mathbb{PGL}_r \longrightarrow 1.$$

If X is a projective variety, we get

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \mathbb{GL}_{r+1}(\Gamma(X, \mathcal{O}_X)) \longrightarrow \mathbb{PGL}_r(\Gamma(X, \mathcal{O}_X)) \longrightarrow 0,$$

because $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$ and $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Consequently, we also have

$$0 \longrightarrow \check{H}^1(X, \mathcal{O}_X^*) \longrightarrow \check{H}^1(X, \mathbb{GL}_{r+1}) \longrightarrow \check{H}^1(X, \mathbb{PGL}_r) \longrightarrow \check{H}^2(X, \mathcal{O}_X^*) = \mathrm{Br}(X),$$

where the last group, $\mathrm{Br}(X)$, is the cohomological *Brauer group* of X of type T. By our theorem, $\check{H}^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ classifies type T line bundles, $\check{H}^1(X, \mathbb{GL}_{r+1})$ classifies type T rank $r+1$ vector bundles and $\check{H}^1(X, \mathbb{PGL}_r)$ classifies type T fibre bundles with fibre $\mathbb{P}^r_{\mathbb{C}}$ (all on X).

Let X and Y be two topological spaces and let $\pi\colon Y \to X$ be a surjective continuous map. Say we have sheaves of rings $\mathcal{O}_X$ on X and $\mathcal{O}_Y$ on Y; we have a homomorphism of sheaves of rings, $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$. Then, each $\mathcal{O}_Y$-module (or $\mathcal{O}_Y$-algebra), $\mathcal{F}$, gives us the $\mathcal{O}_X$-module (or algebra), $\pi_*\mathcal{F}$ on X (and more generally, $R^q\pi_*\mathcal{F}$) as follows: For any open subset, $U \subseteq X$,

$$\Gamma(U, \pi_*\mathcal{F}) = \Gamma(\pi^{-1}(U), \mathcal{F}).$$

So, $\Gamma(\pi^{-1}(U), \mathcal{O}_Y)$ acts on $\Gamma(\pi^{-1}(U), \mathcal{F})$ and commutes to restriction to smaller opens. Consequently, $\pi_*\mathcal{F}$ is a $\pi_*\mathcal{O}_Y$-module (or algebra) and then $\mathcal{O}_X$ acts on it *via* $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$. Recall also, that $R^q\pi_*\mathcal{F}$ is the sheaf on X generated by the presheaf

$$\Gamma(U, R^q\pi_*\mathcal{F}) = H^q(\pi^{-1}(U), \mathcal{F}).$$

If $\mathcal{F}$ is an algebra (not commutative), then only π_* and $R^1\pi_*$ are so-far defined.

Let's look at $\mathcal{F}$ and $\Gamma(Y, \mathcal{F}) = \Gamma(\pi^{-1}(X), \mathcal{F}) = \Gamma(X, \pi_*\mathcal{F})$. Observe that

$$\Gamma(Y, -) = \Gamma(X, -) \circ \pi_*.$$

So, if π_* maps an injective resolution to an exact sequence, then the usual homological algebra gives the spectral sequence of composed functors (Leray spectral sequence)

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{F}) \Longrightarrow H^\bullet(Y, \mathcal{F}).$$

We get the exact sequence of terms of low degree (also called edge sequence)

$$1 \longrightarrow H^1(X, \pi_*\mathcal{F}) \longrightarrow H^1(Y, \mathcal{F}) \longrightarrow H^0(X, R^1\pi_*\mathcal{F}) \xrightarrow{\delta_0} H^2(X, \pi_*\mathcal{F}) \longrightarrow H^2(Y, \mathcal{F}) \longrightarrow$$

In the non-commutative case, we get only

$$1 \longrightarrow H^1(X, \pi_*\mathcal{F}) \longrightarrow H^1(Y, \mathcal{F}) \longrightarrow H^0(X, R^1\pi_*\mathcal{F}).$$

Application: Let X be an algebraic variety with the Zariski topology, let $\mathcal{O}_X$ be the sheaf of germs of algebraic functions and let $Y = X_{\mathbb{C}}$ also with $\mathcal{O}_Y =$ the sheaf of germs of algebraic functions. The map $\pi\colon Y \to X$ is just the identity, which is continuous since the Zariski topology is coarser than the $\mathbb{C}$-topology. Take $\mathcal{F} =$ (possibly noncommutative) $\mathbb{GL}_r$.

Claim: $R^1\mathrm{id}_*\mathbb{GL}_r = (0)$, for all $r \geq 1$.

Proof. It suffices to prove that the stalks are zero. But these are the stalks of the corresponding presheaf

$$\varinjlim_{U \ni x} H^1_{\mathbb{C}}(U, \mathbb{GL}_r)$$

where U runs over Z-opens and H^1 is taken in the $\mathbb{C}$-topology. Pick $x \in X$ and some $\xi \in H^1_{\mathbb{C}}(U, \mathbb{GL}_r)$ for some Z-open, $U \ni x$. So, ξ consists of a vector bundle on U, locally trivial in the $\mathbb{C}$-topology. There is some open in the $\mathbb{C}$-topology, call it U_0, with $x \in U_0$ and $U_0 \subseteq U$ where $\xi \upharpoonright U_0$ is trivial iff there exists some sections, $\sigma_1, \ldots, \sigma_r$, of ξ over U_0, and $\sigma_1, \ldots, \sigma_r$ are linearly independent everywhere on U_0. The σ_j are algebraic functions on U_0 to $\mathbb{C}^r$. Moreover, they are l.i. on U_0 iff $\sigma_1 \wedge \cdots \wedge \sigma_r$ is everywhere nonzero on U_0. But, $\sigma_1 \wedge \cdots \wedge \sigma_r$ is an algebraic function and its zero set is a Z-closed subset in X. So, its complement, V, is Z-open and $x \in U_0 \subseteq V \cap U$. It follows that $\xi \upharpoonright V \cap U$ is trivial (since the σ_j are l.i. everywhere); so, ξ indeed becomes trivial on a Z-open, as required. $\square$

Apply our exact sequence and get

Theorem 3.2 *(Comparison Theorem) If X is an algebraic variety, then the canonical map*

$$\mathrm{Vect}^r_{\mathrm{Zar}}(X) \cong \check{H}^1(X_{\mathrm{Zar}}, \mathbb{GL}_r) \longrightarrow \check{H}^1(X_{\mathbb{C}}, \mathbb{GL}_r) \cong \mathrm{Vect}^r_{\mathbb{C}}(X)$$

is an isomorphism for all $r \geq 1$ (i.e., a bijection of pointed sets).

Thus, to give a rank r *algebraic* vector bundle in the $\mathbb{C}$-topology is the same as giving a rank r *algebraic* vector bundle in the Zariski topology.

If we use $\mathcal{O}_X$ = holomorphic (analytic) functions, then for many X, we get only an injection $\mathrm{Vect}^r_{\mathrm{Zar}}(X) \hookrightarrow \mathrm{Vect}^r_{\mathbb{C}}(X)$.

Connection with the geometry inside X:

First, assume X is smooth and irreducible (thus, connected). Let V be an irreducible subvariety of codimension 1. We know from Chapter 1 that locally on some open, U, there is some $f \in \Gamma(U, \mathcal{O}_X) = \mathcal{O}_U$ such that $f = 0$ cuts out V in U. Furthermore, f is analytic if V is, algebraic if V is. Form the free abelian group on the V's (we can also look at "locally finite" $\mathbb{Z}$-combinations in the analytic case); call these objects *Weil divisors* (W-divisors), and denote the corresponding group, $\mathrm{WDiv}(X)$.

A divisor $D \in \mathrm{WDiv}(X)$ is *effective* if $D = \sum_\alpha a_\alpha V_\alpha$, with $a_\alpha \geq 0$ for all α. This gives a cone inside $\mathrm{WDiv}(X)$ and partially orders $\mathrm{WDiv}(X)$.

Say g is a holomorphic (or algebraic) function near x. If V passes through x, in $\mathcal{O}_{X,x}$–which is a UFD (by Zariski) we can write

$$g = f^a \widetilde{g}, \quad \text{where } (\widetilde{g}, f) = 1.$$

(The equation $f = 0$ defines V near x so f is a prime of $\mathcal{O}_{X,x}$.) Notice that if $\mathfrak{p} = (f)$ in $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_U$, then $g = f^a\widetilde{g}$ iff $g \in \mathfrak{p}^a$ and $g \notin \mathfrak{p}^{a+1}$ iff $g \in \mathfrak{p}^a(\mathcal{O}_U)_\mathfrak{p}$ and $g \notin \mathfrak{p}^{a+1}(\mathcal{O}_U)_\mathfrak{p}$. The ring $(\mathcal{O}_U)_\mathfrak{p}$ is a local ring of dimension 1 and is regular as X is a manifold (can be regular even if X is singular). Therefore, a is independent of x. The number a is by definition the *order of vanishing of g along V*, denoted $\mathrm{ord}_V(g)$. If g is a meromorphic function near x, we write $g = g_1/g_2$ locally in $(\mathcal{O}_U)_\mathfrak{p}$, with $(g_1, g_2) = 1$ and set

$$\mathrm{ord}_V(g) = \mathrm{ord}_V(g_1) - \mathrm{ord}_V(g_2).$$

We say that g has a zero of order a along V iff $\mathrm{ord}_V(g) = a > 0$ and a pole of order a iff $\mathrm{ord}_V(g) = -a < 0$. If $g \in \Gamma(X, \mathcal{M}\mathrm{er}(X)^*)$, set

$$(g) = \sum_{V \in \mathrm{WDiv}(X)} \mathrm{ord}_V(g) \cdot V.$$

Claim. The above sum is finite, under suitable conditions:

(a) We use algebraic functions.

(b) We use holomorphic functions and restrict X (DX).

Look at g, then $1/g$ vanishes on a Z-closed, W_0. Look at $X - W_0$. Now, $X - W_0$ is Z-open so it is a variety and $g \restriction X - W_0$ is holomorphic. Look at $V \subseteq X$ and $\mathrm{ord}_V(g) = a \neq 0$, i.e., $V \cap U \neq \emptyset$. Thus, $(g) = \mathfrak{p}^a$ in $(\mathcal{O}_U)_{\mathfrak{p}}$, which yields $(g) \subseteq \mathfrak{p}$ and then $V \cap (X - W_0) = V(\mathfrak{p}) \subseteq V((g))$. But, $V(g)$ is a union of irreducible components (algebraic case) and V is codimension 1, so V is equal to one of these components. Therefore, there are only finitely many V's arising from $X - W_0$.

The function $1/g$ vanishes on W_0, so write W_0 as a union of irreducible components. Again, there are only finitely many V arising from W_0. So, altogether, there are only finitely many V's associated with g where g has a zero or a pole. We call $(g) \in \mathrm{WDiv}(X)$ a *principal divisor*. Given any two divisors $D, E \in \mathrm{WDiv}(X)$, we define *linear* (or *rational*) *equivalence* by

$$D \sim E \quad \text{iff} \quad (\exists g \in \mathcal{M}\mathrm{er}(X))(D - E = (g)).$$

The equivalence classes of divisors modulo $\sim$ is the *Weil class group*, $\mathrm{WCl}(X)$.

Remark: All goes through for any X (of our sort) for which, for all primes, $\mathfrak{p}$, of height 1, the ring $(\mathcal{O}_U)_{\mathfrak{p}}$ is a regular local ring (of dimension 1, i.e., a P.I.D.) This is, in general, hard to check (but, OK if X is normal).

Cartier had the idea to use a general X but consider only the V's given locally as $f = 0$. For every open, $U \subseteq X$, consider $A_U = \Gamma(U, \mathcal{O}_X)$. Let S_U be the set of all non-zero divisors of A_U, a multiplicative set. We get a presheaf of rings, $U \mapsto S_U^{-1}A_U$, and the corresponding sheaf, $\mathcal{M}\mathrm{er}(X)$, is the *total fraction sheaf of* $\mathcal{O}_X$. We have an embedding $\mathcal{O}_X \longrightarrow \mathcal{M}\mathrm{er}(X)$ and we let $\mathcal{M}\mathrm{er}(X)^*$ be the sheaf of invertible elements of $\mathcal{M}\mathrm{er}(X)$. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}\mathrm{er}(X)^* \longrightarrow \mathcal{D}_X \longrightarrow 0,$$

where $\mathcal{D}_X$ is the sheaf cokernel.

We claim that if we define $\mathcal{D}_X = \mathrm{Coker}\,(\mathcal{O}_X^* \longrightarrow \mathcal{M}\mathrm{er}(X)^*)$ in the $\mathbb{C}$-topology, then it is also the kernel in the Z-topology.

Take $\sigma \in \Gamma(U, \mathcal{D}_X)$ and replace X by U, so that we may assume that $U = X$. Then, as σ is liftable locally in the $\mathbb{C}$-topology, there exist a $\mathbb{C}$-open cover, U_α and some $\sigma_\alpha \in \Gamma(U, \mathcal{M}\mathrm{er}(X)^*)$ so that $\sigma_\alpha \mapsto \sigma \restriction U_\alpha$. Make the U_α small enough so that $\sigma_\alpha = f_\alpha/g_\alpha$, where f_α, g_α are holomorphic. It follows that σ_α is defined on a Z-open, $\widetilde{U}_\alpha \supseteq U_\alpha$. Look at $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \supseteq U_\alpha \cap U_\beta$. We know $\sigma_\alpha/\sigma_\beta$ is invertible holomorphic on $U_\alpha \cap U_\beta$ and so,

$$\frac{\sigma_\alpha}{\sigma_\beta} \cdot \frac{\sigma_\beta}{\sigma_\alpha} \equiv 1 \quad \text{on } U_\alpha \cap U_\beta.$$

It follows that $\sigma_\alpha/\sigma_\beta$ is invertible on $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$ and then, restricting slightly further we get a Z-open cover and σ_α's on it lifting σ. $\square$

Definition 3.1 A *Cartier divisor* (for short, C-divisor) on X is a global section of $\mathcal{D}_X$. Two Cartier divisors, σ, τ are *rationally equivalent*, denoted $\sigma \sim \tau$, iff $\sigma/\tau \in \Gamma(X, \mathcal{M}\mathrm{er}(X)^*)$. Of course, this means there is a $\mathbb{C}$ or Z-open cover, U_α, of X and some $\sigma_\alpha, \tau_\alpha \in \Gamma(U_\alpha, \mathcal{M}\mathrm{er}(X)^*)$ with $\sigma_\alpha/\tau_\alpha$ invertible holomorphic on $U_\alpha \cap U_\beta$. The group of Cartier divisors is denoted by $\mathrm{CDiv}(X)$ and the corresponding group of equivalence classes modulo rational equivalence by $\mathrm{Cl}(X)$ (the *class group*).

The idea is that if $\{(U_\alpha, \sigma_\alpha)\}_\alpha$ defines a C-divisor, then we look on U_α at

$$\sigma_\alpha^0 - \sigma_\alpha^\infty = (\mathrm{locus}\, \sigma_\alpha = 0) - (\mathrm{locus}\, \frac{1}{\sigma_\alpha} = 0).$$

When we have the situation where $\mathrm{WDiv}(X)$ exists, then the map

$$\{(U_\alpha, \sigma_\alpha)\}_\alpha \mapsto \{\sigma_\alpha^0 - \sigma_\alpha^\infty\}$$

takes C-divisors to Weil divisors. Say σ_α and σ'_α are both liftings of the same σ, then on U_α we have

$$\sigma'_\alpha = \sigma_\alpha g_\alpha \quad \text{where } g_\alpha \in \Gamma(X, \mathcal{O}_X^*).$$

Therefore,

$$\sigma_\alpha'^0 - \sigma_\alpha'^\infty = \sigma_\alpha^0 - \sigma_\alpha^\infty$$

and the Weil divisors are the same (provided they make sense). If $\sigma, \tau \in \mathrm{CDiv}(X)$ and $\sigma \sim \tau$, then there is a global meromorphic function, f, with $\sigma = f\tau$. Consequently

$$\sigma_\alpha^0 - \sigma_\alpha^\infty = (f)^0 - (f)^\infty + \tau_\alpha^0 - \tau_\alpha^\infty,$$

which shows that the corresponding Weil divisors are linearly equivalent. We get

Proposition 3.3 *If X is an algebraic variety, the sheaf $\mathcal{D}_X$ is the same in either the Zariski or $\mathbb{C}$-topology and if X allows Weil divisors (non-singular in codimension 1), then the map $\mathrm{CDiv}(X) \longrightarrow \mathrm{WDiv}(X)$ given by $\sigma \mapsto \sigma_\alpha^0 - \sigma_\alpha^\infty$ is well-defined and we get a commutative diagram with injective rows*

$$\begin{tikzcd}
\mathrm{CDiv}(X) \arrow[r, hook] \arrow[d] & \mathrm{WDiv}(X) \arrow[d] \\
\mathrm{Cl}(X) \arrow[r, hook] & \mathrm{WCl}(X).
\end{tikzcd}$$

If X is a manifold then our rows are isomorphisms.

Proof. We only need to prove the last statement. Pick $D = \sum_\alpha n_\alpha V_\alpha$, a Weil divisor, where each V_α is irreducible of codimension 1. As X is manifold, each V_α is given by $f_\alpha = 0$ on a small enough open, U; take for $\sigma \upharpoonright U$, the product $\prod_\alpha f_\alpha^{n_\alpha}$ and this gives our C-divisor.

We can use the following in some computations.

Proposition 3.4 *Assume X is an algebraic variety and $Y \hookrightarrow X$ is a subvariety. Write $U = X - Y$, then the maps*

$$\sigma \in \mathrm{CDiv}(X) \mapsto \sigma \upharpoonright U \in \mathrm{CDiv}(U),$$

resp.

$$\sum_\alpha n_\alpha V_\alpha \in \mathrm{WDiv}(X) \mapsto \sum_\alpha n_\alpha (V_\alpha \cap U) \in \mathrm{WDiv}(U)$$

are surjections from $\mathrm{CDiv}(X)$ or $\mathrm{WDiv}(X)$ to the corresponding object in U. If $\mathrm{codim}_X(Y) \geq 2$, then our maps are isomorphisms. If $\mathrm{codim}_X(Y) = 1$ and Y is irreducible and locally principal, then the sequences

$$\mathbb{Z} \longrightarrow \mathrm{CDiv}(X) \longrightarrow \mathrm{CDiv}(U) \longrightarrow 0 \quad \textit{and} \quad \mathbb{Z} \longrightarrow \mathrm{WDiv}(X) \longrightarrow \mathrm{WDiv}(U) \longrightarrow 0$$

are exact (where the left hand map is $n \mapsto nY$).

Proof. The maps clearly exist. Given an object in U, take its closure in X, then restriction to U gives back the object. For Y of codimension at least 2, all procedures are insensitive to such Y, so we don't change anything by removing Y. A divisor $\xi \in \mathrm{CDiv}(X)$ (or $\mathrm{WDiv}(X)$) goes to zero iff its "support" is contained in Y. But, Y is irreducible and so are the components of ξ. Therefore, $\xi = nY$, for some n. $\square$

Recall that line bundles on X are in one-to-one correspondence with invertible sheaves, that is, rank 1, locally free $\mathcal{O}_X$-modules. If L is a line bundle, we associate to it, $\mathcal{O}_X(L)$, the sheaf of sections (algebraic, holomorphic, C^∞) of L.

In the other direction, if $\mathcal{L}$ is a rank 1 locally free $\mathcal{O}_X$-module, first make $\mathcal{L}^D$ and the $\mathcal{O}_X$-algebra, $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D)$, where

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) = \coprod_{n\geq 0} (\mathcal{L}^D)^{\otimes n}/(a\otimes b - b\otimes a).$$

On a small enough open, U,

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) \restriction U = \mathcal{O}_U[T],$$

so we form $\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) \restriction U) \cong U \prod \mathbb{C}^1$, and glue using the data for $\mathcal{L}^D$. We get the line bundle, $\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D))$.

Given a Cartier divisor, $D = \{(U_\alpha, f_\alpha)\}$, we make the submodule, $\mathcal{O}_X(D)$, of $\mathcal{M}\mathrm{er}(X)$ given on U_α by

$$\mathcal{O}_X(D) \restriction U_\alpha = \frac{1}{f_\alpha}\,\mathcal{O}_X \restriction U_\alpha \subseteq \mathcal{M}\mathrm{er}(X) \restriction U_\alpha.$$

If $\{(U_\alpha, g_\alpha)\}$ also defines D (we may assume the covers are the same by refining the covers if necessary), then there exist $h_\alpha \in \Gamma(U_\alpha, \mathcal{M}\mathrm{er}(X)^*)$, with

$$f_\alpha h_\alpha = g_\alpha.$$

Then, the map $\xi \mapsto \frac{1}{h_\alpha}\,\xi$ takes $\frac{1}{f_\alpha}$ to $\frac{1}{g_\alpha}$; so, $\frac{1}{f_\alpha}$ and $\frac{1}{g_\alpha}$ generate the same submodule of $\mathcal{M}\mathrm{er}(X) \restriction U_\alpha$. On $U_\alpha \cap U_\beta$, we have

$$\frac{f_\alpha}{f_\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*),$$

and as

$$\frac{f_\alpha}{f_\beta} \cdot \frac{1}{f_\alpha} = \frac{1}{f_\beta},$$

we get

$$\frac{1}{f_\alpha}\,\mathcal{O}_{U_\alpha} \restriction U_\alpha \cap U_\beta = \frac{1}{f_\beta}\,\mathcal{O}_{U_\beta} \restriction U_\alpha \cap U_\beta.$$

Consequently, our modules agree on the overlaps and so, $\mathcal{O}_X(D)$ is a rank 1, locally free subsheaf of $\mathcal{M}\mathrm{er}(X)$.

Say D and E are Cartier divisors and $D \sim E$. So, there is a global meromorphic function, $f \in \Gamma(X, \mathcal{M}\mathrm{er}(X)^*)$ and on U_α,

$$f_\alpha f = g_\alpha.$$

Then, the map $\xi \mapsto \frac{1}{f}\,\xi$ is an $\mathcal{O}_X$-isomorphism

$$\mathcal{O}_X(D) \cong \mathcal{O}_X(E).$$

Therefore, we get a map from $\mathrm{Cl}(X)$ to the invertible submodules of $\mathcal{M}\mathrm{er}(X)$.

Given an invertible submodule, $\mathcal{L}$, of $\mathcal{M}\mathrm{er}(X)$, locally, on U, we have $\mathcal{L} \restriction U = \frac{1}{f_U}\,\mathcal{O}_U \subseteq \mathcal{M}\mathrm{er}(X) \restriction U$. Thus, $\{(U, f_U)\}$ gives a C-divisor describing $\mathcal{L}$. Suppose $\mathcal{L}$ and $\mathcal{M}$ are two invertible submodules of $\mathcal{M}\mathrm{er}(X)$ and $\mathcal{L} \cong \mathcal{M}$; say $\varphi\colon \mathcal{L} \to \mathcal{M}$ is an $\mathcal{O}_X$-isomorphism. Locally (possibly after refining covers), on U_α, we have

$$\mathcal{L} \restriction U_\alpha \cong \frac{1}{f_\alpha}\,\mathcal{O}_{U_\alpha} \quad \text{and} \quad \mathcal{M} \restriction U_\alpha \cong \frac{1}{g_\alpha}\,\mathcal{O}_{U_\alpha}.$$

So, $\varphi\colon \mathcal{L} \restriction U_\alpha \to \mathcal{M} \restriction U_\alpha$ is given by some τ_α such that

$$\varphi\Big(\frac{1}{f_\alpha}\Big) = \tau_\alpha \frac{1}{g_\alpha}.$$

Consequently, $\varphi_\alpha \restriction U_\alpha$ is multiplication by τ_α and $\varphi_\beta \restriction U_\beta$ is multiplication by τ_β. Yet $\varphi_\alpha \restriction U_\alpha$ and $\varphi_\beta \restriction U_\beta$ agree on $U_\alpha \cap U_\beta$, so $\tau_\alpha = \tau_\beta$ on $U_\alpha \cap U_\beta$. This shows that the τ_α patch and define a global τ such that

$$\tau \restriction U_\alpha = \tau_\alpha = g_\alpha \varphi\Big(\frac{1}{f_\alpha}\Big) \quad \text{and} \quad \tau \restriction U_\beta = \tau_\beta = g_\beta \varphi\Big(\frac{1}{f_\beta}\Big)$$

on overlaps. Therefore, we can define a global Φ *via*

$$\Phi = g_\alpha \varphi\Big(\frac{1}{f_\alpha}\Big) \in \mathcal{M}\mathrm{er}(X),$$

and we find $\xi \mapsto \frac{1}{\Phi}\,\xi$ gives the desired isomorphism.

Theorem 3.5 *If X is an algebraic variety (or holomorphic or C^∞ variety) then there is a canonical map,* $\mathrm{CDiv}(X) \longrightarrow$ *rank* 1*, locally free submodules of* $\mathcal{M}\mathrm{er}(X)$*. It is surjective. Two Cartier divisors D and E are rationally equivalent iff the corresponding invertible sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ are (abstractly) isomorphic. Hence, there is an injection of the class group,* $\mathrm{Cl}(X)$ *into the group of rank* 1*, locally free $\mathcal{O}_X$-submodules of* $\mathcal{M}\mathrm{er}(X)$ *modulo isomorphism. If X is an algebraic variety and we use algebraic functions and if X is irreducible, then every rank* 1*, locally free $\mathcal{O}_X$-module is an $\mathcal{O}_X(D)$. The map $D \mapsto \mathcal{O}_X(D)$ is just the connecting homomorphism in the cohomology sequence,*

$$H^0(X, \mathcal{D}_X) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^*).$$

Proof. Only the last statement needs proof. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}\mathrm{er}(X)^* \longrightarrow \mathcal{D}_X \longrightarrow 0.$$

Apply cohomology (we may use the Z-topology, by the comparison theorem): We get

$$\Gamma(X, \mathcal{M}\mathrm{er}(X)^*) \longrightarrow \mathrm{CDiv}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^1(X, \mathcal{M}\mathrm{er}(X)^*).$$

But, X is irreducible and in the Z-topology $\mathcal{M}\mathrm{er}(X)$ is a constant sheaf. As constant sheaves are flasque, $\mathcal{M}\mathrm{er}(X)$ is flasque, which implies that $H^1(X, \mathcal{M}\mathrm{er}(X)^*) = (0)$. Note that this shows that there is a surjection $\mathrm{CDiv}(X) \longrightarrow \mathrm{Pic}(X)$.

How is δ defined? Given $D \in H^0(X, \mathcal{D}_X) = \mathrm{CDiv}(X)$, if $\{(U_\alpha, f_\alpha)\}$ is a local lifting of D, the map δ associates the cohomology class $[f_\beta/f_\alpha]$, where f_β/f_α is viewed as a 1-cocycle on $\mathcal{O}_X^*$. On the other hand, when we go through the construction of $\mathcal{O}_X(D)$, we have the isomorphisms

$$\mathcal{O}_X(D) \restriction U_\alpha = \frac{1}{f_\alpha}\,\mathcal{O}_{U_\alpha} \cong \mathcal{O}_{U_\alpha} \supseteq \mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta} \quad (\text{mult. by } f_\alpha)$$

and

$$\mathcal{O}_X(D) \restriction U_\beta = \frac{1}{f_\beta}\,\mathcal{O}_{U_\beta} \cong \mathcal{O}_{U_\beta} \supseteq \mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta} \quad (\text{mult. by } f_\beta)$$

and we see that the transition function, g_α^β, on $\mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta}$ is nonother that multiplication by f_β/f_α. But then, both $\mathcal{O}_X(D)$ and $\delta(D)$ are line bundles defined by the same transition functions (multiplication by f_β/f_α) and $\delta(D) = \mathcal{O}_X(D)$. $\square$

Say $D = \{(U_\alpha, f_\alpha)\}$ is a Cartier divisor on X. Then, the intuition is that the geometric object associated to D is

$$(\text{zeros of } f_\alpha - \text{poles of } f_\alpha) \quad \text{on} \quad U_\alpha.$$

This leads to saying that the Cartier divisor D is an *effective* divisor iff each f_α is holomorphic on U_α. In this case, $f_\alpha = 0$ gives on U_α a locally principal, codimension 1 subvariety and conversely. Now each subvariety, V, has a corresponding sheaf of ideals, $\mathfrak{I}_V$. If V is locally principal, given by the f_α's, then $\mathfrak{I}_V \restriction U_\alpha = f_\alpha \mathcal{O}_X \restriction U_\alpha$. But, $f_\alpha \mathcal{O}_X \restriction U_\alpha$ is exactly $\mathcal{O}_X(-D)$ on U_α if $D = \{(U_\alpha, f_\alpha)\}$. Hence, $\mathfrak{I}_X = \mathcal{O}_X(-D)$. We get

Proposition 3.6 *If X is an algebraic variety, then the effective Cartier divisors on X are in one-to-one correspondence with the locally principal codimension 1 subvarieties of X. If V is one of the latter and if D corresponds to V, then the ideal cutting out V is exactly $\mathcal{O}_X(-D)$. Hence*

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_V \longrightarrow 0 \quad \textit{is exact.}$$

What are the global sections of $\mathcal{O}_X(D)$?

Such sections are holomorphic maps $\sigma\colon X \to \mathcal{O}_X(D)$ such that $\pi \circ \sigma = id$ (where $\pi\colon \mathcal{O}_X(D) \to X$ is the canonical projection associated with the bundle $\mathcal{O}_X(D)$). If D is given by $\{(U_\alpha, f_\alpha)\}$, the diagram

$$\begin{tikzcd}
\mathcal{O}_X(D) \restriction U_\alpha \arrow[r, equal] & f_\alpha\mathcal{O}_X \restriction U_\alpha \arrow[r, "\times f_\alpha"] & \mathcal{O}_X \restriction U_\alpha \\
\mathcal{O}_X(D) \restriction U_\alpha \cap U_\beta \arrow[u, hook] \arrow[d, equal] & & \mathcal{O}_X \restriction U_\alpha \cap U_\beta \arrow[u, hook] \arrow[d, "g_\alpha^\beta"] \\
\mathcal{O}_X(D) \restriction U_\beta \cap U_\alpha \arrow[d, hook] & & \mathcal{O}_X \restriction U_\beta \cap U_\alpha \arrow[d, hook] \\
\mathcal{O}_X(D) \restriction U_\beta \arrow[r, equal] & f_\beta\mathcal{O}_X \restriction U_\beta \arrow[r, "\times f_\beta"] & \mathcal{O}_X \restriction U_\beta
\end{tikzcd}$$

implies that

$$\sigma_\alpha = f_\alpha\sigma\colon U_\alpha \longrightarrow \mathcal{O}_X \restriction U_\alpha \quad \text{and} \quad \sigma_\beta = f_\beta\sigma\colon U_\beta \longrightarrow \mathcal{O}_X \restriction U_\beta.$$

However, we need

$$\sigma_\beta = g_\alpha^\beta\sigma_\alpha,$$

which means that a global section, σ, is a family of local holomorphic functions, σ_α, so that $\sigma_\beta = g_\alpha^\beta\sigma_\alpha$.But, as $g_\alpha^\beta = f_\beta/f_\alpha$, we get

$$\frac{\sigma_\alpha}{f_\alpha} = \frac{\sigma_\beta}{f_\beta} \quad \text{on } U_\alpha \cap U_\beta.$$

Therefore, the meromorphic functions, σ_α/f_α, patch and give a global meromorphic function, F_σ. We have

$$f_\alpha(F_\sigma \restriction U_\alpha) = \sigma_\alpha$$

a holomorphic function. Therefore, $(f_\alpha \restriction U_\alpha) + (F_\sigma \restriction U_\alpha) \geq 0$, for all α and as the pieces patch, we get

$$D + (F_\sigma) \geq 0.$$

Conversely, say $F \in \Gamma(X, \mathcal{M}\mathrm{er}(X))$ and $D+(F) \geq 0$. Locally on U_α, we have $D = \{(U_\alpha, f_\alpha)\}$ and $(f_\alpha F) \geq 0$. If we set $\sigma_\alpha = f_\alpha F$, we get a holomorphic function on U_α. But,

$$g_\alpha^\beta\sigma_\alpha = \frac{f_\beta}{f_\alpha} f_\alpha F = f_\beta F = \sigma_\beta,$$

so the σ_α's give a global section of $\mathcal{O}_X(D)$.

Proposition 3.7 *If X is an algebraic variety, then*

$$H^0(X, \mathcal{O}_X(D)) = \{0\} \cup \{F \in \Gamma(X, \mathcal{M}\mathrm{er}(X)) \mid (F) + D \geq 0\}.$$

in particular,

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D))) = \{E \mid E \geq 0 \quad \textit{and} \quad E \sim D\},$$

the complete linear system of D, is naturally a projective space and $H^0(X, \mathcal{O}_X(D)) \neq (0)$ iff there is some Cartier divisor, $E \geq 0$, and $E \sim D$.

Recall that an $\mathcal{O}_X$-module, $\mathcal{F}$, is a Z-QC (resp. $\mathbb{C}$-QC, here QC = *quasi-coherent*) iff everywhere locally, i.e., for small (Z, resp. $\mathbb{C}$) open, U, there exist sets $I(U)$ and $J(U)$ and some exact sequence

$$(\mathcal{O}_X \restriction U)^{I(U)} \xrightarrow{\varphi_U} (\mathcal{O}_X \restriction U)^{J(U)} \longrightarrow \mathcal{F} \restriction U \longrightarrow 0.$$

Since $\mathcal{O}_X$ is coherent (usual fact that the rings $\Gamma(U_\alpha, \mathcal{O}_X) = A_\alpha$, for U_α open affine, are noetherian) or Oka's theorem in the analytic case, a sheaf, $\mathcal{F}$, is *coherent* iff it is QC and finitely generated iff it is finitely presented, i.e., everywhere locally,

$$(\mathcal{O}_X \restriction U)^q \xrightarrow{\varphi_U} (\mathcal{O}_X \restriction U)^q \longrightarrow \mathcal{F} \restriction U \longrightarrow 0 \quad \text{is exact.} \tag{†}$$

(Here, p, q are functions of U and finite).

In the case of the Zariski topology, $\mathcal{F}$ is QC iff for every affine open, U, the sheaf $\mathcal{F} \restriction U$ has the form $\widetilde{M}$, for some $\Gamma(U, \mathcal{O}_X)$-module, M. The sheaf $\widetilde{M}$ is defined so that, for every open $W \subseteq U$,

$$\Gamma(W, \widetilde{M}) = \left\{ \sigma\colon W \longrightarrow \bigcup_{\xi \in W} M_\xi \;\middle|\; \begin{array}{l} (1)\ \sigma(\xi) \in M_\xi \\ (2)\ (\forall \xi \in W)(\exists V \text{ (open)} \subseteq W, \exists f \in M, \exists g \in \Gamma(V, \mathcal{O}_X))(g \neq 0 \text{ on } V) \\ (3)\ (\forall y \in V)\left(\sigma(y) = \text{image}\left(\frac{f}{g}\right) \text{ in } M_y\right). \end{array} \right\}$$

Proposition 3.8 *Say X is an algebraic variety and $\mathcal{F}$ is an $\mathcal{O}_X$-module. Then, $\mathcal{F}$ is Z-coherent iff $\mathcal{F}$ is $\mathbb{C}$-coherent.*

Proof. Say $\mathcal{F}$ is Z-coherent, then locally Z, the sheaf $\mathcal{F}$ satisfies (†). But, every Z-open is also $\mathbb{C}$-open, so $\mathcal{F}$ is $\mathbb{C}$-coherent.

Now, assume $\mathcal{F}$ is $\mathbb{C}$-coherent, then locally $\mathbb{C}$, we have (†), where U is $\mathbb{C}$-open. The map φ_U is given by a $p \times q$ matrix of holomorphic functions on U. Each is algebraically defined on a Z-open containing U. The intersection of these finitely many Z-opens is a Z-open, $\widetilde{U}$ and $\widetilde{U} \supseteq U$. So, we get a sheaf

$$\widetilde{\mathcal{F}} \restriction \widetilde{U} = \mathrm{Coker}\,((\mathcal{O}_X \restriction \widetilde{U})^q \longrightarrow (\mathcal{O}_X \restriction \widetilde{U})^p).$$

The sheaves $\widetilde{\mathcal{F}} \restriction \widetilde{U}$ patch (easy–DX) and we get a sheaf, $\widetilde{\mathcal{F}}$. On U, the sheaf $\widetilde{\mathcal{F}}$ is equal to $\mathcal{F}$, so $\widetilde{\mathcal{F}} = \mathcal{F}$. $\square$

We have the continuous map $X_{\mathbb{C}} \xrightarrow{\text{id}} X_{\text{Zar}}$ and we get (see Homework)

Theorem 3.9 *(Comparison Theorem for cohomology of coherent sheaves) If X is an algebraic variety and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the canonical map*

$$H^q(X_{\text{Zar}}, \mathcal{F}) \longrightarrow H^q(X_{\mathbb{C}}, \mathcal{F})$$

is an isomorphism for all $q \geq 0$.

Say V is a closed subvariety of $X = \mathbb{P}^n_{\mathbb{C}}$. Then, V is given by a coherent sheaf of ideals of $\mathcal{O}_X$, say $\mathfrak{I}_V$ and we have the exact sequence

$$0 \longrightarrow \mathfrak{I}_V \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_V \longrightarrow 0,$$

where $\mathcal{O}_V$ is the sheaf of germs of holomorphic functions on V and has support on V. If V is a hypersurface, then V is given by $f = 0$, where f is a form of degree d. If D is a Cartier divisor of f, then $\mathfrak{I}_V = \mathcal{O}_X(-D)$. Similarly another hypersurface, W, is given by $g = 0$ and if $\deg(f) = \deg(g)$, then f/g is a global meromorphic function on $\mathbb{P}^n$. Therefore, $(f/g) = V - W$, which implies $V \sim W$. In particular, $g = (\text{linear form})^d$ and so, $V \sim dH$, where H is a hyperplane. Therefore the set of effective Cartier disisors of $\mathbb{P}^n$ is in one-to-one correspondence with forms of varying degrees $d \geq 0$ and

$$\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z},$$

namely, $V \mapsto \deg(V) = \delta(V)$ (our old notation) $= (\deg(f)) \cdot H \in H^2(\mathbb{P}^n, \mathbb{Z})$. We deduce,

$$\mathrm{Pic}^0(\mathbb{P}^n) = (0) \quad \text{and} \quad \mathrm{Pic}(\mathbb{P}^n) = \mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}.$$

Say V is a closed subvariety of $\mathbb{P}^n_{\mathbb{C}}$, then we have the exact sequence

$$0 \longrightarrow \mathfrak{I}_V \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_V \longrightarrow 0.$$

Twist with $\mathcal{O}_{\mathbb{P}^n}(d)$, i.e., tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$ (Recall that by definition, $\mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^n}(dH)$, where H is a hyperplane). We get the exact sequence

$$0 \longrightarrow \mathfrak{I}_V(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_V(d) \longrightarrow 0$$

(with $\mathfrak{I}_V(d) = \mathfrak{I}_V \otimes \mathcal{O}_{\mathbb{P}^n}(d)$ and $\mathcal{O}_V(d) = \mathcal{O}_V \otimes \mathcal{O}_{\mathbb{P}^n}(d)$) and we can apply cohomology, to get

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(V, \mathcal{O}_V(d)) \quad \text{is exact,}$$

as $\mathcal{O}_V(d)$ has support V. Now,

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \{0\} \cup \{E \geq 0,\ E \sim dH\}.$$

If $E = \sum_Q a_Q Q$, where $\dim(Q) = n-1$ and $a_Q \geq 0$, we set $\deg(E) = \sum_Q a_Q \deg(Q)$. If $E \geq 0$, then $\deg(E) \geq 0$, from which we deduce

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} (0) & \text{if } d < 0 \\ \mathbb{C}^{\binom{n+d}{d}} & \text{i.e., all forms of degree } d \text{ in } X_0, \ldots, X_n, \text{ if } d \geq 0. \end{cases}$$

We deduce,

$$H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) = \{\text{all forms of degree } d \text{ vanishing on } V\} \cup \{0\},$$

that is, all hypersurfaces, $Z \subseteq \mathbb{P}^n$, with $V \subseteq Z$ (and 0).

Consequently, to give $\xi \in H^0(\mathbb{P}^n, \mathfrak{I}_V(d))$ is to give a hypersurface of $\mathbb{P}^n$ containing V. Therefore,

$$H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) = (0) \quad \text{iff} \quad \text{no hypersurface of degree } d \text{ contains } V.$$

(In particular, V is nondegenerate iff $H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) = (0)$.)

We now compute the groups $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, for all n, q, d. First, consider $d \geq 0$ and use induction on n. For $\mathbb{P}^0$, we have

$$H^q(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) = \begin{cases} (0) & \text{if } q > 0 \\ \mathbb{C} & \text{if } q = 0. \end{cases}$$

Next, $\mathbb{P}^1$. The sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^0} \longrightarrow 0 \quad \text{is exact.}$$

By tensoring with $\mathcal{O}_{\mathbb{P}^1}(d)$, we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^0}(d) \longrightarrow 0 \quad \text{is exact}$$

by taking cohomology, we get

$$\begin{array}{l} 0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \xrightarrow{\ \alpha\ } H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \xrightarrow{\ \beta\ } H^0(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) \longrightarrow \\ \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow 0 \end{array}$$

since $H^1(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) = (0)$, by hypothesis. Now, if we pick coordinates, the embedding $\mathbb{P}^0 \hookrightarrow \mathbb{P}^1$ corresponds to $x_0 = 0$. Consequently, the map α is multiplication by x_0 and the map β is $x_0 \mapsto 0$. Therefore,

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)), \quad \text{for all } d \geq 0,$$

and we deduce

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}^g = (0),$$

and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = (0)$, too. We know that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \mathbb{C}^{d+1}; \quad d \geq 0;$$

and we just proved that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = (0); \quad d \geq -1.$$

In order to understand the induction pattern, let us do the case of $\mathbb{P}^2$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(d-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^2}(d) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

and by taking cohomology, we get

$$\begin{array}{l}
0 \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \xrightarrow{\alpha} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \xrightarrow{\beta} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow \\
\longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow \\
\longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \longrightarrow 0
\end{array}$$

By the induction hypothesis, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = (0)$ if $d \geq -1$, so

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)), \quad \text{for all } d \geq -1.$$

Therefore,

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}), \quad \text{for all } d \geq -2.$$

But, the dimension of the right hand side is $h^{0,1} = 0$ (the irregularity, $h^{0,1}$, of $\mathbb{P}^2$ is zero). We conclude that

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = (0) \quad \text{for all } d \geq -2.$$

A similar reasoning applied to H^2 shows

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}), \quad \text{for all } d \geq -2.$$

The dimension of the right hand side group is $H^{0,2} = p_g(\mathbb{P}^2) = 0$, so we deduce

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = (0) \quad \text{for all } d \geq -2.$$

By induction, we get

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} \mathbb{C}^{\binom{n+d}{d}} & \text{if } d \geq 0 \\ (0) & \text{if } d < 0 \end{cases}$$

and

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \text{if } d \geq -n, \text{ for all } q > 0.$$

For the rest of the cases, we use Serre duality and the Euler sequence. Serre duality says

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^D \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d) \otimes \Omega^n_{\mathbb{P}^n}).$$

From the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \coprod_{n+1 \text{ times}} \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow T^{1,0}_{\mathbb{P}^n} \longrightarrow 0,$$

by taking the highest wedge, we get

$$\bigwedge^{n+1}\Big(\coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(1)\Big) \cong \bigwedge^{n} T^{1,0}_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n},$$

from which we conclude

$$(\Omega^n_{\mathbb{P}^n})^D \cong \bigwedge^{n+1}\Big(\coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(1)\Big) \cong \mathcal{O}_{\mathbb{P}^n}(n+1).$$

Therefore

$$\omega_{\mathbb{P}^n} = \Omega^n_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-(n+1)) = \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}),$$

where $K_{\mathbb{P}^n}$ is the canonical divisor on $\mathbb{P}^n$, by definition. Therefore, we have

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1))^D.$$

If $1 \leq q \leq n-1$ and $d \geq -n$, then we know that the left hand side is zero. As $1 \leq n-q \leq n-1$, it follows that

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1)) = (0) \quad \text{when } d \geq -n.$$

Therefore,

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \text{for all } d \text{ and all } q \text{ with } 1 \leq q \leq n-1.$$

We also have

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^D \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1)),$$

and the right hand side is (0) if $-d-(n+1) < 0$, i.e., $d \geq -n$. Thus, if $d \leq -(n+1)$, then we have $\delta = -d-(n+1) \geq 0$, so

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\delta))^D = \mathbb{C}^{\binom{n+\delta}{\delta}}, \quad \text{where } \delta = -(d+n+1).$$

The pairing is given by

$$\frac{1}{f} \otimes \frac{f}{x_0x_1\cdots x_n} \mapsto \int_{\mathbb{P}^n} \frac{dx_0 \wedge \cdots \wedge dx_n}{x_0 \cdots x_n},$$

where $\deg(f) = -d$, with $d \leq -n-1$. Summarizing all this, we get

Theorem 3.10 *The cohomology of line bundles on $\mathbb{P}^n$ satisfies*

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \textit{for all } n, d \textit{ and all } q \textit{ with } 1 \leq q \leq n-1.$$

Furthermore,

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}^{\binom{n+d}{d}}, \quad \textit{if } d \geq 0, \textit{ else } (0),$$

and

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}^{\binom{n+\delta}{\delta}}, \quad \textit{where } \delta = -(d+n+1) \textit{ and } d \leq -n-1, \textit{ else } (0).$$

We also proved that

$$\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1)) = \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}).$$

3.2 Chern Classes and Segre Classes

The most important spaces (for us) are the Kähler manifolds and unless we explicitly mention otherwise, X will be Kähler. But, we can make Chern classes if X is worse.

Remark: The material in this Section is also covered in Hirzebruch [8] and under other forms in Chern [4], Milnor and Stasheff [11], Bott and Tu [3], Madsen and Tornehave [9] and Griffith and Harris [6].

Let X be *admissible* iff

(1) X is *σ-compact*, i.e.,

  (a) X is locally compact and

  (b) X is a countable union of compacts.

(2) The *combinatorial dimension* of X is finite.

Note that (1) implies that X is paracompact. Consequently, everthing we did on sheaves goes through.

Say X is an algebraic variety and $\mathcal{F}$ is a QC $\mathcal{O}_X$-module. Then, $H^0(X, \mathcal{F})$ encodes the most important geometric information contained in $\mathcal{F}$. For example, $\mathcal{F}$ = a line bundle or a vector bundle, then

$$H^0(X, \mathcal{F}) = \text{space of global sections of given type.}$$

If $\mathcal{F} = \mathfrak{I}_V(d)$, where $V \subseteq \mathbb{P}^n$, then

$$H^0(X, \mathcal{F}) = \text{hypersurfaces containing } V.$$

This leads to the Riemann-Roch (RR) problem.

Given X and a QC $\mathcal{O}_X$-module, $\mathcal{F}$,

(a) Determine when $H^0(X, \mathcal{F})$ has finite dimension and

(b) If so, compute the dimension, $\dim_{\mathbb{C}} H^0(X, \mathcal{F})$.

Some answers:

(a) Finiteness Theorem: If X is a compact, complex, analytic manifold and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H^q(X, \mathcal{F})$ has finite dimension for every $q \geq 0$.

(b) It was noticed in the fifties (Kodaira and Spencer) that if $\{X_t\}_{t\in S}$ is a reasonable family of compact algebraic varieties ($\mathbb{C}$-analytic manifolds), (S is just a $\mathbb{R}$-differentiable smooth manifold and the X_t are a proper flat family), then

$$\chi(X_t, \mathcal{O}_{X_t}) = \sum_{i=0}^{\dim X_t} (-1)^i \dim(H^i(X_t, \mathcal{O}_{X_t}))$$

was independent of t.

The Riemann-Roch problem goes back to Riemann and the finiteness theorem goes back to Oka, Cartan-Serre, Serre, Grauert, Grothendieck, ...

Examples. (1) Riemann (1850's): If X is a compact Riemann surface, then

$$\chi(X, \mathcal{O}_X) = 1 - g$$

where g is the number of holes of X (as a real surface).

(2) Max Noether (1880's): If X is a compact, complex surface, then

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(K_X^2 + \text{top Euler char.}(X)).$$

(Here, $K_X^2 = \mathcal{O}_X(K_X) \cup \mathcal{O}_X(K_X)$ in the cohomology ring, an element of $H^4(X, \mathbb{Z})$.)

(3) Severi, Eger-Todd (1920, 1937) conjectured:

$$\chi(X, \mathcal{O}_X) = \text{some polynomial in the Euler-Todd class of } X,$$

for X a general compact algebraic, complex manifold.

(4) In the fourties and fifties (3) was reformulated as a statement about Chern classes–no proof before Hirzebruch.

(5) September 29, 1952: Serre (letter to Kodaira and Spencer) conjectured: If $\mathcal{F}$ is a rank r vector bundle over the compact, complex algebraic manifold, X, then

$$\chi(X, \mathcal{F}) = \text{polynomial in the Chern classes of } X \text{ and those of } \mathcal{F}.$$

Serre's conjecture (5) was proved by Hirzebruch a few months later.

To see this makes sense, we'll prove

Theorem 3.11 *(Riemann-Roch for a compact Riemann Surface and for a line bundle) If X is a compact Riemann surface and if $\mathcal{L}$ is a complex analytic line bundle on X, then there is an integer,* $\deg(\mathcal{L})$*, it is* $\deg(D)$ *where $\mathcal{L} \cong \mathcal{O}_X(D)$, where D is a Cartier divisor on X, and*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{L}) - \dim_{\mathbb{C}} H^0(X, \omega_X \otimes \mathcal{L}^D) = \deg(\mathcal{L}) + 1 - g$$

where $g = \dim H^0(X, \omega_X) = \dim H^1(X, \mathcal{O}_X)$ is the genus of X.

Proof. First, we know X is an algebraic variety (a curve), by Riemann's theorem (see Homework). From another Homework (from Fall 2003), X is embeddable in $\mathbb{P}^N_{\mathbb{C}}$, for some N, and by GAGA (yet to come!), $\mathcal{L}$ is an algebraic line bundle. It follows that $\mathcal{L} = \mathcal{O}_X(D)$, for some Cartier divisor, D. Now, if $f \in \mathcal{M}\mathrm{er}(X)$, we showed (again, see Homework) that $f \colon X \to \mathbb{P}^1_{\mathbb{C}} = S^2$ is a branched covering map and this implies that

$$\#(f^{-1}(\infty)) = \#(f^{-1}(0)) = \text{degree of the map},$$

so $\deg(f) = \#(f^{-1}(0)) - \#(f^{-1}(\infty)) = 0$. As a consequence, if $E \sim D$, then $\deg(E) = \deg(D)$ and the first statement is proved. Serre duality says

$$H^0(X, \omega_X \otimes \mathcal{L}^D) \cong H^1(X, \mathcal{L})^D.$$

Thus, the left hand side of the Riemann-Roch formula is just $\chi(X, \mathcal{O}_X(D))$, where $\mathcal{L} = \mathcal{O}_X(D)$. Observe that $\chi(X, \mathcal{O}_X(D))$ is an Euler function in the bundle sense (this is always true of Euler-Poincaré characteristics). Look at any point , P, on X, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \kappa_P \longrightarrow 0,$$

where κ_P is the skyscraper sheaf at P, i.e.,

$$(\kappa_P)_x = \begin{cases} (0) & \text{if } x \neq P \\ \mathbb{C} & \text{if } x = P. \end{cases}$$

If we tensor with $\mathcal{O}_X(D)$, we get the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D-P) \longrightarrow \mathcal{O}_X(D) \longrightarrow \kappa_P \otimes \mathcal{O}_X(D) \longrightarrow 0.$$

When we apply cohomology, we get

$$\chi(X, \kappa_P \otimes \mathcal{O}_X(D)) + \chi(X, \mathcal{O}_X(D-P)) = \chi(X, \mathcal{O}_X(D)).$$

There are three cases.

(a) $D = 0$. The Riemann-Roch formula is a tautology, by definition of g and the fact that $H^0(X, \mathcal{O}_X) = \mathbb{C}$.

(b) $D > 0$. Pick any P appearing in D. Then, $\deg(D-P) = \deg(D) - 1$ and we can use induction. The base case holds, by (a). Using the induction hypothesis, we get

$$1 + \deg(D-P) + 1 - g = \chi(X, \mathcal{O}_X(D)),$$

which says

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g,$$

proving the induction step when $D > 0$.

(c) D is arbitrary. In this case, write $D = D^+ - D^-$, with $D^+, D^- \geq 0$; then

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D^-) \longrightarrow \kappa_{D^-} \longrightarrow 0 \quad \text{is exact}$$

and

$$\deg(\kappa_{D^-}) = \deg(D^-) = \chi(X, \mathcal{O}_X(D^-)).$$

If we tensor the above exact sequence with $\mathcal{O}_X(D)$, we get

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D+D^-) \longrightarrow \kappa_{D^-} \longrightarrow 0 \quad \text{is exact.}$$

When we apply cohomology, we get

$$\chi(X, \mathcal{O}_X(D)) + \deg(D^-) = \chi(X, \mathcal{O}_X(D+D^-)) = \chi(X, \mathcal{O}_X(D^+)).$$

However, by (b), we have $\chi(X, \mathcal{O}_X(D^+)) = \deg(D^+) + 1 - g$, so we deduce

$$\chi(X, \mathcal{O}_X(D)) = \deg(D^+) - \deg(D^-) + 1 - g = \deg(D) + 1 - g,$$

which finishes the proof. $\square$

We will show:

(a) $\mathcal{L}$ possesses a class, $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$.

(b) If X is a Riemann surface and $[X] \in H_2(X, \mathbb{Z}) = \mathbb{Z}$ is its fundamental class, then $\deg(\mathcal{L}) = c(\mathcal{L})[X] \in \mathbb{Z}$. Then, the Riemann-Roch formula becomes

$$\begin{aligned}
\chi(X, \mathcal{L}) &= c_1(\mathcal{L})[X] + 1 - g \\
&= \left[c_1(\mathcal{L}) + \frac{1}{2}(2-2g)\right][X] \\
&= \left[c_1(\mathcal{L}) + \frac{1}{2}c_1(T_X^{1,0})\right][X].
\end{aligned}$$

This is Hirzebruch's form of the Riemann-Roch theorem for Riemann surfaces and line bundles.

What about vector bundles?

Theorem 3.12 *(Atiyah-Serre on vector bundles) Let X be either a compact, complex C^∞-manifold or an algebraic variety. If E is a rank r vector bundle on X, of class C^∞ in case X is just C^∞, algebraic if X is algebraic, in the latter case assume E is generated by its global sections (that is, the map, $\Gamma_{\mathrm{alg}}(X, \mathcal{O}_X(E)) \longrightarrow E_x$, given by $\sigma \mapsto \sigma(x)$, is surjective for all x), then, there is a trivial bundle of rank $r-d$ (where $d = \dim_{\mathbb{C}} X$) denoted $\mathbb{I}^{r-d}$, and a bundle exact sequence*

$$0 \longrightarrow \mathbb{I}^{r-d} \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

and the rank of the bundle E'' is at most d.

Proof. Observe that if $r < d$, there is nothing to prove and $\mathrm{rk}(E'') = \mathrm{rk}(E)$ and also if $r = d$ take (0) for the left hand side. So, we may assume $r > d$. In the C^∞-case, we always have E generated by its global C^∞-sections (partition of unity argument).

Pick x, note $\dim E_x = r$, so there is a finite dimensional subspace of $\Gamma(X, \mathcal{O}_X(E))$ surjecting onto E_x. By continuity (or algebraicity), this holds $\mathbb{C}$-near (resp. Z-near) x. Cover by these opens and so

(a) In the C^∞-case, finitely many of these opens cover X (recall, X is compact).

(b) In the algebraic case, again, finitely many of these opens cover X, as X is quasi-compact in the Z-topology.

Therefore, there exists a finite dimensional space, $W \subseteq \Gamma(X, \mathcal{O}_X(E))$, and the map $W \longrightarrow E_x$ given by $\sigma \mapsto \sigma(x)$ is *surjective* for all $x \in X$. Let

$$\ker(x) = \mathrm{Ker}\,(W \longrightarrow E_x).$$

Consider the projective space $\mathbb{P}(\ker(x)) \hookrightarrow \mathbb{P} = \mathbb{P}(W)$. Observe that $\dim \ker(x) = \dim W - r$ is independent of x. Now, look at $\bigcup_{x \in X} \mathbb{P}(\ker(x))$ and let Z be its Z-closure. We have

$$\dim Z = \dim X + \dim W - r - 1 = \dim W + d - r - 1,$$

so, $\mathrm{codim}(Z \hookrightarrow \mathbb{P}) = r - d$. Thus, there is some projective subspace, T, of $\mathbb{P}$ with $\dim T = r - d - 1$, so that

$$T \cap Z = \emptyset.$$

Then, $T = \mathbb{P}(S)$, for some subspace, S, of W ($\dim S = r - d$). Look at

$$X \prod S = X \prod \mathbb{C}^{r-d} = \mathbb{I}^{r-d}.$$

Send $\mathbb{I}^{r-d}$ to E *via* $(x, s) \mapsto s(x) \in E$. As $T \cap Z = \emptyset$, the value $s(x)$ is *never* zero. Therefore, for any $x \in X$, $\mathrm{Im}(\mathbb{I}^{r-d} \hookrightarrow E)$ has full rank; set $E'' = E/\mathrm{Im}((\mathbb{I}^{r-d} \hookrightarrow E) =$ a vector bundle of rank d, then

$$0 \longrightarrow \mathbb{I}^{r-d} \longrightarrow E \longrightarrow E'' \longrightarrow 0 \quad \text{is exact}$$

as a bundle sequence. $\square$

Remarks:

(a) If $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ is bundle exact, then

$$c_1(E) = c_1(E') + c_1(E'').$$

(b) If E is the trivial bundle, $\mathbb{I}^r$, then $c_j(E) = 0$, for $j = 1, \ldots, r$.

(c) If $\mathrm{rk}(E) = r$, then $c_1(E) = c_1(\bigwedge^r E)$.

In view of (a)–(c), Atiyah-Serre can be reformulated as

$$c_1(E) = c_1\Big(\bigwedge^{\mathrm{rk}\,E} E\Big) = c_1(E'') = c_1\Big(\bigwedge^{\mathrm{rk}\,E''} E''\Big).$$

We now use the Atiyah-Serre theorem to prove a version of Riemann-Roch first shown by Weil.

Theorem 3.13 *(Riemann-Roch on a Riemann surface for a vector bundle) If X is a compact Riemann surface and E is a complex analytic rank r vector bundle on X, then*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(E)) - \dim_{\mathbb{C}} H^1(X, \omega_X \otimes \mathcal{O}_X(E)^D) = \chi(X, \mathcal{O}_X) = c_1(E) + \mathrm{rk}(E)(1-g).$$

Proof. The first equality is just Serre Duality. As before, by Riemann's theorem X is projective algebraic and by GAGA, E is an algebraic vector bundle. Now, as $X \hookrightarrow \mathbb{P}^N$, it turns out (Serre) that for $\delta >> 0$, the "twisted bundle", $E \otimes \mathcal{O}_X(\delta)$ $(= E \otimes \mathcal{O}_X^{\otimes\delta})$ is generated by its global holomorphic sections. We can apply Atiyah-Serre to $E \otimes \mathcal{O}_X(\delta)$. We get

$$0 \longrightarrow \mathbb{I}^{r-1} \longrightarrow E \otimes \mathcal{O}_X(\delta) \longrightarrow E'' \longrightarrow 0 \quad \text{is exact},$$

where $\mathrm{rk}(E'') = 1$. If we twist with $\mathcal{O}_X(-\delta)$, we get the exact sequence

$$0 \longrightarrow \coprod_{r-1} \mathcal{O}_X(-\delta) \longrightarrow E \longrightarrow E''(-\delta) \longrightarrow 0.$$

(Here, $E''(-\delta) = E'' \otimes \mathcal{O}_X(-\delta)$.) Now, use induction on r. The case $r = 1$ is ordinary Riemann-Roch for line bundles. Assume the induction hypothesis for $r-1$. As χ is an Euler function, we have

$$\chi(X, \mathcal{O}_X(E)) = \chi(X, E''(-\delta)) + \chi\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big).$$

The first term on the right hand side is

$$c_1(E''(-\delta)) + 1 - g,$$

by ordinary Riemann-Roch and the second term on the right hand side is

$$c_1\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big) + (r-1)(1-g).$$

by the induction hypothesis. We deduce that

$$\chi(X, \mathcal{O}_X(E)) = c_1(E''(-\delta)) + c_1\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big) + r(1-g).$$

But, we know that

$$c_1(E) = c_1(E''(-\delta)) + c_1\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big),$$

so we conclude that

$$\chi(X, \mathcal{O}_X(E)) = c_1(E) + r(1-g),$$

establishing the induction hypothesis and the theorem. $\square$

Remark: We can write the above as

$$\chi(X, \mathcal{O}_X(E)) = c_1(E) + \frac{\mathrm{rk}(E)}{2} c_1(T_X^{1,0}),$$

which is Hirzebruch's form of Riemann-Roch.

We will need later some properties of $\chi(X, \mathcal{O}_X)$ and $p_g(X)$. Recall that $p_g(X) = \dim_{\mathbb{C}} H^n(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \Omega^n_X)$, where $\Omega^l_X = \bigwedge^l T^{1,0}_X$. (The vector spaces $H^0(X, \Omega^l_X)$ were what the Italian geometers (in fact, all geometers) of the nineteenth century understood.)

Proposition 3.14 *The functions $\chi(X, \mathcal{O}_X)$ and $p_g(X)$ are multiplicative on compact, Kähler manifolds, i.e.,*

$$\begin{aligned} \chi\Big(X \coprod Y, \mathcal{O}_{X \coprod Y}\Big) &= \chi(X, \mathcal{O}_X)\chi(Y, \mathcal{O}_Y) \\ p_g\Big(X \coprod Y\Big) &= p_g(X)p_g(Y). \end{aligned}$$

Proof. Remember that

$$\dim_{\mathbb{C}} H^l(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \Omega^l_X) = h^{0,l} = h^{l,0}.$$

Then,

$$\chi(X, \mathcal{O}_X) = \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^0(X, \Omega^j_X) = \sum_{j=0}^{n} (-1)^j h^{j,0}.$$

Also recall the Künneth formula

$$\coprod_{\substack{p+p'=a \\ q+q'=b}} H^q(X, \Omega^p_X) \otimes H^{q'}(X, \Omega^{p'}_X) \cong H^b\Big(X \coprod Y, \Omega^a_{X \coprod Y}\Big).$$

Set $b = 0$, then $q = q' = 0$ and we get

$$\sum_{p+p'=a} h^{p,0}(X) h^{p',0}(Y) = h^{a,0}\Big(X \coprod Y\Big).$$

Then,

$$\begin{aligned} \chi(X, \mathcal{O}_X)\chi(Y, \mathcal{O}_Y) &= \left(\sum_{r=0}^{m} (-1)^r h^{r,0}(X)\right)\left(\sum_{s=0}^{n} (-1)^s h^{s,0}(Y)\right) \\ &= \sum_{r,s=0}^{m+n} (-1)^{r+s} h^{r,0}(X) h^{s,0}(Y) \\ &= \sum_{k=0}^{m+n} (-1)^k \sum_{r+s=k} h^{r,0}(X) h^{s,0}(Y) \\ &= \sum_{k=0}^{m+n} (-1)^k h^k(X \coprod Y) = \chi\Big(X \coprod Y, \mathcal{O}_{X \coprod Y}\Big). \end{aligned}$$

The second statement is obvious from Künneth. $\square$

Next, we introduce Hirzebruch's axiomatic approach.

Let E be a complex vector bundle on X, where X is one of our spaces (admissible). It will turn out that E is a unitary bundle (a $\mathrm{U}(q)$-bundle, where $q = \mathrm{rk}(E)$).

Chern classes are cohomology classes, $c_l(E)$, satisfying the following axioms:

Axiom (I). (Existence and Chern polynomial). If E is a rank q unitary bundle over X and X is admissible, then there exist cohomology classes, $c_l(E) \in H^{2l}(X, \mathbb{Z})$, the *Chern classes* of E and we set

$$c(E)(t) = \sum_{l=0}^{\infty} c_l(E)t^l \in H^*(X, \mathbb{Z})[[t]],$$

with $c_0(E) = 1$.

As $\dim_{\mathbb{C}} X = d < \infty$, we get $c_l(E) = 0$ for $l > d$, so $C(E)(t)$ is in fact a polynomial in $H^*(X, \mathbb{Z})[t]$ called the *Chern polynomial* of E where $\deg(t) = 2$.

Say $\pi \colon Y \to X$ and E is a $\mathrm{U}(q)$-bundle over X, then we have two maps

$$H^*(X, \mathbb{Z}) \xrightarrow{\pi^*} H^*(Y, \mathbb{Z}) \quad \text{and} \quad H^1(X, \mathrm{U}(q)) \xrightarrow{\pi^*} H^1(Y, \mathrm{U}(q)).$$

Axiom (II). (Naturality). For every E, a $\mathrm{U}(q)$-bundle on X and map, $\pi \colon Y \to X$, (with X, Y admissible), we have

$$c(\pi^* E)(t) = \pi^*(c(E))(t),$$

as elements of $H^*(Y, \mathbb{Z})[[t]]$.

Axiom (III). (Whitney coproduct axiom). If E, a $\mathrm{U}(q)$-bundle is a coproduct (in the $\mathbb{C}$ or C^∞-sense),

$$E = \coprod_{j=1}^{\mathrm{rk}(E)} E_j$$

of $\mathrm{U}(1)$-bundles, then

$$c(E)(t) = \prod_{j=1}^{\mathrm{rk}(E)} c(E_j)(t).$$

Axiom (IV). (Normalization). If $X = \mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{O}_X(1)$ is the $\mathrm{U}(1)$-bundle corresponding to the hyperplane divisor, H, on $\mathbb{P}^n_{\mathbb{C}}$, then

$$c(\mathcal{O}_X(1))(t) = 1 + Ht,$$

where H is considered in $H^2(X, \mathbb{Z})$.

Remark: If $i \colon \mathbb{P}^{n-1}_{\mathbb{C}} \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, then

$$i^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$$

and $i^*(H)$ in $H^2(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$ is $H_{\mathbb{P}^{n-1}_{\mathbb{C}}}$. By Axiom (II) and Axiom (IV)

$$i^*(1 + H_{\mathbb{P}^n_{\mathbb{C}}} t) = i^*(c(\mathcal{O}_{\mathbb{P}^n})(t)) = c(i^*(\mathcal{O}_{\mathbb{P}^n})(t)) = 1 + H_{\mathbb{P}^{n-1}_{\mathbb{C}}}.$$

Therefore, we can use *any* n to normalize.

Some Remarks on bundles. First, on $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$: Geometric models of $\mathcal{O}_{\mathbb{P}^n}(\pm 1)$.

Consider the map

$$\mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n.$$

If we blow up 0 in $\mathbb{C}^{n+1}$, we get $B_0(\mathbb{C}^{n+1})$ as follows: In $\mathbb{C}^{n+1} \prod \mathbb{P}^n$, look at the subvariety given by

$$\{\langle \langle z \rangle ; (\xi) \rangle \mid z_i \xi_j = z_j \xi_i,\ 0 \le i, j \le n\}.$$

By definition, this is $B_0(\mathbb{C}^{n+1})$, an algebraic variety over $\mathbb{C}$. We have the two projections

$$\begin{tikzcd} & B_0(\mathbb{C}^{n+1}) \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{C}^{n+1} & & \mathbb{P}^n. \end{tikzcd}$$

Look at the fibre, $pr_1^{-1}(\langle z\rangle)$ over $z \in \mathbb{C}^{n+1}$. There are two cases:

(a) $\langle z\rangle = 0$, in which case, $pr_1^{-1}(\langle z\rangle) = \mathbb{P}^n$.

(b) $\langle z\rangle \neq 0$, so, there is some j with $z_j \neq= 0$. We get $\xi_i = \frac{z_i}{z_j}\xi_j$, for all i, which implies:

  (α) $\xi_j \neq 0$.

  (β) All ξ_i are determined by ξ_j.

  (γ) $\dfrac{\xi_i}{\xi_j} = \dfrac{z_i}{z_j}$.

This implies

$$(\xi) = \left(\frac{\xi_0}{\xi_j} \colon \frac{\xi_1}{\xi_j} \colon \cdots \colon 1 \colon \cdots \frac{\xi_n}{\xi_j}\right) = \left(\frac{z_0}{z_j} \colon \frac{z_1}{z_j} \colon \cdots \colon 1 \colon \cdots \frac{z_n}{z_j}\right).$$

Therefore, $pr_1^{-1}(\langle z\rangle) = \langle\langle z\rangle; (z)\rangle$, a single point.

Let us now look ar $pr_2^{-1}(\xi)$, for $(\xi) \in \mathbb{P}^n$. Since $(\xi) \in \mathbb{P}^n$, there is some j such that $\xi_j \neq 0$. A point $\langle\langle z\rangle; (\xi)\rangle$ above (ξ) is given by all $\langle z_0 \colon z_1 \colon \cdots \colon z_n\rangle$ so that

$$z_i = \frac{\xi_i}{\xi_j} z_j.$$

Let $z_j = t$, then the fibre above ξ is the complex line

$$z_0 = \frac{\xi_0}{\xi_j}t,\ z_1 = \frac{\xi_1}{\xi_j}t,\ \cdots,\ z_j = t,\ \cdots,\ z_n = \frac{\xi_n}{\xi_j}t.$$

We get a line family over $\mathbb{P}^n$. Thus, $pr_2 \colon B_0(\mathbb{C}^{n+1}) \to \mathbb{P}^n$ is a line family.

(A) What kinds of maps, $\sigma \colon \mathbb{P}^n \to B_0(\mathbb{C}^{n+1})$, exist with σ holomorphic and $pr_2 \circ \sigma = \mathrm{id}$?

If σ exists, then $pr_1 \circ \sigma \colon \mathbb{P}^n \to \mathbb{C}^{n+1}$ is holomorphic; this implies that $pr_1 \circ \sigma$ is a constant map. But, $\sigma(\xi)$ belongs to a line through $(\xi) = (\xi_0 \colon \cdots \colon \xi_n)$, for all (ξ), yet $pr_1 \circ \sigma = \mathrm{const}$, so this point must lie on all line. This can only happen if $\sigma(\xi) = 0$ in the line through ξ.

(B) I claim $B_0(\mathbb{C}^{n+1})$ is locally trivial, i.e., a line bundle. If so, (A) says $B_0(\mathbb{C}^{n+1})$ has *no global holomorphic* sections and we will know that $B_0(\mathbb{C}^{n+1}) = \mathcal{O}_{\mathbb{P}^n}(-q)$, for some $q > 0$.

To show that $B_0(\mathbb{C}^{n+1})$ is locally trivial over $\mathbb{P}^n$, consider the usual cover, $U_0, \ldots, U_n$, of $\mathbb{P}^n$ (recall, $U_j = \{(\xi) \in \mathbb{P}^n \mid \xi_j \neq 0\}$). If $v \in B_0(\mathbb{C}^{n+1}) \upharpoonright U_j$, then $v = \langle\langle z\rangle; (x)\rangle$, with $\xi_j \neq 0$. Define φ_j as the map

$$v \mapsto \langle(\xi); z_j\rangle \in U_j \prod \mathbb{C}$$

and the backwards map

$$\langle(\xi); t\rangle \in U_j \prod \mathbb{C} \mapsto \langle\langle z\rangle; (\xi)\rangle, \quad \text{where} \quad z_i = \frac{\xi_i}{\xi_j}t, \quad i = 0, \ldots, n.$$

The reader should check that the point of $\mathbb{C}^{n+1} \prod \mathbb{P}^n$ so constructed is in $B_0(\mathbb{C}^{n+1})$ and that the maps are inverses of one another.

We can make a section, σ_j, of $B_0(\mathbb{C}^{n+1}) \restriction U_j$, *via*

$$\sigma((\xi)) = \left\langle \left\langle \frac{\xi_0}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, 1, \ldots, \frac{\xi_n}{\xi_j} \right\rangle ; (\xi) \right\rangle,$$

and we see that $\varphi(\sigma((\xi))) = \langle ((\xi); 1\rangle \in U_j \prod \mathbb{C}$, which shows that σ is a holomorphic section which is never zero. The transition function, g_i^j, renders the diagram

$$\begin{tikzcd}
B_0 \restriction U_i \arrow[r, "\varphi_i"] & U_i \prod \mathbb{C} \arrow[dd, "g_i^j"] \\
B_0 \restriction U_i \cap U_j \arrow[u, hook] \arrow[d, hook] & \\
B_0 \restriction U_j \arrow[r, "\varphi_j"] & U_j \prod \mathbb{C}
\end{tikzcd}$$

commutative. It follows that

$$\varphi_j(v) = g_i^j(\varphi_i(v) = g_i^j(\langle (\xi); z_i\rangle) = \langle (\xi); z_j\rangle$$

and we conclude that $g_i^j(z_i) = z_j$, which means that g_i^j is multiplication by $z_j/z_i = \xi_j/\xi_i$.

We now make another bundle on $\mathbb{P}^n$, which will turn out to be $\mathcal{O}_{\mathbb{P}^n}(1)$. Embed $\mathbb{P}^n$ in $\mathbb{P}^{n+1}$ by viewing $\mathbb{P}^n$ as the hyperplane defined by $z_{n+1} = 0$ and let $P = (0\colon \cdots : 1) \in \mathbb{P}^{n+1}$. Clearly, $P \notin \mathbb{P}^n$. We have the projection, $\pi\colon (\mathbb{P}^{n+1} - \{P\}) \to \mathbb{P}^n$, from P onto $\mathbb{P}^r$, where

$$\pi(z_0\colon \cdots : z_n : z_{n+1}) = (z_0\colon \cdots : z_n).$$

We get a line family over $\mathbb{P}^n$, where the fibre over $Q \in \mathbb{P}^n$ is just the line l_{PQ} (since $P \notin \mathbb{P}^n$, this line is always well defined). The parametric equations of this line are

$$(u\colon t) \mapsto (uz_0\colon \cdots : uz_n : t),$$

where $(u\colon t) \in \mathbb{P}^1$ and $Q = (z_0\colon \cdots : z_n)$. When $t = 0$, we get Q and hen $u = 0$, we get P. Next, we prove that $\mathbb{P}^{n+1} - \{P\}$ is locally trivial. Make a section, σ_j, of π over $U_j \subseteq \mathbb{P}^n$ by setting

$$\sigma_j((\xi)) = (\xi\colon \xi_j).$$

This points corresponds to the point $(1\colon \xi_j)$ on l_{PQ} and $\xi_j \neq 0$, so it is well-defined. As Q is the point of l_{PQ} for which $t = 0$, we have $\sigma_j((\xi)) \neq Q$. We make an isomorphism, $\psi_j\colon (\mathbb{P}^{n+1} - \{P\}) \restriction U_j \to U_j \prod \mathbb{C}$, *via*

$$(z_0\colon \cdots : z_{j-1} : z_j : z_{j+1} : \cdots : z_{n+1}) \mapsto \left(z_0\colon \cdots : z_n : \frac{z_{n+1}}{z_j} \right).$$

Observe that

$$s_j((\xi)) = \psi_j \circ \sigma_j((\xi)) = \psi_j(\xi\colon \xi_j) = (\xi\colon 1) \in U_j \prod \mathbb{C}.$$

For any $(z_0\colon \cdots : z_{n+1}) \in (\mathbb{P}^{n+1} - \{P\}) \restriction U_i \cap U_j$, we have $z_i \neq 0$ and $z_j \neq 0$; moreover

$$\psi_i(z_0\colon \cdots : z_{n+1}) = \left(z_0\colon \cdots : z_n : \frac{z_{n+1}}{z_i} \right) \quad \text{and} \quad \psi_j(z_0\colon \cdots : z_{n+1}) = \left(z_0\colon \cdots : z_n : \frac{z_{n+1}}{z_j} \right).$$

This means that the transition function, h_i^j, on $U_i \cap U_j$, is multiplication by z_i/z_j. These are the inverses of the transition functions of our previous bundle, $B_0(\mathbb{C}^{n+1})$, which means that the bundle $\mathbb{P}^{n+1} - \{P\}$ is the dual bundle of $B_0(\mathbb{C}^{n+1})$. We will use geometry to show that the bundle $\mathbb{P}^{n+1} - \{P\}$ is in fact $\mathcal{O}_{\mathbb{P}^n}(1)$.

Look at the hyperplanes, H, of $\mathbb{P}^{n+1}$. They are given by linear forms,

$$H\colon \sum_{j=0}^{n+1} a_j Z_j = 0.$$

The hyperplanes through P form a $\mathbb{P}^n$, since $P \in H$ iff $a_{n+1} = 0$. The rest of the hyperplanes are in the affine space, $\mathbb{C}^{n+1} = \mathbb{P}^{n+1} - \mathbb{P}^n$. Indeed such hyperplanes, $H_{(\alpha)}$, are given by

$$H_{(\alpha)}\colon \sum_{j=0}^{n} \alpha_j Z_j + Z_{n+1} = 0, \quad (\alpha_0, \ldots, a_n) \in \mathbb{C}^{n+1}.$$

Given any hyperplane, $H_{(\alpha)}$ (with $\alpha \in \mathbb{C}^{n+1}$), find the intersection, $\sigma_{(\alpha)}(Q)$, of the line l_{PQ} with $H_{(\alpha)}$. Note that $\sigma_{(\alpha)}$ is a global section of $\mathbb{P}^{n+1} - \{P\}$. The affine line obtained from l_{PQ} by deleting P is given by

$$\tau \mapsto (z_0\colon \cdots \colon z_n \colon \tau),$$

where $Q = (z_0\colon \cdots \colon z_n)$. This lines cuts $H_{(\alpha)}$ iff

$$\sum_{j=0}^{n} \alpha_j z_j + \tau = 0,$$

so we deduce $\tau = -\sum_{j=0}^n \alpha_j z_j$ and

$$\sigma_{(\alpha)}(z_0\colon \cdots \colon z_n) = \left(z_0\colon \cdots \colon z_n \colon -\sum_{j=0}^{n} \alpha_j z_j\right),$$

which means that $\sigma_{(\alpha)}$ is a holomorphic section. Now, consider a holomorphic section, $\sigma\colon \mathbb{P}^n \to (\mathbb{P}^{n+1} - \{P\}) \hookrightarrow \mathbb{P}^{n+1}$, of $\pi\colon (\mathbb{P}^{n+1} - \{P\}) \to \mathbb{P}^n$. As σ is an algebraic map and $\mathbb{P}^r$ is proper, $\sigma(\mathbb{P}^n)$ is Z-closed, irreducible and has dimension n in $\mathbb{P}^{n+1}$. Therefore, $\sigma(\mathbb{P}^n)$ is a hypersurface. But, our map factors through $\mathbb{P}^{n+1} - \{P\}$, so $\sigma(\mathbb{P}^n) \subseteq \mathbb{P}^{n+1} - \{P\}$. This hypersurface has some degree, d, but all the lines l_{PQ} cut $\sigma(\mathbb{P}^n)$ in a single point, which implies that $d = 1$, i.e., $\sigma(\mathbb{P}^n)$ is a hyperplane *not* through P. Putting all these facts together, we have shown that space of global sections $\Gamma(\mathbb{P}^n, \mathbb{P}^{n+1} - \{P\})$ is in one-to-one correspondence with the hyperplanes $H_{(\alpha)}$, i.e., the linear forms $\sum_{j=0}^n \alpha_j z_j$ (a $\mathbb{C}^{n+1}$). Therefore, we conclude that $\mathbb{P}^{n+1} - \{P\}$ *is* $\mathcal{O}_{\mathbb{P}^n}(1)$. Since $B_0(\mathbb{C}^{n+1})$ is the dual of $\mathbb{P}^{n+1} - \{P\}$, we also conclude that $B_0(\mathbb{C}^{n+1}) = \mathcal{O}_{\mathbb{P}^n}(-1)$.

In order to prove that Chern classes exist, we need to know more about bundles. The reader may wish to consult Atiyah [2], Milnor and Stasheff [11], Hirsh [7], May [10] or Morita [12] for a more detailed treatment of bundles.

Recall that if G is a group, then $H^1(X, G)$ classifies the G-torsors over X, e.g., (in our case) the fibre bundles, fibre F, over X (your favorite topology) with $\mathrm{Aut}(F) = G$. When $F = G$ and G acts by left translation to make it $\mathrm{Aut}(F)$, the fibre bundle is called a *principal bundle*. Look at $\varphi\colon G' \to G$, a homomorphism of groups. Now, we know that we get a map

$$H^1(X, G') \longrightarrow H^1(X, G).$$

We would like to see this geometrically and we may take as representations principal bundles. Say $E' \in H^1(X, G')$ a principal bundle with fibre G' and group G'. Consider $G \prod E'$ and make an equivalence relation $\sim$ *via*: For all $\sigma \in G'$, all $g \in G$, all $e' \in E'$

$$(g\varphi(\sigma), e') \sim (g, e'\sigma^{-1}).$$

Set $E'_{G' \longrightarrow G} = \varphi_*(E') = G \prod E' / \sim$.

Let us check that the fibre over $x \in X$ is G. Since E' is locally trivial, we have $E' \upharpoonright U \cong U \prod G'$, for some small enough open, U. The action of G' is such that: For $\sigma \in G'$ and $(u, \tau) \in U \prod G'$,

$$\sigma(u, \tau) = (u, \sigma\tau).$$

Over U, we have $(G \prod E') \upharpoonright U = G \prod U \prod G'$, so our $\varphi_*(E')$ is still locally trivial and the action is on the left on G, its fibre. It follows that

$$E' \mapsto \varphi_*(E')$$

is our map $H^1(X, G') \longrightarrow H^1(X, G)$.

Next, say $\theta \colon Y \to X$ is a map (of spaces), then we get a map

$$H^1(X, G) \xrightarrow{\theta^*} H^1(Y, G).$$

Given $E \in H^1(X, G)$, we have the commutative diagram

$$\begin{tikzcd}
E \prod\limits_X Y \arrow[r] \arrow[d] & E \arrow[d, "\pi_E"] \\
Y \arrow[r, "\theta"] & X,
\end{tikzcd}$$

so we get a space, $\theta^*(E) = E \prod\limits_X Y$, over Y. Over a "small" open, U, of X, we have $E \upharpoonright U \cong G \prod U$ and

$$\theta^*(E) \upharpoonright \theta^{-1}(U) \cong G \prod \theta^{-1}(U),$$

and this gives

$$H^1(X, G) \xrightarrow{\theta^*} H^{(}Y, G).$$

Say G is a (Lie) group and we have a linear representation, $\varphi \colon G \to \mathrm{GL}(r, \mathbb{C})$. By the above, we get a map

$$E \mapsto E_{G \longrightarrow \mathrm{GL}(r,\mathbb{C})} = \varphi_*(E)$$

from principal G-bundles over X to principal $\mathrm{GL}(r, \mathbb{C})$-bundles over X. But if V is a fixed vector space of dimension r, the construction above gives a rank r vector bundle $\mathrm{GL}(r, \mathbb{C}) \prod V / \sim$. If $\mathcal{V}$ is a rank r vector bundle over $\mathbb{C}$, then look at the sheaf, $\mathcal{I}som(\mathbb{I}^r, \mathcal{V})$, whose fibre at x is the space $\mathrm{Isom}(\mathbb{C}^r, \mathcal{V}_x)$. This sheaf defines a $\mathrm{GL}(r, \mathbb{C})$-bundle.

Say $G' \subseteq G$ is a closed subgroup of the topological group, G.

If G is a real Lie group and G' is a closed subgroup, then G' is also a real Lie group (E. Cartan). But, if G is a complex Lie group and G' is a closed subgroup, then G' need *not* be a complex Lie group. For example, look at $G = \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$ and $G' = \mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$.

Convention: If G is a complex Lie group, when we say G' is a closed subgroup we mean a complex Lie group, closed in G.

Say G is a topological group and G' is a closed subgroup of G. Look at the space G/G' and at the continuous map, $\pi \colon G \to G/G'$. We say π has a local section iff there is some some $V \subseteq G/G'$ with $1_G \cdot G' \in V$ and a continuous map

$$s \colon V \to G, \quad \text{such that} \quad \pi \circ s = \mathrm{id}_V.$$

When we untwist this definition we find that it means $s(v) \in v$, where v is viewed as a coset. Generally, one must assume the existence of a local section–this is not true in general.

Theorem 3.15 *If G and G' are topological groups and G' is a closed subgroup of G, assume a local section exists. Then*

(1) The map $G \longrightarrow G/G'$ makes G a continuous principal bundle with fibre and group G' and base G/G'.

(2) If G is a real Lie group and G' is a closed subgroup, then a local smooth section always exists and G is a smooth principal bundle over G/G', with fibre (and group) G'.

(3) If G is a complex Lie group and G' is a closed complex Lie subgroup, then a complex analytic local section always exists and makes G is a complex holomorphic principal bundle over G/G', with fibre (and group) G'.

Proof. The proof of (1) is deferred to the next theorem.

(2) & (3). Use local coordinates, choosing coordinates trasnverse to G' after choosing coordinates in G' near $1_{G'}$. The rest is (DX)– because we get a local section and we repeat the proof for (1) to prove the bundle assertion. $\square$

Now, say E is a fibre bundle, with group G over X (and fibre F) and say G' is a closed subgroup of G. Then, we have a new bundle, E/G'. The bundle E/G' is obtrained from E by identifying in each fibre the elements x and $x\sigma$, where $\sigma \in G'$. Then, the group of E/G' is still G and the fibre is F/G'. In particular, if E is principal, then the group of E/G' is G and its fibre is G/G'. We have a map $E \longrightarrow E/G'$ and a diagram

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "\pi_E"'] & & E/G' \arrow[dl] \\ & X & \end{tikzcd}$$

Theorem 3.16 *If $G \longrightarrow G/G'$ possesses a local section, then for a principal G-bundle E over X*

(1) E/G' is a fibre bundle over X, with fibre G/G'.

(2) $E \longrightarrow E/G'$ is in a natural way a principal bundle (over E/G') with group and fibre G'. If $\xi \in H^1(X, G)$ represents E, write $\xi_{G'}$ for the element of $H^1(E/G', G')$ whose bundle is just $E \longrightarrow E/G'$.

(3) From the diagram of bundles

$$\begin{tikzcd} E \arrow[rr, "\pi_{E\to E/G'}"] \arrow[dr, "\pi_E"'] & & E/G' \arrow[dl, "\pi_{E/G'}"] \\ & X & \end{tikzcd}$$

we get the commutative diagram

$$\begin{tikzcd} H^1(X, G') \arrow[r] \arrow[d, "\pi^*_{E/G'}"'] & H^1(X, G) \ni \xi \arrow[d, "\pi^*_{E/G'}"] \\ \xi_{G'} \in H^1(E/G', G') \arrow[r, "i_*"] & H^1(E/G', G) \end{tikzcd}$$

(Here $i\colon G' \hookrightarrow G$ is the inclusion map) and $i_*(\xi_{G'}) = \pi^*_{E/G'}(\xi)$, that is, when E is pulled back to the new base E/G', it arises from a bundle whose structure group is G'.

Figure 3.1: The fibre bundle E over E/G'

Proof. (1) is already proved (there is no need for our hypothesis on local sections).

(2) Pick a cover $\{U_\alpha\}$, of C where $E \restriction U_\alpha$ is trivial so that

$$E \restriction U_\alpha \cong U_\alpha \prod G.$$

Now, consider $G \longrightarrow G/G'$ and the local section $s\colon V(\subseteq G/G') \longrightarrow G$ (with $1_{G/G'} \in V$). We know $s(v) \in v$ (as a coset) and look at $\pi^{-1}(V)$. If $x \in \pi^{-1}(V)$, set

$$\theta(x) = (x^{-1}s(\pi(x)), \pi(x)) \in G' \prod V.$$

This gives an isomorphism (in the appropriate category), $\pi^{-1}(V) \cong G' \prod V$. If we translate V around G/G', we get G as a fibre bundle over G/G' and group G' giving (1) of the previous theorem. But, $U_\alpha \prod V$ and the $U_\alpha \prod$(translate of V) give a cover of E/G' and we have

$$E \restriction U_\alpha \cong U_\alpha \prod \pi^{-1}(V) \cong U_\alpha \prod V \prod G',$$

giving E as fibre bundle over E/G' with group and fibre G'. Here, the diagrams are obvious and the picture of Figure 3.1 finishes the proof. Both sides of the last formula are "push into the board" (by definition for i_* and by elementary computation in $\pi^*_{E/G'}(\xi)$). $\square$

Definition 3.2 If E is a bundle over X with group G and if G' is a closed subgroup of G so that the cohomology representative of G, say ξ actually arises as $i_*(\eta)$ for some $\eta \in H^1(X, G')$, then E *can have its structure group reduced to* G'.

If we restate (3) of the previous theorem in this language, we get

Corollary 3.17 *Every bundle E over X with group G when pulled back to E/G' has its structure group reduced to G'.*

Theorem 3.18 *Let E be a bundle over X, with group G and let G' be a closed subgroup of G. Then, E as a bundle over X can have its structure group reduced to G' iff the bundle E/G' admits a global section over X. In this case if $s\colon X \to E/G'$ is the global section of E/G', then $s^*(E)$ where E is considered as bundle over E/G' with group G' is the element $\eta \in H^1(X, G')$ which gives the structure group reduction. In terms of cocycles, E admits a reduction to group G' iff there exists an open cover $\{U_\alpha\}$ of X so that the transition functions*

$$g_\alpha^\beta\colon U_\alpha \cap U_\beta \to G$$

map $U_\alpha \cap U_\beta$ into the subgroup G'. The section of E/G' is given in the cover by maps $s_\alpha\colon U_\alpha \to U_\alpha \prod G/G'$, where $s_\alpha(u) = (u, 1_{G/G'})$. The cocycle g_α^β represents $s^*(E)$ when its values are considered to be in G' and represents E when its values are considered to be in G.

Proof. Consider the picture of Figure 3.1 above. Suppose E can have structure group reduced to G', then there is a principal bundle, F, for G' and its transition functions give E too. This F can be embedded in E, the fibres are G'. Apply $\pi_{E\longrightarrow E/G'}$ to F, we get get a space over X whose points lie in the bundle E/G', one point for each point of X. Thus, the map $s\colon X \longrightarrow$ point of $\pi_{E\longrightarrow E/G'}(F)$ over x, is our section of E/G' over X.

Conversely, given a section, $s\colon X \to E/G'$, we have E as principal bundle over E/G', with fibre and group G'. So, $s^*(E)$ gives a bundle, F, principal for G', lying over X. Note, F is the bundle given by $s^*(\xi_{G'})$, where ξ represents E. This shows the F constructed reduces to the group G'. The rest (with cocycles) is standard. $\square$

Look at $\mathbb{C}^q$ and $\mathrm{GL}(q, \mathbb{C})$. Write $\mathbb{C}^q_r$ for the span of $e_1, \ldots, e_r$ (the first r canonical basis vectors) = Ker π_r, where π_r is projection on the last $q-r$ basis vectors, $e_{r+1}, \ldots, e_q$. Let $\mathcal{G}rass(r, q; \mathbb{C})$ denote the complex Grassmannian of r-dimensional linear subspaces in $\mathbb{C}^q$. There is a natural action of $\mathrm{GL}(q, \mathbb{C})$ on $\mathcal{G}rass(r, q; \mathbb{C})$ and it is clearly transitive. Let us look for the stabilizer of $\mathbb{C}^q_r$. It is the subgroup, $\mathrm{GL}(r, q-r; \mathbb{C})$, of $\mathrm{GL}(q, \mathbb{C})$, consisting of all matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is $r \times r$. It follows that, as a homogeneous space,

$$\mathrm{GL}(q, \mathbb{C})/\mathrm{GL}(r, q-r; \mathbb{C}) \cong \mathcal{G}rass(r, q; \mathbb{C}).$$

If we restrict the action to $\mathrm{U}(q)$, the above matrices must be of the form

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

where $A \in \mathrm{U}(r)$ and $C \in \mathrm{U}(q-r)$, so

$$\mathrm{U}(q)/\mathrm{U}(r) \prod \mathrm{U}(q-r) \cong \mathcal{G}rass(r, q; \mathbb{C}).$$

Remark: Note, in the real case we obtain

$$\mathrm{GL}(q, \mathbb{R})/\mathrm{GL}(r, q-r; \mathbb{R}) \cong \mathrm{O}(q)/\mathrm{O}(r) \prod \mathrm{O}(q-r) \cong \mathcal{G}rass(r, q; \mathbb{R}).$$

If one looks at oriented planes, then this becomes

$$\mathrm{GL}^+(q, \mathbb{R})/\mathrm{GL}^+(r, q-r; \mathbb{R}) \cong \mathrm{SO}(q)/\mathrm{SO}(r) \prod \mathrm{SO}(q-r) \cong \mathcal{G}rass^+(r, q; \mathbb{R}).$$

Theorem 3.19 *(Theorem A) If X is paracompact, f and g are two maps $X \longrightarrow Y$ and E is a bundle over Y, then when f is homotopic to g and not for holomorphic bundles, we have $f^*E \cong g^*E$.*

Theorem 3.20 *(Theorem B) Suppose X is paracompact and E is a bundle over X whose fibre is a cell. If Z is any closed subset of X (even empty) then any section (continuous, smooth, but not holomorphic) of E over Z admits an extension to a global section (continuous or smooth) of E. That is, the sheaf $\mathcal{O}_X(E)$ is a soft sheaf. (Note this holds when E is a vector bundle and it is Tietze's Extension Theorem).*

Theorem 3.21 *(Theorem C) Say G' is a closed subgroup of G and X is paracompact. If G/G' is a cell, then the natural map*

$$H^1_{\mathrm{top}}(X, G') \longrightarrow H^1_{\mathrm{top}}(X, G) \quad \textit{or} \quad H^1_{\mathrm{diff}}(X, G') \longrightarrow H^1_{\mathrm{diff}}(X, G)$$

is a bijection. That is, every principal G-bundle can have its structure group reduced to G' and comes from a unique principal G'-bundle.

Proof. Suppose E is a principal G-bundle and look at E/G' over X. The fibre of E/G' over X is G/G', a cell. Over a small closed set, say Z, the bundle E/G' has a section; so, by Theorem B our section extends to a global section (G/G' is a cell). Then, by Theorem 3.18, the bundle E comes from $H^1(X, G')$ and surjectivity is proved.

Now, say E and F are principal G'-bundles and that they become isomorphic as G-bundles. Take a common covering $\{U_\alpha\}$, where E and F are trivialized. Then $g_\alpha^\beta(E), g_\alpha^\beta(F)$, their transition functions become cohomologous in the G-bundle theory. This means that there exist maps, $h_\alpha\colon U_\alpha \to G$ so that

$$g_\alpha^\beta(F) = h_\beta^{-1} g_\alpha^\beta(E) h_\alpha^{-1}.$$

Consider $X \prod I$ where $I = [0,1]$ and cover $X \prod I$ by the opens

$$U_\alpha^0 = U_\alpha \prod [0,1) \quad \text{and} \quad U_\alpha^1 = U_\alpha \prod (0,1].$$

Make a principal bundle on $X \prod I$ using the following transition functions:

$$g_{\alpha\, 0}^{\beta\, 0}\colon U_\alpha^0 \cap U_\beta^0 \longrightarrow G$$

via $g_{\alpha\, 0}^{\beta\, 0}(x,t) = g_\alpha^\beta(E)(x)$;

$$g_{\alpha\, 1}^{\beta\, 1}\colon U_\alpha^1 \cap U_\beta^1 \longrightarrow G$$

via $g_{\alpha\, 1}^{\beta\, 1}(x,t) = g_\alpha^\beta(F)(x)$;

$$g_{\alpha\, 0}^{\beta\, 1}\colon U_\alpha^0 \cap U_\beta^1 \longrightarrow G$$

via $g_{\alpha\, 0}^{\beta\, 1}(x,t) = h_\beta(x) g_\alpha^\beta(F)(x) = g_\alpha^\beta(E)(x) h_\alpha(x)$. Call this new bundle (E,F) and let

$$Z = X \prod \{0\} \cup X \prod \{1\} \hookrightarrow X \prod I$$

a closed subset. Note that (E,F) over Z is a G'-bundle. Thus, Theorem 3.18 says $(E,F)/G'$ has a global section over Z. But, its fibre is G/G', a cell. Therefore, by Theorem B, the bundle $(E,F)/G'$ has a global section over all of X. By Theorem 3.18, again, the bundle (E,F) comes from a G'-bundle, $\widetilde{(E,F)}$. Write $f_0\colon X \to X \prod I$ for the function given by

$$f_0(x) = (x,0)$$

and $f_1\colon X \to X \prod I$ for the function given by

$$f_1(x) = (x,1).$$

If $\widetilde{(E,F)} \restriction X \prod \{0\} = \widetilde{(E,F)}_0$, then $f_0^*(\widetilde{(E,F)}_0) = E$, i.e., $f_0^*\widetilde{(E,F)} = E$ and similarly, $f_1^*\widetilde{(E,F)} = F$; and f_0 is homotopic to f_1. By Theorem A, we get $E \cong F$ as G'-bundles. $\square$

There is a theorem of Steenrod stating: If X is a differentiable manifold and E is a fibre bundle over X, then every continuous section of E may be approximated (with arbitrary ϵ) on compact subsets of X by a smooth section. When E is a vector bundle, this is easy to prove by an argument involving a partition of unity and approximation techniques using convolution. This proves

Theorem 3.22 *(Theorem D) If X is a differentiable manifold and G is a Lie group, then the map*

$$H^1_{\mathrm{diff}}(X,G) \longrightarrow H^1_{\mathrm{cont}}(X,G)$$

is a bijection.

We get the

Corollary 3.23 *If X is a differentiable manifold, then in the diagram below, for the following pairs (G',G)*

(α) $G' = \mathrm{U}(q)$, $G = \mathrm{GL}(q,\mathbb{C})$.

(β) $G' = \mathrm{U}(r) \prod \mathrm{U}(q-r)$, $G = \mathrm{GL}(r, q-r; \mathbb{C})$ or $G = \mathrm{GL}(r,\mathbb{C}) \prod \mathrm{GL}(q-r,\mathbb{C})$.

(γ) $G' = \mathbb{T}^q = S^1 \times \cdots \times S^1$ (the real q-torus), $G = \Delta(q, \mathbb{C})$ or $G = \mathbb{G}_m \prod \cdots \prod \mathbb{G}_m = \mathbb{C}^* \prod \cdots \prod \mathbb{C}^*$ $(= \mathrm{GL}(1, \mathbb{C}) \prod \cdots \prod \mathrm{GL}(1, \mathbb{C}))$ (the complex q-torus)

all the maps are bijective

$$\begin{tikzcd} H^1_{\mathrm{cont}}(X, G') \arrow[r] & H^1_{\mathrm{cont}}(X, G) \\ H^1_{\mathrm{diff}}(X, G') \arrow[r] \arrow[u] & H^1_{\mathrm{diff}}(X, G). \arrow[u] \end{tikzcd}$$

Here,

$$\Delta(q, \mathbb{C}) = \bigcap_{r=1}^{q} \mathrm{GL}(r, q-r; \mathbb{C})$$

the upper triangular matrices.

Proof. Observe that G/G' is a cell in all cases and that $\Delta(q, \mathbb{C}) \cap \mathrm{U}(q) = \mathbb{T}^q$. $\square$

Suppose ξ corresponds to a $\mathrm{GL}(q)$-bundle which has group reduced to $\mathrm{GL}(r, q-r; \mathbb{C})$. Then, the maps

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto A \quad \text{and} \quad M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto C$$

give surjections $\mathrm{GL}(r, q-r; \mathbb{C}) \longrightarrow \mathrm{GL}(r, \mathbb{C})$ and $\mathrm{GL}(r, q-r; \mathbb{C}) \longrightarrow \mathrm{GL}(q-r, \mathbb{C})$, so ξ comes from $\widetilde{\xi}$ and $\widetilde{\xi}$ gives rise to ξ' and ξ'' which are $\mathrm{GL}(r, \mathbb{C})$ and $\mathrm{GL}(q-r, \mathbb{C})$-bundles, respectively. In this case one says: the $\mathrm{GL}(q, \mathbb{C})$-bundle ξ *admits a reduction to a (rank r) subbundle ξ' and a (rank $q-r$) quotient bundle ξ''*. When we use $\Delta(q, \mathbb{C})$ and $\mathrm{GL}(q, \mathbb{C})$ then we get q maps, $\varphi_l \colon \Delta(q, \mathbb{C}) \to \mathbb{C}^*$, namely

$$\varphi_j \colon \begin{pmatrix} a_1 & * & \cdots & * & * \\ 0 & a_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{q-1} & * \\ 0 & 0 & \cdots & 0 & a_q \end{pmatrix} \mapsto a_l.$$

So, if $\widetilde{\xi}$ is our $\Delta(q, \mathbb{C})$-bundle, we get q *line bundles* $\xi_1, \ldots, \xi_q$ from $\widetilde{\xi}$ and one says ξ has $\xi_1, \ldots, \xi_q$ *as diagonal line bundles*.

Set

$$\mathbb{F}_q = \mathrm{GL}(q; \mathbb{C})/\Delta(q; \mathbb{C}) = \mathrm{GL}(q; \mathbb{C})/\bigcap_{r=1}^{q} \mathrm{GL}(r, q-r; \mathbb{C}),$$

the *flag manifold*, i.e., the set of all flags

$$\{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_q = V \mid \dim(V_j) = j\}.$$

Since $\mathbb{F}_q = \mathrm{GL}(q; \mathbb{C})/\bigcap_{r=1}^{q} \mathrm{GL}(r, q-r; \mathbb{C})$, we see that $\mathbb{F}_q$ is embedded in $\prod_{r=1}^{1} \mathcal{G}rass(r, q; \mathbb{C})$. Thus, as the above is a closed immersion, $\mathbb{F}_q$ is an algebraic variety, even a projective variety (by Segre). If V is a rank q vector bundle over X, say $E(V)$ ($\cong \mathrm{Isom}(\mathbb{C}^q, V)$) is the associated principal bundle, then write

$$[r]V = E(V)/\mathrm{GL}(r, q-r; \mathbb{C}),$$

a bundle over X whose fibres are $\mathcal{G}rass(r, q; \mathbb{C})$ and

$$[\Delta]V = E(V)/\Delta(q; \mathbb{C})$$

a bundle over X whose fibres are the $\mathbb{F}(q)$'s. We have maps $\rho_r \colon [r]V \to X$ and $\rho_\Delta \colon [\Delta]V \to X$. Now we apply our theorems to the pairs

(a) $G' = \mathrm{U}(q)$, $G = \mathrm{GL}(q, \mathbb{C})$.

(b) $G' = \mathrm{U}(r) \prod \mathrm{U}(q-r)$ and $G = \mathrm{GL}(r, q-r, \mathbb{C})$ or $G = \mathrm{GL}(r, \mathbb{C}) \prod \mathrm{GL}(q-r, \mathbb{C})$.

(c) $G' = \mathbb{T}^q$ and $G = \mathrm{U}(q)$ or $G = \mathbb{C}^* \prod \cdots \prod \mathbb{C}^* = (\mathbb{G}_m)^q$.

(d) $G' = \Delta(q, \mathbb{C})$ and $G = \mathrm{GL}(q, \mathbb{C})$

and then we get, (for example) every rank r vector bundle over X is "actually" a rank r unitary bundle over X and we also have

Theorem 3.24 *If X is paracompact or a differentiable manifold or a complex analytic manifold or an algebraic variety and V is a rank q vector bundle of the appropriate category on X, then*

(1) V reduces to a rank r subbundle, V', and rank $q-r$ quotient bundle, V'', over X iff $[r]V$ possesses an appropriate global section over X.

(2) V reduces to diagonal bundles over X iff $[\Delta]V$ possesses an appropriate global section over X.

(3) For the maps ρ_r in case (1), resp. ρ_Δ in case (2), the bundle $\rho_r^* V$ reduces to a rank r subbundle and rank $q-r$ quotient bundle over $[r]V$ (resp. reduces to diagonal bundles over $[\Delta]V$).

Remark: The sub, quotient, diagonal bundles are continuous, differentiable, analytic, algebraic, respectively.

Say $s\colon X \to [r]V$ is a global section. For every $x \in X$, we have $sx \in \mathcal{G}rass(r, q; V_x)$; i.e., $s(x)$ is an r-plane in V_v and so, $\bigcup_{x \in X} s(x)$ gives an "honest" rank r subbundle or V. It is isomorphic to the subbundle, V', of the reduction. Similarly, $\bigcup_{x \in X} V_x/s(x)$ is an "honest" rank $q-r$ quotient bundle of V; it is just V''.

Hence, we get

Corollary 3.25 *If the hypotheses of the previous theorem hold, then*

(1) $[r]V$ has a section iff there is an exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

of vector bundles on X.

(2) $[\Delta]V]$ has a section iff there exist exact sequences

$$\begin{array}{l}
0 \longrightarrow L_1 \longrightarrow V \longrightarrow V_1'' \longrightarrow 0 \\
0 \longrightarrow L_2 \longrightarrow V_1'' \longrightarrow V_2'' \longrightarrow 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \longrightarrow L_{j+1} \longrightarrow V_j'' \longrightarrow V_{j+1}'' \longrightarrow 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\qquad\qquad L_q \cong V_{q-1}''
\end{array}$$

where the L_j's are line bundles, in fact, the diagonal bundles.

Theorem 3.26 *In the continuous and differentiable categories, when V has an exact sequence as in (1) of Corollary 3.25 or diagonal bundles as in (2) of Corollary 3.25, then*

(1) $V \cong V' \amalg V''$.

(2) $V \cong L_1 \amalg \cdots \amalg L_q$.

The above is false if we need splitting analytically!

All we need to prove is (1) as (2) follows by induction. We know V comes from $H^1(X, \mathrm{GL}(r, q-r; \mathbb{C})$. By (b) above, V comes from $H^1(X, \mathrm{U}(r) \prod \mathrm{U}(q-r))$ and by (b) again, V comes from $H^1(X, \mathrm{GL}(r) \prod \mathrm{GL}(q-r)) \cong H^1(X, \mathrm{GL}(r)) \amalg H^1(X, \mathrm{GL}(q-r))$ and we get (1). $\square$

Corollary 3.27 *(Splitting Principle) Given V, a continuous, differentiable, analytic, algebraic rank q vector bundle over X, then $\rho_r^* V$ is in the continuous or differentiable category a coproduct $V = V' \amalg V''$ ($\mathrm{rk}(V') = r$, $\mathrm{rk}(V'') = q - r$) or $\rho_\Delta^* V$ is $V = L_1 \amalg \cdots \amalg L_q$.*

Note that $[r]V$ and $[\Delta]V$ are fibre bundles over X; consequently, there is a relation between $H^j(X, \mathbb{Z})$ and $H^j([r]V, \mathbb{Z})$ (resp. $H^j([\Delta]V, \mathbb{Z})$. This is the *Borel spectral sequence*. Under the condition that (E, X, F, G) is a fibre space over X (admissible), group G, fibre F, total space E, there is a spectral sequence whose $E_2^{p,q}$-term is

$$H^p(X, H^q(F, \mathbb{Z}))$$

and whose ending is $\mathrm{gr}(H^\bullet(E, \mathbb{Z}))$,

$$H^p(X, H^q(F, \mathbb{Z})) \underset{p}{\Longrightarrow} H^\bullet(E, \mathbb{Z}).$$

Borel proves that in our situation: The map

$$\rho^* \colon H^\bullet(X, \mathbb{Z}) \to H^\bullet([r]V, \mathbb{Z})$$

(resp. $\rho^* \colon H^\bullet(X, \mathbb{Z}) \to H^\bullet([\Delta]V, \mathbb{Z})$) is an injection. From the hand-out, we also get the following: Write

$$\mathrm{BU}(q) = \varinjlim_N \mathcal{G}rass(q, N; \mathbb{C}).$$

Note,

$$\mathrm{BU}(1) = \varinjlim_N \mathbb{P}_{\mathbb{C}}^{N-1} = \mathbb{P}_{\mathbb{C}}^{\infty}.$$

Theorem 3.28 *If X is admissible (locally compact, σ-compact, finite dimensional) then $\mathrm{Vect}_q(X)$ (isomorphism classes of rank q vector bundles over X) in the continuous or differentiable category is in one-to-one correspondence with homotopy classes of maps $X \longrightarrow \mathrm{BU}(q)$. In fact, if X is compact and $N \geq 2\dim(X)$ then already the homotopy classes of maps $X \longrightarrow \mathcal{G}rass(q, N; \mathbb{C})$ classify rank q vector bundles on X (differentiably). Moreover, on $\mathrm{BU}(q)$, there exists a bundle, the "universal quotient", W_q, it has rank q over $\mathrm{BU}(q)$ (in fact, it is algebraic) so that the map is*

$$f \in [X \longrightarrow \mathrm{BU}(q)] \mapsto f^* W_q.$$

We are now in the position where we can prove the uniqueness of Chern classes.

Uniqueness of Chern Classes:

Assume existence (Axiom (I)) and good behavior (Axioms (II)–(IV)). First, take a line bundle, L, on X. By the classification theorem there is a map

$$f \colon X \to \mathrm{BU}(1)$$

so that $f^*(H) = L$ (here, H is the universal quotient line bundle). By Axiom (II),

$$f^*(c(H)(t)) = c(f^*(H))(t) = c(L)(t)$$

and the left hand side is $f^*(1 + Ht)$, by Axiom (IV) (viewing H as a cohomology class). It follows that the left hand side is $1 + f^*(H)t$ and so,

$$c_1(L) = f^*(H), \quad \text{and} \quad c_j(L) = 0, \quad \text{for all} \quad j \geq 2.$$

This is independent of f as homotopic maps agree cohomologically.

Now, let V be a rank q vector bundle on X and make the bundle $[\Delta]V$ whose fibre is $\mathbb{F}(q)$. Take $\rho^*(V)$, where $\rho\colon [\Delta]V \to X$. We know

$$\rho^* V = \coprod_{j=1}^{q} L_j,$$

where the L_j's are line bundes and by Axiom (II),

$$c(\rho^*(V))(t) = \prod_{j=1}^{1} (1 + c_1(L_j)(t)).$$

Now, the left hand side is $\rho^*(c(V)(t))$, by Axiom (II); then, ρ^* being an injection implies $c(V)(t)$ is uniquely determined.

Remark: Look at $\mathrm{U}(q) \supseteq \mathrm{U}(1) \coprod \mathrm{U}(q-1) \supseteq \mathbb{T}^q$. Then,

$$\mathrm{U}(1) \coprod \mathrm{U}(q-1)/\mathbb{T}^q \hookrightarrow \mathrm{U}(q)/\mathbb{T}^q = \mathbb{F}(q)$$

and the left hand side is $\mathrm{U}(q-1)/\mathbb{T}^{q-1} = \mathbb{F}(q-1)$. So, we have an injection $\mathbb{F}(q-1) \hookrightarrow \mathbb{F}(q)$ over the base $\mathrm{U}(q)/\mathrm{U}(1) \coprod \mathrm{U}(q-1)$, which is just $\mathbb{P}^{q-1}$. Thus, we can view $\mathbb{F}(q)$ as a fibre bundle over $\mathbb{P}^{q-1}$ and the fibre is $\mathbb{F}(q-1)$.

Take a principal $\mathrm{U}(q)$-bundle, E, over X and make $E/\mathbb{T}^q$, a fibre space whose fibre is $\mathbb{F}(q)$. Let E_1 be $E/\mathrm{U}(1) \coprod \mathrm{U}(q-1)$, a fibre space whose fibre is $\mathbb{P}^{q-1}$. Then, we have a map

$$E/\mathbb{T}^q \longrightarrow E_1,$$

where the fibre is $\mathrm{U}(1) \coprod \mathrm{U}(q-1)/\mathbb{T}^q = \mathbb{F}(q-1)$. We get

$$\begin{tikzcd}
E/\mathbb{T}^q = [\Delta]E \arrow[d, "\text{fibre } \mathbb{F}(q-1)"'] \\
E_1 \arrow[d, "\text{fibre } \mathbb{P}^{q-1}"'] \\
X.
\end{tikzcd}$$

If we repeat this process, we get the tower

$$\begin{tikzcd}
E/\mathbb{T}^q = [\Delta]E \arrow[d, "\text{fibre } \mathbb{P}^1"'] \arrow[dddd, bend left, "\rho"] \\
E_{q-1} \arrow[d, "\text{fibre } \mathbb{P}^2"'] \\
E_{q-2} \arrow[d, dashed] \\
E_1 \arrow[d, "\text{fibre } \mathbb{P}^{q-1}"'] \\
X.
\end{tikzcd}$$

So, to show ρ^* is injective, all we need to show is the same fact when the fibre i $\mathbb{P}^n$ and the $\mathbb{P}^r$-bundle comes from a vector bundle.

Suggestion: Look in Hartshorne in Chapter III, Section ? on projective fibre bundles and Exercise ?? about

$$\rho^*(\mathcal{O}_{\mathbb{P}(E)}(l)) = \mathcal{S}^l(\mathcal{O}_X(E)).$$

Sup up to tangent bundles and wedges and use Hodge:

$$H^\bullet(X, \mathbb{C}) = \text{in term of the holomorphic cohomology of } \bigwedge^{\text{top}} T.$$

We get that ρ^* is injective on $H^\bullet(X, \mathbb{C})$, not $H^\bullet(X, \mathbb{Z})$.

Existence of Chern Classes:

Start with L, a line bundle over X. Then, there is a map (continuous, diff.), $f\colon X \to \mathbb{P}^N_{\mathbb{C}}$, for $N >> 0$ and $L = f^*(H)$. Then, set $c_1(L) = f^*(H)$, where H is the cohomology class of the hyperplane bundle in $H^2(\mathbb{P}^N, \mathbb{Z})$ and $c_j(L) = 0$ if $j \geq 2$. If another map, g, is used, then $f^*(H) = L = h^*(L)$ implies that f and g are homotopic, so f^* and g^* agree on cohomology and $c_1(L)$ is independent of f. It is also independent of N, we we already proved. Clearly, Axiom (II) and Axiom (IV) are built in.

Now, let V be a rank q vector bundle over X. Make $[\Delta]V$ and let ρ be the map $\rho\colon [\Delta]V \to X$. Look at ρ^*V. We know that

$$\rho^*V = \coprod_{j=1}^{q} L_j,$$

where the L_j's are line bundles. By the above,

$$c_j(L_j)(t) = 1 + c_1(L_j)t = 1 + \gamma_j t.$$

Look at the polynomial

$$\prod_{j=1}^{q} (1 + \gamma_j t) \in H^\bullet([\Delta]V, \mathbb{Z})[t].$$

If we show this polynomial (whose coefficients are the symmetric functions $\sigma_l(\gamma_1, \ldots, \gamma_q)$) is in the image of $\rho^*\colon H^\bullet(X, \mathbb{Z})[t] \longrightarrow H^\bullet([\Delta]V, \mathbb{Z})[t]$, then there is a unique polynomial $c(V)(t)$ so that

$$\rho^*(c(V)(t)) = \prod_{j=1}^{q} (1 + \gamma_j t).$$

(Then, $c_l(V) = \sigma_l(\gamma_1, \ldots, \gamma_q)$.) Look at the normalizer of $\mathbb{T}^q$ in $\mathrm{U}(q)$. Some a belongs to this normalizer iff $a\mathbb{T}^q a^{-1} = \mathbb{T}^q$. As the new diagonal matrix, $a\theta a^{-1}$ (where $a \in \mathbb{T}^q$ has the same characteristic polynomial as θ, it follows that $a\theta a^{-1}$ is just θ, but with its diagonal entries permuted. This gives a map

$$\mathcal{N}_{\mathrm{U}(q)}(\mathbb{T}^q) \longrightarrow \mathfrak{S}_q.$$

What is the kernel of this map? We have $a \in \mathrm{Ker}$ iff $a\theta a^{-1} = \theta$, i.e., $a\theta = \theta a$, for all $\theta \in \mathbb{T}^q$. This means (see the 2×2 case) $a \in \mathbb{T}^q$ and thus, we have an injection

$$\mathcal{N}_{\mathrm{U}(q)}(\mathbb{T}^q)/\mathbb{T}^q \hookrightarrow \mathfrak{S}_q.$$

The left hand side, by definition, is the *Weyl group*, W, of $\mathrm{U}(q)$. In fact (easy DX), $W \cong \mathfrak{S}_q$.

Look at $[\Delta]V$ and write a covering of X trivializing $[\Delta]V$, call it $\{U_\alpha\}$. We have

$$[\Delta]V \restriction U_\alpha \cong U_\alpha \prod \mathrm{U}(q)/\mathbb{T}^q.$$

Make the element a act on the latter *via*

$$a(u, \xi\mathbb{T}^q) = (u, \xi\mathbb{T}^q a^{-1}) = (u, \xi a^{-1}\mathbb{T}^q).$$

These patch as the transition functions are *left* translations. This gives an automorphism of $[\Delta]V$, call it $\widetilde{a}$, determined by each $a \in W$. We get a map

$$\widetilde{a}^*\colon H^\bullet([\Delta]V, -) \to H^\bullet([\Delta]V, -).$$

Now, as $a \in W$ acts on $\mathbb{T}^q$ by permuting the diagonal elements it acts on $H^1([\Delta]V, \mathbb{T}^q)$ by permuting the diagonal bundles, say L_j, call this action $a^\#$. Moreover, ρ^*V comes from a unique element of $H^1([\Delta]V, \mathbb{T}^q)$, which implies that $\widetilde{a}$ acts on ρ^*V by permuting its cofactors. But, $\widetilde{a}^*$ also acts on $H^1([\Delta]V, \mathbb{T}^q)$ and one should check (by a Čech cohomology argument) that

$$\widetilde{a}^* = a^\#.$$

Now associate to the L_j's their Chern classes, γ_j, and $\widetilde{a}^*(\gamma_j)$ goes over to $a^\#(\gamma_j)$, i.e., permute the $|gamma_j$s's. Thus, W acts on the L_j and γ_j by permuting them. Our polynomial

$$\prod_{j=1}(1 + \gamma_j t)$$

goes to itself *via* the action of W. But, Borel's Theorem is that an element of $H^\bullet([\Delta]V, \mathbb{Z})$ lies in the image of $\rho^*\colon H^\bullet(X, \mathbb{Z}) \to H^\bullet([\Delta]V, \mathbb{Z})$ iff W fixes it. So, by the above, our elementary symmetric functions lie in $\operatorname{Im}\rho^*$; so, Chern classes exist. Furthermore, it is clear that they satisfy Axioms (I), (II), (IV).

Finally, consider Axiom (III). Suppose V splits over X as

$$V = \coprod_{j=1}^q L_j.$$

We need to show that $c(V)(t) = \prod_{j=1}^1 (1 + c_1(L_j)t)$.

As V splits over X, the fibre bundle $\rho\colon [\Delta]V \longrightarrow X$ has a section; call it s. So, $s^*\rho^* = \mathrm{id}$ and

$$c(V)(t) = s^*\rho^*(c(V)(t)) = s^*(\rho^*(c(V)(t))).$$

By Axiom (II), $s^*(\rho^*(c(V)(t))) = s^*(c(\rho^*(V))(t))$. Since $\rho^* = \coprod_{j=1}^q \rho^* L_j$ and we know that if we set $\gamma_j = c_1(\rho^*(L_j))$, then

$$\rho^*(c(V)(t)) = c(\rho^*(V)(t)) = \prod_{j=1}^q (1 + \gamma_j t).$$

But then,

$$c(V)(t) = s^* \prod_{j=1}^q (1 + \gamma_j t) = \prod_{j=1}^q (1 + s^*(\gamma_j)t). \tag{$\dagger$}$$

However, $L_j = s^*(\rho^*(L_j))$ implies

$$c_1(L_j) = s^*(c_1(\rho^*(L_j))) = s^*(\gamma_j).$$

The above plus (†) yields

$$c(V)(t) = \prod_{j=1}^q (1 + c_1(L_j)t),$$

as required. $\square$

Eine kleine Vektorraumbündel Theorie:

Say V (rank q) and W (rank q') have diagonal bundles $L_1, \ldots, L_q$ and $M_1, \ldots, M_{q'}$ over X. Then, the following hold:

(1) V^D has $L_1^D, \ldots, L^D$ as diagonal line bundles;

(2) $V \amalg W$ has $L_1, \ldots, L_q, M_1, \ldots, M_{q'}$ as diagonal line bundles;

(3) $V \otimes W$ has $L_i \otimes M_j$ (all i, j) as diagonal line bundles;

(4) $\bigwedge^r V$ has $L_{i_1} \otimes \cdots \otimes L_{i_r}$, where $1 \leq i_1 < \cdots < i_r \leq q$, as diagonal line bundles;

(4) $\mathcal{S}^r V$ has $L_1^{m_1} \otimes \cdots \otimes L_q^{m_q}$, where $m_i \geq 0$ and $m_1 + \cdots + m_q = r$, as diagonal line bundles.

Application to the Chern Classes.

(0) (Splitting Principle) Given a rank q vector bundle, V, make believe V splits as $V = \coprod_{j=1}^q L_j$ (for some line bundles, L_j), write $\gamma_j = c_1(L_j)$, the γ_j are the *Chern roots* of V. Then,

$$c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t).$$

(1) $c(V^D)(t) = \prod_{j=1}^q (1 - \gamma_j t)$ when $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$. That is, $c_i(V^D) = (-1)^i c_i(V)$.

(2) If $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is exact, then $c(V)(t) = c(V')(t)c(V'')(t)$.

(3) If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$ and $c(W)(t) = \prod_{j=1}^{q'} (1 + \delta_j t)$, then $c(V \otimes W)(t) = \prod_{j,k=1}^{q,q'} (1 + (\gamma_j + \delta_k) t)$.

(4) If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$, then

$$c\Big(\bigwedge^r V\Big)(t) = \prod_{1 \leq i_1 < \cdots < i_r \leq q} (1 + (\gamma_{i_1} + \cdots + \gamma_{i_r}) t).$$

In particular, when $r = q$, there is just one factor in the polynomial, it has degree 1, it is $1 + (\gamma_1 + \cdots + \gamma_q)t$. By (2). we get

$$c_1\Big(\bigwedge^q V\Big)(t) = c_1(V) \quad \text{and} \quad c_l\Big(\bigwedge^q V\Big)(t) = 0 \quad \text{if} \quad l \geq 2.$$

(5) If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$, then

$$c(\mathcal{S}^r V)(t) = \prod_{\substack{m_j \geq 0 \\ m_1 + \cdots + m_q = r}} (1 + (m_1 \gamma_1 + \cdots + m_q \gamma_q) t).$$

(6) If $\mathrm{rk}(V) \leq q$, then $\deg(c(V)(t)) \leq q$ (where $\deg(c(V)(t)$ is the degree of $c(V)(t)$ as a polynomial in t).

(7) Suppose we know $c(V)$, for some vector bundle, V, and L is a line bundle. Write $c = c_1(L)$. Then, the Chern classes of $V \otimes L$ are

$$c_l(V \otimes L) = \sigma_l(\gamma_1 + c, \gamma_2 + c, \cdots, \gamma_r + c),$$

where $r = \mathrm{rk}(V)$ and the γ_j are the Chern roots of V. This is because the Chern polynomial of $V \otimes L$ is

$$c(V \otimes L)(t) = \prod_{i=1}^{r}(1 + (\gamma_i + c)t).$$

Examples. (1) If $\mathrm{rk}(V) = 2$, then

$$c(V \otimes L)(t) = (1 + (\gamma_1 + c)t)(1 + (\gamma_2 + c)t) = 1 + (\gamma_1 + \gamma_2 + 2c)t + (\gamma_1\gamma_2 + c(\gamma_1 + \gamma_2) + c^2)t^2,$$

so

$$\begin{aligned} c_1(V \otimes L) &= c_1(V) + 2c \\ c_2(V \otimes L) &= c_2(V) + c_1(V)c + c^2. \end{aligned}$$

(2) If $\mathrm{rk}(V) = 3$, then

$$c(V \otimes L)(t) = (1 + (\gamma_1 + c)t)(1 + (\gamma_2 + c)t)(1 + (\gamma_3 + c)t)$$

and so,

$$\begin{aligned} c(V \otimes L)(t) &= 1 + (\gamma_1 + \gamma_2 + \gamma_3 + 3c)t \\ &\quad + (\sigma_2(\gamma_1, \gamma_2, \gamma_3) + 2\sigma_1(\gamma_1, \gamma_2, \gamma_3)c + 3c^2)t^2 \\ &\quad + (\sigma_3(\gamma_1, \gamma_2, \gamma_3) + \sigma_1(\gamma_1, \gamma_2, \gamma_3)c^2 + \sigma_2(\gamma_1, \gamma_2, \gamma_3)c + c^3)t^3. \end{aligned}$$

We deduce

$$\begin{aligned} c_1(V \otimes L) &= c_1(V) + 3c_1(L) \\ c_2(V \otimes L) &= c_2(V) + 2c_1(V)c_1(L) + 3c_1(L)^2 \\ c_3(V \otimes L) &= c_3(V) + c_2(V)c_1(L) + c_1(V)c_1(L)^2 + c_1(L)^3. \end{aligned}$$

In the case of $\mathbb{P}^n$, it is easy to compute the Chern classes. By definition,

$$c(\mathbb{P}^n)(t) = c(T^{1,0}_{\mathbb{P}^n})(t).$$

We can use the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(H) \longrightarrow \mathcal{T}^{1,0}_{\mathbb{P}^n} \longrightarrow 0$$

to deduce that

$$c(\mathcal{O}_{\mathbb{P}^n})(t)c(T^{1,0}_{\mathbb{P}^n})(t) = c(\mathcal{O}_{\mathbb{P}^n}(H)(t))^{n+1}.$$

It follows that

$$c(T^{1,0}_{\mathbb{P}^n})(t) = (1 + Ht)^{n+1} \pmod{t^{n+1}} = \sum_{j=0}^{n} \binom{n+1}{j} H^j t^j$$

and so,

$$c_j(T^{1,0}_{\mathbb{P}^n}) = \binom{n+1}{j} H^j \in H^{2j}(\mathbb{P}^n, \mathbb{Z}).$$

(Here $H^j = H \cdot \ldots \cdot H$, the cup-product in cohomology). In particular,

$$c_1(T^{1,0}_{\mathbb{P}^n}) = (n+1)H = c\Big(\bigwedge^n T^{1,0}_{\mathbb{P}^n}\Big).$$

Now, if $\omega_{\mathbb{P}^n}$ is the canonical bundle on $\mathbb{P}^n$, i.e., $\omega_{\mathbb{P}^n} = \bigwedge^n T^{0,1\,D}_{\mathbb{P}^n} = \Big(\bigwedge^n T^{1,0}_{\mathbb{P}^n}\Big)^D$, we get

$$c_1(\omega_{\mathbb{P}^n}) = -(n+1)H.$$

Say a variety X sits inside $\mathbb{P}^n_{\mathbb{C}}$ and assume X is a manifold. Let $\mathfrak{I}$ be the ideal sheaf of X. By definition, $\mathfrak{I}$ is the kernel in the exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

If X is a hypersurface of degree d, we know

$$\mathfrak{I} = \mathcal{O}_{\mathbb{P}^n}(-d) = \mathcal{O}_{\mathbb{P}^n}(-dH).$$

We also have the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^n} \restriction X \longrightarrow \mathcal{N}_{X\hookrightarrow\mathbb{P}^n} \longrightarrow 0,$$

where $\mathcal{N}_{X\hookrightarrow\mathbb{P}^n}$ is a rank $n-q$ bundle on X, with $q = \dim X$ (the *normal bundle*). If we write $i\colon X \to \mathbb{P}^n$, we get

$$\Big(\bigwedge^n T_{\mathbb{P}^n}\Big) \restriction X = \bigwedge^n T_X \otimes \bigwedge^{n-q} \mathcal{N}_{X\hookrightarrow\mathbb{P}^n},$$

and so,

$$i^*(1 + c_1\Big(\bigwedge^n T_{\mathbb{P}^n}\Big)t) = (1 + c_1\Big(\bigwedge^n T_X\Big)t)(1 + c_1\Big(\bigwedge^{n-q} \mathcal{N}_{X\hookrightarrow\mathbb{P}^n}\Big)t),$$

which yields

$$1 + i^*((n+1)H)t = 1 + c_1(T_X)t + c_1(\mathcal{N}_{X\hookrightarrow\mathbb{P}^n})t.$$

For the normal bundle, we can compute using $\mathfrak{I}$. Look at a small open, then we have the usual case of $\mathbb{C}$-algebras

$$\mathbb{C} \hookrightarrow A \longrightarrow B$$

where A corresponds to local functions on $\mathbb{P}^n$ restricted to X and B to local functions on X and we have the exact sequence of relative Kähler differentials

$$\Omega^1_{A/C} \otimes_A B \longrightarrow \Omega^1_{B/C} \longrightarrow \Omega^1_{B/A} \longrightarrow 0.$$

If A mapping onto B is given, then $\Omega^1_{B/A} = (0)$, $B = A/\mathfrak{A}$ (globally, $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n}/\mathfrak{I}$), and we get

$$0 \longrightarrow \mathrm{Ker} \longrightarrow \Omega^1_A \otimes_A A/\mathfrak{A} \longrightarrow \Omega^1_{A/\mathfrak{A}} \longrightarrow 0.$$

Now, $\mathfrak{I} \longrightarrow \Omega^1_A \otimes_A A/\mathfrak{A}$, *via* $\xi d\xi \mapsto \otimes 1$ and in fact, $\mathfrak{I} \longrightarrow 0$. We conclude that

$$i^*(\mathfrak{I}) = \mathfrak{I}/\mathfrak{I}^2 \longrightarrow i^*(\Omega^1_{\mathbb{P}^n}) \longrightarrow \Omega^1_X \longrightarrow 0.$$

Because X is a manifold, the arrow on the left is an injection. To see this we need only look locally at x. We can take completions and then use either the C^1-implicit function theorem or the holomorphic implicit function theorem or the formal implicit function theorem and get the result (DX). If we dualize, from

$$0 \longrightarrow \mathfrak{I}/\mathfrak{I}^2 = i^*(\mathfrak{I}) \longrightarrow i^*\Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_X \longrightarrow 0$$

we get

$$0 \longrightarrow T_X \longrightarrow i^*T_{\mathbb{P}^n} = T_{\mathbb{P}^n} \upharpoonright X \longrightarrow (\mathfrak{I}/\mathfrak{I}^2)^D \longrightarrow 0$$

Therefore,

$$\mathcal{N}_{X\hookrightarrow\mathbb{P}^n} = (\mathfrak{I}/\mathfrak{I}^2)^D = i^*(\mathfrak{I})^D = (\mathfrak{I} \upharpoonright X)^D.$$

Thus,

$$c_1(\mathcal{N}_{X\hookrightarrow\mathbb{P}^n}) = -c_1(\mathfrak{I}/\mathfrak{I}^2),$$

and

$$(n+1)i^*(H) + c_1(\mathfrak{I}/\mathfrak{I}^2) = c_1(T_X).$$

We obtain a version of the *adjunction formula*:

$$c_1(\omega_X) = -(n+1)i^*(H) - c_1(\mathfrak{I}/\mathfrak{I}^2).$$

When X is a hypersurface of degree d, then $\mathfrak{I} = \mathcal{O}_{\mathbb{P}^n}(-dH)$ and

$$\mathfrak{I}/\mathfrak{I}^2 = i^*(\mathfrak{I}) = \mathcal{O}_X(-d \cdot i^*H).$$

We deduce that $-c_1(\mathfrak{I}/\mathfrak{I}^2) = d(i^*H)$ and

$$c_1(\omega_X) = (d-n-i)i^*H,$$

Say $n = 2$, and $\dim X = 1$, a curve in $\mathbb{P}^2$. When X is smooth, we have

$$c_1(\omega_X) = (d-n-1)i^*(H).$$

Facts soon to be proved:

(a) $i^*(H) = H \cdot X$, in the sense of intersection theory (that is, $\deg X$ points on X).

(b) $c_1(L)$ on a curve is equal to the degree of the divisor of L.

It follows from above that

$$\deg(\omega_X) = (d-2-1)d = d(d-3).$$

However, from Riemann-Roch on a curve, we know $\deg(\omega_X) = 2g-2$, so we conclude that for a smooth algebraic curve, its genus, g, is given by

$$g = \frac{1}{2}(d-1)(d-2).$$

In particular, observe that $g = 2$ is missed.

We know from the theory that if we know all c_1's then we can determine all c_n's for all n by the splitting principle.

There are three general methods for determining c_1;

(I) The exponential sequence.

(II) Curvature.

(III) Degree of a divisor.

Proposition 3.29 *Say X is an admissible, or a differentiable manifold, or a complex analytic manifold or an algebraic manifold. In each case, write $\mathcal{O}_X$ for the sheaf of germs of appropriate functions on X. Then, from the exponential sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e} \mathcal{O}_X^* \longrightarrow 0,$$

where $e(f) = \exp(2\pi i f)$, we get in each case the connecting map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \tag{$\dagger$}$$

and all obvious diagrams commute

*** Steve, what are these obvious diagrams? ***

and as the group $H^1(X, \mathcal{O}_X^*)$ classifies the line bundles of appropriate type, we get $\delta(L)$, a cohomology class in $H^2(X, \mathbb{Z})$ and we have

$$c_1(L) = \delta(L).$$

In the continuous and differentiable case, δ is an isomorphism. Therefore, a continuous or differentiable line bundle is completely determined by its first Chern class.

Proof. That the diagrams commute is clear. For the isomorphism statement, we have the cohomology sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

But, in the continuous or C^∞-case, $\mathcal{O}_X$ is a fine sheaf, so $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = (0)$ and we get

$$H^1(X, \mathcal{O}_X^*) \cong H^2(X, \mathbb{Z}).$$

First, we show that ($\dagger$) can be reduced to the case $X = \mathbb{P}^1_{\mathbb{C}} = S^2$.

** Steve, in this case, are we assuming that X is projective? **

Take a line bundle, L on X (continuous or C^∞), then, for $N >> 0$, there is a function, $f\colon X \to \mathbb{P}^N_{\mathbb{C}}$, so that $f^*H = L$. Now, we have the diagram

$$\begin{tikzcd} H^1(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}^*_{\mathbb{P}^N}) \arrow[r, "\delta"] \arrow[d] & H^2(\mathbb{P}^N_{\mathbb{C}}, \mathbb{Z}) \arrow[d] \\ H^1(X, \mathcal{O}_X^*) \arrow[r, "\delta"'] & H^2(X, \mathbb{Z}) \end{tikzcd}$$

which commutes by cofunctoriality of cohomology. Consequently, the existence of ($\dagger$) on the top line implies the existence of ($\dagger$) in general. Now, consider the inclusions

$$\mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^2_{\mathbb{C}} \hookrightarrow \cdots \hookrightarrow \mathbb{P}^N_{\mathbb{C}},$$

and H on $\mathbb{P}^N_{\mathbb{C}}$ pulls back at each stage to H and Chern classes have Axiom (II). Then, one sees that we are reduced to $\mathbb{P}^1_{\mathbb{C}}$.

Recall how simplicial cohomology is isomorphic (naturaly) to Čech cohomology: Take a triangulation of X and v, a vertex of a simplex, Δ. Write

$$U_v = \mathrm{st}(v) = \overset{\circ}{\bigcup} \{\sigma \mid v \in \sigma\}$$

the open star of the vertex v. The U_σ form an open cover and we have:

$$U_{v_0} \cap \cdots \cap U_{v_p} = \begin{cases} \emptyset & \text{if } (v_0, \ldots, v_p) \text{ is not a simplex;} \\ \text{a connected nonempty set} & \text{if } (v_0, \ldots, v_p) \text{ is a simplex.} \end{cases}$$

Given a Čech p-cochain, τ, then

$$\tau(U_{v_0}\cap\cdots\cap U_{v_p}) = \begin{cases} 0 & \text{if } (v_0,\ldots,v_p) \text{ is not a simplex;} \\ \text{some integer} & \text{if } (v_0,\ldots,v_p) \text{ is a simplex.} \end{cases}$$

Define

$$\tau(v_0,\ldots,v_p) = \tau(U_{v_0}\cap\cdots\cap U_{v_p}).$$

Take a simplex, $\Delta = (v_0,\ldots,v_p)$ and define linear functions $\theta(\tau)$ by

$$\theta(\tau)(v_0,\ldots,v_p) = \tau(v_0,\ldots,v_p) = \tau(U_{v_0}\cap\cdots\cap U_{v_p})$$

and extend by linearity. We get a map,

$$\check{H}^p(X,\mathbb{Z}) \cong H^p_{\mathrm{simp}}(X,\mathbb{Z})$$

via $\tau\mapsto\theta(\tau)$, which is an isomorphism.

We are down to the case of $\mathbb{P}^1_{\mathbb{C}} = S^2$ and we take H as the North pole. The Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ has coordinates $(Z_0\colon Z_1)$, say $Z_1 = 0$ is the north pole ($Z_0 = 0$ is the south pole) and let

$$z = \frac{Z_0}{Z_1}, \quad w = \frac{Z_1}{Z_0}.$$

We have the standard opens, $V_0 = \{(Z_0\colon Z_1) \mid Z_0 \neq 0\}$ and $V_1 = \{(Z_0\colon Z_1) \mid Z_1 \neq 0\}$. The local equations for H are $f_0 = w = 0$ and $f_1 = 1$. The transitions functions g^β_α are f_β/f_α, i.e.,

$$g^1_0 = \frac{f_1}{f_0} = z \quad\text{and}\quad g^0_1 = \frac{f_0}{f_1} = w.$$

Now, we triangulate S^2 using four triangles whose vertices are: $o = z$; $z = 1$; $z = i$ and $z = -1$. Note that H is represented by a point which is in the middle of a face of the simplex $(1, i, -1)$ We have U_0, U_1, U_i, U_{-1}, the four open stars and $U_0 \subseteq V_1$; $U_1 \subseteq V_0$; $U_i \subseteq V_0$; $U_{-1} \subseteq V_0$. The U-cover refined the V-cover and on it, $g^s_r \equiv 1$ iff both $r, s \neq 0$. Also, $g^t_0 = w$, for all $t \neq 0$. To lift back the exponential, $\mathcal{O}_{\mathbb{P}^1} \overset{\exp(2\pi i -)}{\longrightarrow} \mathcal{O}^*_{\mathbb{P}^1}$, we form $\frac{1}{2\pi i}\log(g^s_r)$, a one-cochain with values in $\mathcal{O}_{\mathbb{P}^1}$. Since the intersections $U_r\cap U_s$ are all simply-connected, on each, we can define a single-valued branch of the log. Consistently do this on these opens *via*: Start on $U_1\cap U_i$ and pick any single-valued branch of the log. Continue analytically to $U_i \cap U_{-1}$. Then, continue analytically to $U_{-1}\cap U_1$, we get $2\pi i + \log$ on $U_1 \cap U_i$. Having defined the $\log g^s_r$, we take the Čech δ of the 1-cochain, that is

$$c_{rst} = \frac{1}{2\pi i}[\log g^t_s - \log g^t_r + \log g^s_r] = \frac{1}{2\pi i}[\log g^s_r + \log g^t_s + \log g^r_t].$$

If none of r, s, t are 0, then $c_{rst} = 0$. So, look at $c_{0\,-1\,1}$. We have

$$c_{0\,-1\,1} = \frac{1}{2\pi i}[\log g^{-1}_0 + \log g^1_{-1} + \log g^0_1] = \frac{1}{2\pi i}[\log w - \text{“}\log\text{''} w].$$

As $w = 1/z$, the second log is $-2\pi i + \log w$, so we get

$$c_{0\,-1\,1} = +1.$$

For every even permutation σ of $(0, -1, 1)$, we have $c_{\sigma(0),\sigma(-1),\sigma(1)} = +1$ and for every odd permutation σ of $(0, -1, 1)$, we have $c_{\sigma(0),\sigma(-1),\sigma(1)} = -1$. Yet, the orientation of the simplex $(0, -1, 1)$ is positive, so we get $\delta(H) =$ the class represented by the cocycle on one simplex (positively oriented) by 1, i.e, $c_1(H)$. $\square$

Proposition 3.30 *Say X is a complex manifold and L is a C^∞ line bundle on it. Let ∇ be an arbitrary connection on X and write Θ for the curvature of Δ. Then, the 2-form $\frac{i}{2\pi}\Theta$ is real and it represents in $H^2_{\mathrm{DR}}(X,\mathbb{R})$ the image of $c_1(L)$ under the map*

$$H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{R}).$$

Proof. Pick a trivializing cover for L, say $\{U_\alpha\}$. Then, $\nabla \upharpoonright L$ on U_α comes from its connection matrix, θ_α, this is a 1×1 matrix (L is a line bundle). We know (gauge transformation)

$$\theta_\alpha = g^\alpha_\beta \theta_\beta (g^\alpha_\beta)^{-1} + dg^\alpha_\beta (g^\alpha_\beta)^{-1},$$

where the g^α_β are the transition functions. But, we have scalars here, so

$$\theta_\alpha = \theta_\beta + d\log(g^\alpha_\beta),$$

that is

$$\theta_\beta - \theta_\alpha = -d\log(g^\alpha_\beta). \tag{$\dagger$}$$

By Cartan-Maurer, the curvature, Θ, (a 2-form) is given locally by

$$\Theta = d\theta - \theta\wedge\theta = d\theta_\alpha = d\theta_\beta.$$

We get the de Rham isomorphism in the usual way by splicing exact sequences. We begin with

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty \xrightarrow{d} \mathrm{cok}_1 \longrightarrow 0 \tag{$*$}$$

and

$$0 \longrightarrow \mathrm{cok}_1 \longrightarrow \bigwedge^1 \xrightarrow{d} \mathrm{cok}_2 \longrightarrow 0 \tag{$**$}$$

It follows that

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathcal{C}^\infty \arrow[rr, "d"] \arrow[dr] & & \bigwedge^1 \arrow[rr, "d"] \arrow[dr] & & \bigwedge^2 \arrow[r] & \cdots \\
& & & \mathrm{cok}_1 \arrow[ur] \arrow[dr] & & \mathrm{cok}_2 \arrow[ur] \arrow[dr] & & \\
& & & & 0 & & 0 &
\end{tikzcd}$$

Apply cohomology to $(*)$ and $(**)$ and get

$$H^0(X,\bigwedge^1) \xrightarrow{d} H^0(X,\mathrm{cok}_2) \xrightarrow{\delta'} H^1(X,\mathrm{cok}_1) \longrightarrow H^1(X,\bigwedge^1) = (0)$$

and

$$H^1(X,\mathcal{C}^\infty) \longrightarrow H^1(X,\mathrm{cok}_1) \xrightarrow{\delta''} H^2(X,\mathbb{R}) \longrightarrow H^2(X,C^\infty) = (0)$$

because $\bigwedge^1$ and $\mathcal{C}^\infty$ are fine. We get

$$H^1(X,\mathrm{cok}_1) \cong H^2(X,\mathbb{R}) \quad \text{and} \quad H^0(X,\mathrm{cok}_2)/dH^0(X,\bigwedge^1) \cong H^1(X,\mathrm{cok}_1).$$

Therefore,

$$\delta' \circ \delta' \colon H^0(X, \mathrm{cok}_2) \longrightarrow H^2(X, \mathbb{R}) \longrightarrow 0.$$

We know from the previous proof that

$$c_{\alpha,\beta,\gamma} = \frac{1}{2\pi i}[\log g_\alpha^\beta + \log g_\beta^\gamma + \log g_\gamma^\alpha]$$

represents $c_1(L)$ *via* the δ from the exponential sequence. So,

$$c_{\alpha,\beta,\gamma} = \frac{1}{2\pi i}[\log g_\beta^\alpha + \log g_\alpha^\gamma + \log g_\gamma^\beta]$$

and

$$\delta'[\Theta] = \text{cohomology class of } \Theta = \text{class of cocycle } (\theta_\beta - \theta_\alpha).$$

Now, $\frac{1}{2\pi i}(\theta_\beta - \theta_\alpha)$ can be lifted back to $-\frac{1}{2\pi i}\log g_\beta^\alpha$ under δ'' and we deduce that

$$\delta''\delta'\left(\frac{1}{2\pi i}\Theta\right) = \text{class of } \; -\frac{1}{2\pi i}[\log g_\beta^\alpha + \log g_\alpha^\gamma + \log g_\gamma^\beta] = -\text{class of } c_{\alpha\beta\gamma} = -c_1(L).$$

** There may be a problem with the sign! **

The next way of looking at $c_1(L)$ works when L comes from a divisor. Say X is a complex algebraic manifold and $L = \mathcal{O}_X(D)$, where D is a divisor,

$$D = \sum_j a_j W_j$$

on X. Then, D gives a cycle in homology, so $[D] \in H_{2n-2}(X, \mathbb{Z})$ (here $n = \dim_{\mathbb{C}} X$). By Poincaré duality, our $[D]$ is in $H^2(X, \mathbb{Z})$ and it is $\sum a_j[W_j]$.

Theorem 3.31 *If X is a compact, complex algebraic manifold and D is a divisor on X, then*

$$c_1(\mathcal{O}_X(D)) = [D] \quad \textit{in } H^2(X, \mathbb{Z}),$$

that is, $c_1(\mathcal{O}_X(D))$ is carried by the $(2n-2)$-cycle, D.

Proof. Recall that Poincaré duality is given by: For $\xi \in H^r(X, \mathbb{R})$ and $\eta \in H^s(X, \mathbb{R})$ (where $r + s = 2n$), then

$$(\xi, \eta) = \int_X \xi \wedge \eta.$$

The homology/cohomology duality is given by: For $\omega \in H^s(X, \mathbb{R})$ and $Z \in H_s(X, \mathbb{R})$, then

$$(Z, \omega) = \int_Z \omega.$$

We know that the cocyle (= 2-form) representing $c_1(L)$ is $\left[\frac{i}{2\pi}\Theta\right]$, for any connection on X. We must show that for every closed, real $(2n-2)$-form, ω,

$$\frac{i}{2\pi}\int_X \Theta \wedge \omega = \int_D \omega.$$

We compute Θ for a convenient connection, namely, the uniholo connection. Pick a local holomorphic frame, $e(z)$, for L, then if L has a section, s, we know $s(z) = \lambda(z)e(z)$, locally. For θ, the connection matrix in this frame, we have

(a) $\theta = \theta^{1,0}$ (holomorphic)

(b) $d(|s|^2) = (\nabla s, s) + (s, \nabla s)$ (unitary)

We have

$$\nabla s = \nabla \lambda e = (d\lambda + \theta\lambda)e.$$

Thus, the right hand side of (b) is

$$\begin{aligned} d(|s|^2) &= ((d\lambda + \theta\lambda)e, \lambda e) + (\lambda e, (d\lambda + \theta\lambda)e) \\ &= \overline{\lambda} d\lambda(e,e) + \theta|\lambda|^2(e,e) + \lambda d\overline{\lambda}(e,e) + \overline{\theta}|\lambda|^2(e,e). \end{aligned}$$

Write $h(z) = |e(z)|^2 = (e,e) > 0$; So, the right hand side of (b) is $\overline{\lambda} h d\lambda + \lambda h d\overline{\lambda} + (\theta + \overline{\theta})|\lambda|^2 h$. Now, $|s|^2 = \lambda\overline{\lambda} h$, so

$$d(|s|^2) = \lambda\overline{\lambda} dh + h(\lambda d\overline{\lambda} + \overline{\lambda} d\lambda).$$

From (b), we deduce $dh = (\theta + \overline{\theta})h$, and so,

$$\theta + \overline{\theta} = \frac{dh}{h} = d(\log h) = \partial(\log h) + \overline{\partial}(\log h).$$

Using (a) and the decomposition by type, we get

$$\theta = \partial(\log h) = \partial \log(|e|^2).$$

As $\Theta = d\theta - \theta \wedge \theta$, we get

$$\Theta = d\theta = (\partial + \overline{\partial})(\partial \log(|e|^2)),$$

i.e.,

$$\Theta = \overline{\partial}\partial \log(|e|^2).$$

Now, recall

$$d^c = \frac{i}{4\pi}(\overline{\partial} - \partial),$$

so that

$$dd^c = -d^c d = \frac{i}{2\pi}\partial\overline{\partial} = -\frac{i}{2\pi}\overline{\partial}\partial,$$

and $2\pi i dd^c = \overline{\partial}\,\partial$. Consequently,

$$\Theta = \pi i dd^c \log(|e|^2).$$

This holds for any local frame, e, and has nothing to do with the fact that L comes from a divisor.

Now, $L = \mathcal{O}_X(D)$ and assume that the local equations for D are $f_\alpha = 0$ (on U_α, some open in the trivializing cover for L on X). We know the transition functions are

$$g_\alpha^\beta = \frac{f_\beta}{f_\alpha};$$

Therefore, the local vectors $s_\alpha = f_\alpha e_\alpha$ form a global section, s, of $\mathcal{O}_X(D)$. The zero locus of this section is exactly D. As the bundle L is unitary, $g_\alpha^\beta \in \mathrm{U}(1)$, which implies $|f_\beta| = |f_\alpha|$ and so, $|f_\alpha e_\alpha|$ is well defined. Thus for small $\epsilon > 0$,

$$D(\epsilon) = \{z \in X \mid |s(z)|^2 < \epsilon\}$$

is a tubular neighborhood of D.

Look at $X - D(\epsilon)$, then $\mathcal{O}_X(D) \upharpoonright X - D(\epsilon)$ is trivial as the section s is never zero there. Therefore, s_α will also do as a local frame for $\mathcal{O}_X(D)$ on $X - D(\epsilon)$.

We need to compute $\int_X \Theta \wedge \omega$. By linearity, we may assume D is one of the W's. Then, by definition,

$$\int_X \Theta \wedge \omega = \lim_{\epsilon \downarrow 0} \int_{X - D(\epsilon)} 2\pi i d d^c \log |s|^2 \wedge \omega$$

If we apply Stokes, we find

$$\int_X \Theta \wedge \omega = -\lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} 2\pi i d^c \log |s|^2 \wedge \omega$$

that is,

$$\int_X \Theta \wedge \omega = \frac{2\pi}{i} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} d^c \log |s|^2 \wedge \omega. \tag{$\dagger$}$$

Now $\mathrm{Vol}(D(\epsilon)) \longrightarrow 0$ as $\epsilon \downarrow 0$, as we can see by using the Zariski stratification to reduce to the case where D is non-singular. Also,

$$|s|^2 = |f_\alpha|^2 |e_\alpha|^2 = f_\alpha \overline{f}_\alpha h,$$

where $h = |e_\alpha|^2$ is positive bounded. We have

$$\log |s|^2 = \log f_\alpha + \log \overline{f}_\alpha + \log h$$

and as $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$,

$$d^c \log |s|^2 = \frac{i}{4\pi}[-\partial \log f_\alpha + \overline{\partial} \log \overline{f}_\alpha + (\overline{\partial} - \partial) \log h].$$

It follows that

$$\frac{2\pi}{i} d^c \log |s|^2 \wedge \omega = \frac{1}{2}[-\partial \log f_\alpha \wedge \omega + \overline{\partial} \log \overline{f}_\alpha \wedge \omega + (\overline{\partial} - \partial) \log h \wedge \omega].$$

In the right hand side of ($\dagger$), the third term is

$$\frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} (\overline{\partial} - \partial) \log h \wedge \omega.$$

Now, $(\overline{\partial} - \partial) \log h$ is bounded (X is compact) and $\mathrm{Vol}(\partial D(\epsilon)) \longrightarrow 0$ as $\epsilon \downarrow 0$. So, this third term vanishes in the limit. But, $\overline{\partial} \log \overline{f}_\alpha = \overline{\partial \log f_\alpha}$ and $\omega = \overline{\omega}$, as ω is real. Consequently,

$$\overline{\partial} \log \overline{f}_\alpha \wedge \omega = \overline{\partial \log f_\alpha \wedge \omega}.$$

From ($\dagger$), we get

$$\begin{aligned}
\int_X \Theta \wedge \omega &= \frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} -\partial \log f_\alpha \wedge \omega + \overline{\partial \log f_\alpha \wedge \omega} \\
&= -\frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega - \overline{\partial \log f_\alpha \wedge \omega} \\
&= -i \lim_{\epsilon \downarrow 0} \Im \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega.
\end{aligned}$$

Now, $f_\alpha = 0$ is the local equation of D and we can compute the integral on the right hand side away from the singularities of D as the latter have measure 0. The divisor D is compact, so we can cover it by polydics centered at nonsingular points of D, say ζ_0 is a such a point. By the local complete intersection then, there exist local coordinates for X near ζ_0, of the form

$$z_1 = f_\alpha, \quad \underbrace{z_2, \ldots, z_n}_{\text{rest}},$$

on $\Delta \cap U_\alpha$ (where Δ is a polydisc). Break up ω as

$$\omega = g(z_1, \ldots, z_n) \underbrace{dz_2 \wedge \cdots \wedge d\overline{z}_2 \wedge \cdots}_{\text{rest}} + \kappa,$$

where κ is a form involving dz_1 and $d\overline{z}_1$ in each summand. Also, as

$$\partial \log f_\alpha = (\partial + \overline{\partial}) \log f_\alpha = d \log f_\alpha = \frac{df_\alpha}{f_\alpha} = \frac{dz_1}{z_1},$$

we get

$$\partial \log f_\alpha \wedge \omega = \frac{dz_1}{z_1} g(z_1, \ldots, z_n) \underbrace{dz_2 \wedge \cdots \wedge d\overline{z}_2 \wedge \cdots}_{\text{rest}} + \text{ terms } \frac{dz_1 \wedge d\overline{z}_1}{z_1} \text{stuff}.$$

Furthermore, $dz_1 \wedge d\overline{z}_1 = 2i dx \wedge dy = 2i r dr_1 d\theta$ (in polar coordinates), so

$$\left| \frac{dz_1 \wedge d\overline{z}_1}{z_1} \right| = 2|dr_1||d\theta_1|,$$

and when $\epsilon \downarrow 0$, this term goes to 0. Therefore

$$\lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon) \cap \Delta} \frac{dz_1}{z_1} g(z_1, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}}) = \lim_{\epsilon \downarrow 0} \int_{(|z_1| = C\epsilon) \prod \text{rest of polydisc}} \frac{dz_1}{z_1} g(z_1, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}})$$

and by Cauchy's integral formula, this is

$$\lim_{\epsilon \downarrow 0} \int_{\text{rest of poly} \cap \partial D(\epsilon)} 2\pi i g(0, z_2, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}}) = 2\pi i \int_{D \cap \Delta} \omega.$$

Adding up the contributions from the finite cover of polydics, we get

$$\Im \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega = \Im\, 2\pi i \int_D \omega = 2\pi \int_D \omega,$$

as ω is real. But then,

$$-i \Im \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \log f_\alpha \wedge \omega = -2\pi i \int_X \omega$$

from which we finally deduce $\int_X \Theta \wedge \omega = -2\pi i \int_D \omega$, that is,

$$\int_X \frac{i}{2\pi} \Theta \wedge \omega = \int_D \omega,$$

as required. $\square$

Corollary 3.32 *Suppose V is a $\mathrm{U}(q)$-bundle on our compact X (so that differentiably, V is generated by its sections). Or, if V is a holomorphic bundle, assume it is generated by its holomorphic sections. Take a generic section, s, of V and say V has rank r. Then, the set $s = 0$ has complex codimension r (in homology) and is the carrier of $c_r(V)$.*

Proof. The case $r = 1$ is exactly the theorem above. Differentiably,

$$V = L_1 \coprod L_2 \coprod \cdots \coprod L_r,$$

for the diagonal line bundles of V. Holomorphically, this is also OK but over the space $[\Delta]V$. So, the transition matrix is a diagonal matrix

$$\mathrm{diag}(g_{1\,\alpha}^{\beta}, \ldots, g_{r\,\alpha}^{\beta}) \quad \text{on } U_\alpha \cap U_\beta$$

and $s_\alpha = (s_{1\,\alpha}, \ldots, s_{r\,\alpha})$. So,

$$\mathrm{diag}(g^\beta_\alpha)s_\alpha = (g^\beta_{1\,\alpha}s_{1\,\alpha}, \ldots, g^\beta_{r\,\alpha}s_{r\,\alpha}) = s_\beta$$

which shows that each $s_{j\,\alpha}$ *is* a section of L_j. Note that $s = 0$ iff all $s_j = 0$. But, the locus $s_j = 0$ carries $c_1(L_j)$, by the previous theorem. Therefore, $s = 0$ corresponds to the intersection in homology of the carriers of $c_1(L), \ldots, c_1(L_r)$. But, intersection in homology is equivalent to product in cohomology, so the cohomology class for $s = 0$ is

$$c_1(L_1)c_1(L_2)\cdots c_1(L_r) = c_r(V)$$

as desired. $\square$

General Principle for Computing $c_q(V)$, for a rank r vector bundle, V.

(1) Let L be an ample line bundle, then $V \otimes L^{\otimes m}$ is generated by its sections for $m >> 0$.

(2) Pick r generic sections, $s_1, \ldots, s_r$, of $V \otimes L^{\otimes m}$. Form $s_1 \wedge \cdots \wedge s_{r-q+1}$, a section of $\bigwedge^{r-q+1}(V \otimes L^{\otimes m})$. Then, the zero locus of $s_1 \wedge \cdots \wedge s_{r-q+1}$ carries the Chern class, $c_q(V \otimes L^{\otimes m})$, of $V \otimes L^{\otimes m}$.

$\Big[$ When $q = r$, this is the corollary. When $q = 1$, we have $s_1 \wedge \cdots \wedge s_r$, a section of $\bigwedge^r V \otimes L^{\otimes m}$, and it is generic (as the fibre dimension is 1). We get $c_1(\bigwedge^r V \otimes L^{\otimes m})$ and we know that it is equal to $c_1(V \otimes L^{\otimes m})$.$\Big]$

(3) Use the relation from the Chern polynomial

$$c(V \otimes L^{\otimes m})(t) = \prod(1 + (\gamma_j + mc_1(L))t)$$

to get the elementary symmetric functions of the γ_j's, i.e., $c_q(V)$.

Remark: if $1 < q < r$, our section $s_1 \wedge \cdots \wedge s_{r-q+1}$ is *not* generic but it works.

Theorem 3.33 *Say X is a complex analytic or algebraic, compact, smooth, manifold and $j\colon W \hookrightarrow X$ is a smooth, complex, codimension q submanifold. Write $\mathcal{N}$ for the normal bundle of W in X; this is rank q ($\mathrm{U}(q)$) vector bundle on W. The subspace W corresponds to a cohomology class, ξ, in $H^{2q}(X, \mathbb{Z})$ (in fact, in $H^{q,q}(X, \mathbb{C})$) and so we get $j^*\xi \in H^{2q}(W, \mathbb{Z})$. Then, we have*

$$c_q(\mathcal{N}) = j^*W.$$

Proof. We begin with the case $q = 1$. In this case, W is a divisor and we know $\mathcal{N} = \mathcal{O}_X(W) \restriction W$. By Corollary 3.32, the Chern class $c_1(\mathcal{N})$ is carried by the zero locus of a section, s,of $\mathcal{N}$. Now, $W \cdot W$ in X as a cycle is just a moving of W by a small amount and then an ordinary intersection of W and the new moved cycle. We see that $W \cdot W = c_1(\mathcal{N})$ as cycle on W. But, j^*W is just $W \cdot W$ as cycle (by Poincaré duality). So, the result holds when $q = 1$. If $q > 1$ and if W is a complete intersection in X, then since $c_q(\mathcal{N})$ is computed by repeated pullbacks and each pullback gives the correct formula (by the case $q = 1$), we get the result. In the general case, we have two classes j^*W and $c_q(\mathcal{N})$. If there exists an open cover, $\{U_\alpha\}$, of W so that

$$j^*W \restriction U_\alpha = c_q(\mathcal{N}) \restriction U_\alpha \quad \text{for all } \alpha,$$

then we are done. But, W is smooth so it is a local complete intersection (LCIT). Therefore, we get the result by the previous case. $\square$

Corollary 3.34 *If X is a compact, complex analytic manifold and if T_X = holomorphic tangent bundle has rank $q = \dim_{\mathbb{C}} X$, then*

$$c_q(T_X) = \chi_{\mathrm{top}} = \sum_{i=0}^{2q} (-1)^i b_i$$

(Here, $b_i = \dim_{\mathbb{R}} H^i(X, \mathbb{Z})$.)

Proof. (Essentially due to Lefschetz). Look at $X \prod X$ and the diagonal embedding, $\Delta\colon X \to X \prod X$. So, $X \hookrightarrow X \prod X$ is a smooth codimension q submanifold. An easy argument shows that

$$T_X \cong \mathcal{N}_{X \hookrightarrow X \prod X} = \mathcal{N}$$

and the previous theorem implies

$$c_q(T_X) = c_q(\mathcal{N}) = X \cdot X$$

in $X \prod X$. Now, look at the map $f\colon X \to X$ given by

$$pr_2 \circ \epsilon\sigma,$$

where ϵ is small and σ is a section of $\mathcal{N}$. The fixed points of our map give the cocycle $X \cdot X$. The Lefschetz fixed point Theorem says the cycle of fixed points is given by

$$\sum_{i=0}^{2q} (-1)^i \mathrm{tr}\, f^* \ \text{ on } \ H^i(X, \mathbb{Z}).$$

But, for ϵ small, the map f is homotopic to id, so $f^* = \mathrm{id}^*$. Now, tr id* = dimension of space $= b_i(X)$ if we are on $H^i(X)$. So the right hand side of the Lefschetz formula is χ_{top}, as claimed. $\square$

Segre Classes.

Let V be a vector bundle on X, then we have classes $s_j(V)$, and they are defined by

$$1 + \sum_{j=1}^{\infty} s_j(V) t^j = \frac{1}{c(V)(t)}.$$

As $c(V)(t)$ is nilpotent, we have

$$\frac{1}{c(V)(t)} = 1 - (c_1(V)t + c_2(V)t^2 + \cdots) + (c_1(V)t + c_2(V)t^2 + \cdots)^2 + \cdots$$

and so,

$$\begin{aligned} s_1(V) &= -c_1(V) \\ s_2(V) &= c_1^2(V) - c_2(V), \end{aligned}$$

etc.

Pontrjagin Classes.

Pontrjagin classes are defined for real O(q)-bundles over real manifolds. We have the commutative diagrams

$$\begin{tikzcd} \mathrm{U}(q) \arrow[r, hook, "\zeta"] \arrow[d, hook] & \mathrm{O}(2q) \arrow[d, hook] \\ \mathrm{GL}(q, \mathbb{C}) \arrow[r, hook] & \mathrm{GL}(2q, \mathbb{R}) \end{tikzcd}$$

where $\zeta(z_1, \ldots, z_q) = (x_1, y_1, \ldots, x_q, y_q)$, with $z_j = x_j + iy_j$ and

$$\begin{tikzcd} \mathrm{O}(q) \arrow[r, hook, "\psi"] \arrow[d, hook] & \mathrm{U}(q) \arrow[d, hook] \\ \mathrm{GL}(q, \mathbb{R}) \arrow[r, hook] & \mathrm{GL}(q, \mathbb{C}) \end{tikzcd}$$

where $\psi(A)$ is the real matrix now viewed as a complex matrix. Given ξ, an $\mathrm{O}(q)$-bundle, we have $\psi(q)$, a $\mathrm{U}(q)$-bundle. Define

The *Pontrjagin classes,* $p_i(\xi)$, are defined by

$$p_i(\xi) = (-1)^i c_{2i}(\psi(\xi)) \in H^{4i}(X, \mathbb{Z}).$$

The *generalized Pontrjagin classes,* $P_i(\xi)$ and the *generalized Pontrjagin polynomial,* $P(\xi)(t)$, are defined by

$$P(\xi)(t) = c(\psi(\xi))(t), \quad \text{and} \quad P_j(\xi) = c_j(\psi(xi)).$$

(Observe: $P_{2l}(\xi) = (-1)^p p_l(\xi)$.)

Now, ξ corresponds to map, $X \longrightarrow \mathrm{BO}(q)$. Then, for i even, $P_{i/2}(\xi)$ is the pullback of something in $H^i(\mathrm{BO}(q), \mathbb{Z})$. It is known that for $i \equiv 2(4)$, the cohomology ring $H^i(\mathrm{BO}(q), \mathbb{Z})$ is 2-torsion, so $2P_{\mathrm{odd}}(\xi) = 0$. So, with rational coefficients, we get

$$P_{\mathrm{odd}}(\xi) = 0 \quad \text{and} \quad P_{\mathrm{even}}(\xi) = \pm P_{\mathrm{even}/2}(\xi).$$

We have the following properties:

(0) $P(\xi)(t) = 1 + \text{stuff}$.

(1) $f^*P(\xi)(t) = P(f^*\xi)(t)$, so $f^*P_i(\xi) = P_i(f^*\xi)$.

(2) Suppose ξ, η are bundle of rank q', q'', respectively, then

$$P(\xi \amalg \eta)(t) = P(\xi)(t)P(\eta)(t)$$

and if we set $p(\xi)(t) = \sum_{j=0}^{\infty} p_j(\xi)t^{2j}$, then

$$p(\xi \amalg \eta)(t) = p(\xi)(t)p(\eta)(t), \quad \text{mod} \;\; \text{elements of order 2 in } H^{\bullet}(X, \mathbb{Z}).$$

(3) Suppose $c(\psi(\xi))(t)$ has Chern roots γ_i. Then, the polynomial $\sum_{j=0}^{\infty}(-1)^j p_j(\xi)t^{2j}$ has roots γ_i^2; in fact,

$$\sum_{j=0}^{\infty}(-1)^j p_j(\xi)t^{2j} = \Big(\sum_l c_j(\xi)t^l\Big)\Big(\sum_m (-1)^m c_m(\xi)t^m\Big).$$

Proposition 3.35 *Say ξ is a $\mathrm{U}(q)$-bundle and make $\zeta(\xi)$, an $\mathrm{O}(2q)$-bundle. Then*

$$\sum_{j=0}^{\infty}(-1)^j p_j(\zeta(\xi))t^{2j} = \Big(\sum_l c_j(\xi)t^l\Big)\Big(\sum_m c_m(\xi^D)t^m\Big).$$

Proof. Consider the maps $\mathrm{U}(q) \hookrightarrow \mathrm{O}(2q) \hookrightarrow \mathrm{U}(2q)$. By linear algebra, if $A \in \mathrm{U}(q)$, its image in $\mathrm{U}(2q)$ by this map is

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$$

after an automorphism of $\mathrm{U}(2q)$, which automorphism is independent of A. By Skolem-Noether, the automorphism is of the form

$$H^{-1}(\psi\zeta(A))H,$$

for some $H \in \mathrm{GL}(2q, \mathbb{C})$. For an inner automorphism, the cohomology class of the vector bundle stays the same. Thus, this cohomology class is the class of

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}.$$

Now, we know the transition matrix of ξ^D is the transpose inverse of that for ξ. But, A is unitary, so

$$\overline{A} = (A^{-1})^\top = A^D$$

and we deduce that $\psi\zeta(A)$ has as transition matrix

$$\begin{pmatrix} A & 0 \\ 0 & A^D \end{pmatrix}.$$

Consequently, the right hand side of our equation is

$$\Big(\sum_l c_j(\xi)t^l\Big)\Big(\sum_m c_m(\xi^D)t^m\Big),$$

as required. $\square$

3.3 The L-Genus and the Todd Genus

The material in this section and the next two was first published in Hirzebruch [8].

Let B be a commutative ring with 1, and let $Z, \alpha_1, \ldots, \alpha_n, \ldots$ be some independent indeterminates, all of degree 1; make new independent indeterminates

$$q_j = \sigma_j(\alpha\text{'s}).$$

(The σ_j are the symmetric functions in the α's; for example, $q_1 = \alpha_1 + \cdots + \alpha_n$.) All computations are carried out in the ring $\mathcal{B} = B[[Z; \alpha_1, \ldots, \alpha_n, \ldots]]$. We have the subring $\mathcal{P} = B[[Z; q_1, \ldots, q_n, \ldots,]]$ and in $\mathcal{P}$, we have certain units (so-called *one-units*), namely

$$1 + \sum_{j \geq 1} b_j Z^j, \quad \text{where } b_j \in B.$$

If $Q(z)$ is a one-unit, $1 + \sum_{j\geq 1} b_j Z^j$, write

$$Q(z) = \prod_{j=1}^{\infty} (1 + \beta_j Z)$$

and call the β_j's the "roots" of Q. In the product $\prod_{l=1}^{\infty} Q(\alpha_j Z)$, the coefficient of Z^k is independent of the order of the α's and is a formal series in the elementary symmetric functions, q_j, of the α's. In fact, this coefficient has weight k and begins with $b_k q_1^k + \cdots$, call the coefficients $K_k^Q(q_1, q_2, \ldots, q_k)$. We deduce that

$$1 + \sum_{l=1}^{\infty} K_l^Q(q_1, q_2, \ldots, q_l) z^l = \prod_{l=1}^{\infty} Q(\alpha_j Z).$$

We see that a 1-unit, $Q(Z) = 1 + \sum_{j\geq 1} b_j Z^j$, yields a sequence of polynomials (in the elementary symmetric functions $q_1, \ldots, q_k$) of weights, $1, 2, \ldots$, say $\{K_l^Q\}_{l=1}^{\infty}$, called the *multiplicative sequence* of the 1-unit.

Conversely, given some sequence of polynomials, $\{K_l\}_{l=1}^{\infty}$, it defines an operator on 1-units to 1-units, call it K. Namely,

$$K(1 + \sum_{j\geq 1} q_j Z^j) = 1 + \sum_{l=1}^{\infty} K_l(q\text{'s}) Z^l.$$

So, Q gives the operator K^Q; namely,

$$K(1 + \sum_{j\geq 1} q_j Z^j) = 1 + \sum_{l=1}^{\infty} K_l^Q(q\text{'s}) Z^l.$$

Claim. When Q is given, the operator K^Q is multiplicative:

$$K^Q(1 + \sum_{j\geq 1} q_j' Z^j) K^Q(1 + \sum_{j\geq 1} q_j'' Z^j) = K^Q((1 + \sum_{j\geq 1} q_j' Z^j)(1 + \sum_{j\geq 1} q_j'' Z^j)).$$

Now, to see this, the left hand side is

$$[1 + \sum_{l=1}^{\infty} K_l^Q(q'\text{s}) Z^l][1 + \sum_{m=1}^{\infty} K_m^Q(q''\text{s}) Z^m] = \prod_{r=1}^{\infty} Q(\alpha_r' Z) \prod_{s=1}^{\infty} Q(\alpha_s'' Z) = \prod_{t=1}^{\infty} Q(\alpha_t Z),$$

where we have chosen some enumeration of the α's and the α''s, say $\alpha_1, \ldots, \alpha_t, \ldots = \alpha_1', \alpha_1'', \alpha_2', \alpha_2'', \ldots$. But,

$$\prod_{t=1}^{\infty} Q(\alpha_t Z) = 1 + \sum_{n=1}^{\infty} K_n^Q(\text{elem. symm. functions in } \alpha'\text{s and } \alpha''\text{s})Z^n,$$

which is the right hand side of the assertion.

If conversely, we have some endomorphism of the 1-units under multiplication, say K, look at $K(1+Z) = 1 + \sum_{j\geq 1} a_j Z^j = Q(Z)$, some power series. Compute K^Q. We have

$$K^Q(1 + \sum_{j\geq 1} q_j Z^j) = \prod_{l=1}^{\infty} Q(\alpha_l Z),$$

where $1 + \sum_{j\geq 1} q_j Z^j = \prod_{j=1}^{\infty}(1 + \alpha_j Z)$. So, as K is multiplicative,

$$K(1 + \sum_{j\geq 1} q_j Z^j) = K(\prod_{j=1}^{\infty}(1 + \alpha_j Z)) = \prod_{j=1}^{\infty} K(1 + \alpha_j Z).$$

By definition of Q, the right hand side of the latter is

$$\prod_{l=1}^{\infty} Q(\alpha_l Z) = K^Q(1 + \sum_{j\geq 1} q_j Z^j)$$

and this proves:

Proposition 3.36 *The endomorphisms (under multiplication) of the* 1*-units are in one-to-one correspondence with the* 1*-units. The correspondence is*

$$\text{endo } K \rightsquigarrow 1\text{-}\textit{unit } K(1+Z),$$

and

$$1\text{-}\textit{unit } Q \rightsquigarrow \text{endo } K^Q.$$

We can repeat the above with new variables: X (for Z); c_j (for q_j); γ_j (for α_j); and connect with the above by the relations

$$Z = X^2; \alpha_l = \gamma_l^2.$$

This means

$$\sum_{i=0}^{\infty}(-1)^i q_i Z^i = \Big(\sum_{j=0}^{\infty} c_j X^j\Big)\Big(\sum_{r=0}^{\infty} c_r(-X)^r\Big) \tag{$*$}$$

and if we set $\widetilde{Q}(X) = Q(X^2) = Q(Z)$, then

$$K_l^Q(q_1, \ldots, q_l) = K_{2l}^{\widetilde{Q}}(c_1, \ldots, c_{2l}) \quad \text{and} \quad K_{2l+1}^{\widetilde{Q}}(c_1, \ldots, c_{2l+1}) = 0.$$

For example, $(*)$ implies that $q_1 = c_1^2 - 2c_2$, etc.

Proposition 3.37 *If $B \supseteq \mathbb{Q}$, then there is one and only one power series, $L(Z)$, so that for all $k \geq 0$, the coefficient of Z^k in $L(Z)^{2k+1}$ is* 1*. In fact,*

$$L(Z) = \frac{\sqrt{Z}}{\tanh\sqrt{Z}} = 1 + \sum_{l=1}^{\infty}(-1)^{l-1}\frac{2^{2l}}{(2l)!}B_l Z^l.$$

Proof. For $k = 0$, we see that $L(Z)$ must be a 1-unit, $L(Z) = 1 + \sum_{j=1}^{\infty} b_j Z^j$. Consider $k = 1$; then, $L(Z)^3 = (1 + b_1 Z + O(Z^2))^3$, so

$$(1 + b_1 Z)^3 + O(Z^2) = 1 + 3b_1 Z + O(Z^2),$$

which implies $b_1 = 1/3$. Now, try for b_2: We must have

$$\begin{aligned}
\left(1 + \frac{1}{3}Z + b_2 Z + O(Z^3)\right)^5 &= \left(1 + \frac{1}{3}Z + b_2 Z\right)^5 + O(Z^3) \\
&= \left(1 + \frac{1}{3}Z\right)^5 + 5\left(1 + \frac{1}{3}Z\right)^4 b_2 Z + O(Z^3) \\
&= \text{junk} + \left(\frac{10}{9} + 5b_2\right) Z^2 + O(Z^3).
\end{aligned}$$

Thus,

$$5b_2 = 1 - \frac{10}{9} = -\frac{1}{9},$$

i.e., $b_2 = -1/45$. It is clear that we can continue by induction and obtain the existence and uniqueness of the power series.

Now, let

$$M(Z) = \frac{\sqrt{Z}}{\tanh \sqrt{Z}}.$$

Then, $M(Z)^{2k+1}$ is a power series and the coefficient of Z^k is (by Cauchy)

$$\frac{1}{2\pi i} \int_{|Z|=\epsilon} \frac{M(Z)^{2k+1}}{Z^{k+1}} dZ.$$

Let $t = \tanh \sqrt{Z}$. Then,

$$dt = \mathrm{sech}^2 \sqrt{Z} \left(\frac{1}{2\sqrt{Z}}\right) dZ,$$

so

$$\frac{M(Z)^{2k+1}}{z^{k+1}} dZ = \frac{\sqrt{Z} 2\sqrt{Z} dt}{t^{2k+1} Z \mathrm{sech}^2 \sqrt{Z}} = \frac{2dt}{t^{2k+1} \mathrm{sech}^2 \sqrt{Z}}.$$

However, $\mathrm{sech}^2 Z = 1 - \tanh^2 Z = 1 - t^2$, so

$$\frac{M(Z)^{2k+1}}{z^{k+1}} dZ = \frac{2dt}{t^{2k+1}(1 - t^2)}.$$

When t goes once around the circle $|t| = \text{small}(\epsilon)$, Z goes around twice around, so

$$\frac{1}{2\pi i} \int_{|t|=\text{small}(\epsilon)} \frac{2dt}{t^{2k+1}(1 - t^2)} = \text{twice what we want}$$

and our answer is

$$\frac{1}{2\pi i} \int_{|t|=\text{small}(\epsilon)} \frac{dt}{t^{2k+1}(1 - t^2)} = \frac{1}{2\pi i} \int_{|t|=\text{small}(\epsilon)} \frac{t^{2k} dt}{t^{2k+1}(1 - t^2)} + \text{other zero terms} = 1,$$

as required. $\square$

Recall that

$$L(Z) = 1 + \frac{1}{3}Z - \frac{1}{45}Z^2 + O(Z^3).$$

Let us find $L_1(q_1)$ and $L_2(q_1, q_2)$. We have

$$\begin{aligned}
1 + L_1(q_1)Z + L_2(q_1, q_2)Z^2 + \cdots &= L(\alpha_1 Z)L(\alpha_2 Z) \\
&= \left(1 + \frac{1}{3}\alpha_1 Z - \frac{1}{45}\alpha_1^2 Z^2 + \cdots\right)\left(1 + \frac{1}{3}\alpha_2 Z - \frac{1}{45}\alpha_2^2 Z^2 + \cdots\right) \\
&= 1 + \frac{1}{3}(\alpha_1 + \alpha_2)Z + -\left(\frac{1}{45}(\alpha_1^2 + \alpha_2^2) + \frac{1}{9}\alpha_1\alpha_2\right)Z^2 + O(Z^3).
\end{aligned}$$

We deduce that

$$L_1(q_1) = \frac{1}{3}q_1$$

and since $\alpha_1^2 + \alpha_2^2 = (\alpha_1 + \alpha_2)^2 - 2\alpha_1\alpha_2 = q_1^2 - 2q_2$, we get

$$L_2(q_1, q_2) = -\frac{1}{45}(7q_2 - q_1^2) = -\frac{1}{3^2 \cdot 5}(7q_2 - q_1^2).$$

Here are some more L-polynomials:

$$\begin{aligned}
L_3 &= \frac{1}{3^3 \cdot 5 \cdot 7}(62q_3 - 13q_1q_2 + 2q_1^3) \\
L_4 &= \frac{1}{3^4 \cdot 5^2 \cdot 7}(381q_4 - 71q_3q_1 - 19q_2^2 + 22q_2q_1^2 - 3q_1^4) \\
L_5 &= \frac{1}{3^5 \cdot 5^2 \cdot 7 \cdot 11}(5110q_5 - 919q_4q_1 - 336q_3q_2 + 237q_3q_1^2 + 127q_2^2q_1 - 83q_2q_1^3 + 10q_1^5).
\end{aligned}$$

Geometric application: Let X be an oriented manifold and let T_X be its tangent bundle. Take a multiplicative sequence, $\{K_l\}$, in the Pontrjagin classes of T_X: $p_1, p_2, \ldots$.

Definition 3.3 The *K-genus* (or *K-Pontrjagin genus*) of X is

$$\begin{cases} 0 & \text{if } \dim_{\mathbb{R}} X \not\equiv 0 \pmod 4, \\ K_n(p_1, \ldots, p_n)[X] & \text{if } \dim_{\mathbb{R}} X = 4n. \end{cases}$$

(a $4n$ rational cohomology class applied to a $4n$ integral homology class gives a rational number). When $K_l = L_l$ (our unique power series, $L(Z)$), we get the *L-genus* of X, denoted $L[X]$.

Look at $\mathbb{P}^{2n}_{\mathbb{C}}$, of course, we mean its tangent bundle, to compute characteristic classes. Write temporarily

$$\Theta = T_{\mathbb{P}^{2n}_{\mathbb{C}}}$$

a $\mathrm{U}(2n)$-bundle. We make $\zeta(\Theta)$ (remember, $\zeta \colon \mathrm{U}(2n) \to \mathrm{O}(4n)$), then we know

$$\sum_i p_i(\zeta(\Theta))(-Z)^i = \Big(\sum_j c_j(\Theta)X^j\Big)\Big(\sum_k c_k(\Theta)(-X)^k\Big),$$

with $Z = X^2$. Now, for projective space, $\mathbb{P}^{2n}_{\mathbb{C}}$,

$$1 + c_1(\Theta)t + \cdots + c_{2n}(\Theta)t^{2n} + t^{2n+1} = (1+t)^{2n+1}.$$

Therefore,

$$\sum_{i=0}^{2n} p_i(\zeta(\Theta))(-X^2)^i + \text{terms in } X^{4n+1}, X^{4n+2} = (1+X)^{2n+1}(1-X)^{2n+1} = (1-X^2)^{2n+1}.$$

Hence, we get

$$p_i(\zeta(\Theta)) = \binom{2n+1}{i} H^{2i}, \quad 1 \le i \le n.$$

Let K^L be the multiplicative homomorphism coming from the 1-unit, L. Then

$$\begin{aligned} K^L(1 + \sum_i p_i(-X^2)^i) &= \sum_j L_l(p_1, \ldots, p_l)(-X^2)^l \\ &= K^L((1-X^2)^{2n+1}) \\ &= K^L(1-X^2)^{2n+1} \\ &= L(-X^2)^{2n+1} = L(-Z)^{2n+1}. \end{aligned}$$

The coefficient of Z^n in the latter is $(-1)^n$ and by the first equation, it is $(-1)^n L_n(p_1, \ldots, p_n)$. Therefore, we have

$$L_n(p_1, \ldots, p_n) = 1, \quad \text{for every } n \ge 1.$$

Thus, we've proved

Proposition 3.38 *On the sequence of real 4n-manifolds: $\mathbb{P}^{2n}_{\mathbb{C}}$, $n = 1, 2, \ldots$, the L-genus of each, namely $L_n(p_1, \ldots, p_n)$, is 1. The L-genus is the unique genus having this property. Alternate form: If we substitute $p_j = \binom{2n+1}{j}$ in the L-polynomials, we get*

$$L_n\left(\binom{2n+1}{1}, \ldots, \binom{2n+1}{n}\right) = 1.$$

Now, for the Todd genus.

Proposition 3.39 *If $B \supseteq \mathbb{Q}$, then there is one and only one power series, $T(X)$, having the property: For all $k \ge 0$, the coefficient of X^k in $T(X)^{k+1}$ is 1. In fact this power series defines the holomorphic function*

$$\frac{X}{1 - e^{-X}}.$$

Proof. It is the usual induction, but we'll compute the first few terms. We see that $k = 0$ implies that T is a 1-unit, ie.,

$$T(X) = 1 + b_1 X + b_2 X^2 + O(X^3).$$

For $k = 1$, we have

$$T(X)^2 = (1 + b_1 X)^2 + O(X^2) = 1 + 2b_1 X + O(X^2),$$

so

$$b_1 = \frac{1}{2}.$$

For $k = 2$, we have

$$\begin{aligned} T(X)^3 &= \left(1 + \frac{1}{2}X + b_2 X^2\right)^3 + O(X^3) \\ &= \left(1 + \frac{1}{2}X\right)^3 + 3\left(1 + \frac{1}{2}X\right)^2 b_2 X^2 + O(X^3) \\ &= \text{stuff} + \frac{3}{4}X^2 + 3b_2 X^2 + O(X^3). \end{aligned}$$

Therefore, we must have

$$\frac{3}{4} + 3b_2 = 1,$$

that is,

$$b_2 = \frac{1}{12}.$$

So,

$$T(X) = 1 + \frac{1}{2}X + \frac{1}{12}X^2 + \cdots.$$

That

$$T(X) = \frac{X}{1 - e^{-X}}$$

comes from Cauchy's formula. $\square$

From $T(X)$, we make the opertor K^T, namely,

$$K^T(1 + c_1X + c_2X^2 + \cdots) = 1 + \sum_{j=1}^{\infty} T_j(c_1, \ldots, c_j)X^j = \prod_{i=0}^{\infty} T(\gamma_i X),$$

where

$$1 + c_1X + c_2X^2 + \cdots = \prod_{i=0}^{\infty}(1 + \gamma_i X).$$

Let's work out $T_1(c_1)$ ans $T_2(c_1, c_2)$. From

$$1 + c_1X + c_2X^2 = (1 + \gamma_1X)(1 + \gamma_2X),$$

we get

$$\begin{aligned} 1 + T(c_1)X + T_2(c_1, c_2)X^2 + \cdots &= T(\gamma_1X)T(\gamma_2X) \\ &= \left(1 + \frac{1}{2}\gamma_1X + \frac{1}{12}\gamma_1^2X^2 + \cdots\right)\left(1 + \frac{1}{2}\gamma_2X + \frac{1}{12}\gamma_2^2X^2 + \cdots\right) \\ &= 1 + \frac{1}{2}(\gamma_1 + \gamma_2) + \left(\frac{1}{12}(\gamma_1^2 + \gamma_2^2) + \frac{1}{4}\gamma_1\gamma_2\right)X^2 + \cdots. \end{aligned}$$

We get

$$T_1(c_1) = \frac{1}{2}c_1$$

and

$$T_2(c_1, c_2) = \frac{1}{12}(\gamma_1^2 + \gamma_2^2) + \frac{1}{4}\gamma_1\gamma_2 = \frac{1}{12}(c_1^2 - 2c_2) + \frac{1}{4}c_2 = \frac{1}{12}(c_1^2 + c_2).$$

i.e.,

$$T_2(c_1, c_2) = \frac{1}{12}(c_1^2 + c_2).$$

From this T, we make for a complex manifold, X, its *Todd genus*,

$$T_n(X) = T_n(c_1, \ldots, c_n)[X],$$

where $c_1, \ldots, c_n$ = Chern classes of T_X (the holomorphic tangent bundle) and $[X]$ = the fundamental homology class on $H_{2n}(X, \mathbb{Z})$. This is a rational number.

Suppose X and Y are two real oriented manifolds of dimensions n and r. Then

$$T_{X \prod Y} = pr_1^* T_X \amalg pr_2^* T_Y.$$

So, we have

$$1 + p_1(X \prod Y)Z + \cdots = pr_1^*(1 + p_1(X)Z + \cdots)pr_2^*(1 + p_1(Y)Z + \cdots). \qquad (\dagger)$$

Further observe that if ξ, η are cohomology classes for X, resp. Y, then $\xi \otimes 1$, $1 \otimes \eta$ are $pr_1^*(\xi)$, $pr_2^*(\eta)$, by Künneth and we have

$$\xi \otimes \eta[X \prod Y] = \xi[X]\eta[Y]. \qquad (\dagger\dagger)$$

Now, say K is an endomorphism of the 1-units from a given 1-unit, so it gives the K-genera of $X \prod Y$, X, Y. We have

$$\begin{aligned} K(1 + p_1(X \prod Y)Z + \cdots) &= K((1 + p_1(X)Z + \cdots)(1 + p_1(Y)Z + \cdots)) \\ &= K(1 + p_1(X)Z + \cdots)K(1 + p_1(Y)Z + \cdots). \end{aligned}$$

Now, evaluate on $[X \prod Y]$, find a cycle of $X \prod Y]$ in $H_{n+r}(X \prod Y, \mathbb{Z})$. By $(\dagger\dagger)$, we get

$$K_{n+r}(p_1, \ldots, p_{n+r})[X \prod Y] = K_n(p_1, \ldots, p_n)[X]K_r(p_1, \ldots, p_r)[Y]$$

and

Proposition 3.40 *If K is an endomorphism of 1-units, then the K-genus is multiplicative, i.e.,*

$$K(X \prod Y) = K(X)K(Y).$$

Interpolation among the genera (of interest).

Let y be a new variable (the interpolation variable). Make a new function, with coefficients in $B \supseteq \mathbb{Q}[y]$,

$$Q(y; x) = \frac{x(y+1)}{1 - e^{-x(y+1)}} - xy$$

(First form of $Q(y; x)$). We can also write

$$\begin{aligned} Q(y; x) &= \frac{x(y+1)e^{x(y+1)}}{e^{x(y+1)} - 1} - xy \\ &= \frac{x(y+1)(e^{x(y+1)} - 1 + 1)}{e^{x(y+1)} - 1} - xy \\ &= x(y+1) + \frac{x(y+1)}{e^{x(y+1)} - 1} - xy \\ &= \frac{x(y+1)}{e^{x(y+1)} - 1} + x. \end{aligned}$$

(Second form of $Q(y; x)$).

Let us compute the first three terms of $Q(y; x)$. As

$$e^{-x(y+1)} = 1 - x(y+1) + \frac{(x(y+1))^2}{2!} + \cdots + (-1)^k \frac{(x(y+1))^k}{k!} + \cdots,$$

we have

$$1 - e^{-x(y+1)} = x(y+1) - \frac{(x(y+1))^2}{2!} + \cdots + (-1)^{k-1} \frac{(x(y+1))^k}{k!} + \cdots$$

and so,

$$\frac{x(y+1)}{1 - e^{-x(y+1)}} = \left[1 + \cdots + (-1)^{k-1} \frac{(x(y+1))^{k-1}}{k!} + \cdots\right]^{-1}.$$

If we denote this power series by $1 + \alpha_1 x + \alpha_2 x^2 + \cdots$, we can solve for α_1, α_2, etc., by solving the equation

$$1 = (1 + \alpha_1 x + \alpha_2 x^2 + \cdots)\left[1 - \frac{x(y+1)}{2} + \cdots + (-1)^{k-1}\frac{(x(y+1))^{k-1}}{k!} + \cdots\right].$$

This implies

$$\alpha_1 = \frac{(y+1)}{2}$$

and

$$\alpha_2 = \frac{1}{4}(y+1)^2 - \frac{1}{6}(y+1)^2 = \frac{1}{12}(y+1)^2.$$

Consequently,

$$Q(y;x) = 1 + \frac{x(y+1)}{2} + \frac{1}{12}x^2(y+1)^2 + O(x^3(y+1)^3) - xy,$$

i.e.,

$$Q(y;x) = 1 + \frac{x(1-y)}{2} + \frac{1}{12}x^2(y+1)^2 + O(x^3(y+1)^3).$$

Make the corresponding endomorphisms, $\mathcal{T}_y$. Recall,

$$\mathcal{T}_y(1 + c_1 X + \cdots + c_n X^n + \cdots) = \begin{cases} \prod_{j=1}^{\infty} Q(y;\gamma_j X) \\ \sum_{j=0}^{\infty} T_j(y;c_1,\ldots,c_j)X^j, \end{cases}$$

where, of course,

$$1 + c_1 X + \cdots + c_n X^n + \cdots = \prod_{j=1}^{\infty}(1 + \gamma_i X).$$

We obtain the $\mathcal{T}_y$-genus. The 1-unit, $Q(y;x)$, satisfies

Proposition 3.41 *If $B \supseteq \mathbb{Q}[y]$, then there exists one and only one power series (it is our $Q(y;x)$) in $B[[x]]$ (actually, $\mathbb{Q}[y][[x]]$) so that, for all $k \geq 0$, the coefficient of X^k in $Q(y;x)^{k+1}$ is $\sum_{i=0}^{k}(-1)^i y^i$.*

Proof. The usual (by induction). Let us check for $k = 1$. We have

$$Q(y;x)^2 = \left(1 + \frac{x(1-y)}{2}\right)^2 + O(x^2) = 1 + (1-y)x + O(x^2).$$

The coefficient of x is indeed $1 - y = \sum_{i=0}^{1}(-1)^i y^i$. $\square$

Look at $Q(y;x)$ for $y = 1, -1, 0$. Start with -1. We have

$$Q(-1;x) = 1 + x.$$

Now, for $y = 0$, we get

$$Q(0;y) = T(X) = \frac{x}{1 - e^{-x}}.$$

Finally, consider $y = 1$. We have

$$\begin{aligned} Q(1;x) &= \left(\frac{2}{1 - e^{-2x}} - 1\right)x \\ &= \left(\frac{2e^{2x}}{e^{2x} - 1} - 1\right)x \\ &= \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right)x \\ &= \frac{x}{\tanh x} = L(x^2). \end{aligned}$$

We proved $Q(y;x)$ is the unique power series in $Q[y][[x]]$ so that the coefficient of x^k in $Q(y;x)^{k+1}$ is $\sum_{i=0}^{k}(-1)^i y^i$. Therefore, we know (once again) that $Q(0;x) = Q(x) =$ the unique power series in $\mathbb{Q}[x]$ so that the coefficient of x^k in $Q(x)^{k+1}$ is 1. Since, for projective space, $\mathbb{P}^k_{\mathbb{C}}$, we have

$$1 + c_1X + \cdots + c_kX^K + X^{k+1} = (1+X)^{k+1}$$

and since

$$K^Q((1+X)^{k+1}) = \begin{cases} K^Q(1+X)^{k+1} = Q(X)^{k+1} \\ \sum_{l=0}^{\infty} T_l(c_1, \cdots, c_l)X^l \end{cases}$$

we get

$$T_k(c_1, \ldots, c_k) = 1$$

when the c's come from $\mathbb{P}^k_{\mathbb{C}}$ and if $T_k(y; c_1, \ldots, c_k)$ means the corresponding object for $Q(y;x)$, we get

Proposition 3.42 *The Todd genus, $T_n(c_1, \ldots, c_n)$, and the T_y-genus, $T_n(y; c_1, \ldots, c_n)$, are the only genera so that on* **all** *$\mathbb{P}^n_{\mathbb{C}}$ ($n = 0, 1, 2, \ldots$) they have values 1, resp. $\sum_{i=0}^{\infty}(-1)^i y^i$.*

Write $\mathcal{T}_y$ for the multiplicative operator obtained from $Q(y;x)$, i.e.,

$$\mathcal{T}_y(1 + c_1X + \cdots + c_jX^j + \cdots) = \sum_{n=0}^{\infty} T_n(y; c_1, \ldots, c_n)X^n.$$

Equivalently,

$$\mathcal{T}_y(1 + c_1X + \cdots + c_jX^j + \cdots) = \prod_{j=1}^{\infty} Q(y; \gamma_j X),$$

where

$$(1 + c_1X + \cdots + c_jX^j + \cdots = \prod_{j=1}^{\infty}(1 + \gamma_j X).$$

Now, for all n, the expression $T_n(y; c_1, \ldots, c_n)$ is some polynomial (with coefficients in the c's) of degree at most n in y. Thus, we can write

$$T_n(y; c_1, \ldots, c_n) = \sum_{l=0}^{n} T_n^{(l)}(y; c_1, \ldots, c_n)y^l,$$

and this is new polynomial invariants, the $T_n^{(l)}(y; c_1, \ldots, c_n)$.

We have

$$T_n(-1; c_1, \ldots, c_n) = \sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = c_n,$$

by the fact that $Q(-1;x) = 1 + x$. Next, when $y = 0$,

$$T_n(0; c_1, \ldots, c_n) = T_n^{(0)}(c_1, \ldots, c_n) = T_n(c_1, \ldots, c_n).$$

When $y = 1$, then

$$T_n(1; c_1, \ldots, c_n) = \sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ L_{\frac{n}{2}}(p_1, \ldots, p_{\frac{n}{2}}) & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we get

Proposition 3.43 *If $B \supseteq \mathbb{Q}[y]$, then we have*

(A) $\sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = c_n$, *for all* n.

(B) $T_n^{(0)}(c_1, \ldots, c_n) = \mathrm{td}(c_1, \ldots, c_n)$ $(= T_n(c_1, \ldots, c_n))$.

(C)
$$\sum_{l=0}^{n} T_n^{(l)}(c_1, \ldots, c_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ L_{\frac{n}{2}}(p_1, \ldots, p_{\frac{n}{2}}) & \text{if } n \text{ is even.} \end{cases}$$

The *total Todd class* of a vector bundle, ξ, is

$$\mathrm{td}(\xi)(t) = \sum_{j=0}^{\infty} \mathrm{td}_j(c_1, \ldots, c_j)t^j = 1 + \frac{1}{2}c_1(\xi) + \frac{1}{12}(c_1^2(\xi) + c_2(\xi))t^2 + \frac{1}{24}(c_1(\xi)c_2(\xi))t^3 + \cdots .$$

Here some more Todd polynomials:

$$\begin{aligned} T_4 &= \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4) \\ T_5 &= \frac{1}{1440}(-c_4c_1 + c_3c_1^2 + 3c_2^2c_1 - c_2c_1^3) \\ T_6 &= \frac{1}{60480}(2c_6 - 2c_5c_1 - 9c_4c_2 - 5c_4c_1^2 - c_3^3 + 11c_3c_2c_1 + 5c_3c_1^3 + 10c_2^3 + 11c_2^2c_1^2 - 12c_2c_1^4 + 2c_1^6). \end{aligned}$$

Say
$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0$$
is an exact sequence of vector bundles. Now,
$$(1 + c_1't + \cdots + c_{q'}'t^{q'})(1 + c_1''t + \cdots + c_{q''}''t^{q''}) = 1 + c_1t + \cdots + c_qt^q,$$
and td is a multiplicative sequence, so
$$\mathrm{td}(\xi')(t)\mathrm{td}(\xi'')(t) = \mathrm{td}(\xi)(t).$$

Let us define the *K-ring* of vector bundles. As a group, this is the free abelian group of isomorphism classes of vector bundles modulo the equivalence relation
$$[V] = [V'] + [V'']$$
iff
$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0 \quad \text{is exact.}$$
For the product, define
$$[V] \cdot [W] = [V \otimes W].$$
The ring K is a graded ring by rank (the rank of the vb).

Say ξ is a vector bundle and
$$1 + c_1t + \cdots + c_qt^q + \cdots = \prod(1 + \gamma_jt).$$
Remember,
$$1 + \mathrm{td}_1(c_1)t + \cdots + \mathrm{td}_n(c_1, \ldots, c_n)t^n + \cdots = \prod T(\gamma_jt) = \prod \frac{\gamma_jt}{1 - e^{-\gamma_jt}}.$$

Now, we define the *Chern character* of a vector bundle. For

$$1 + c_1 t + \cdots + c_q t^q + \cdots = \prod (1 + \gamma_j t)$$

set

$$\mathrm{ch}(\xi)(t) = \sum_j e^{\gamma_j t} = \mathrm{ch}_0(\xi) + \mathrm{ch}_1(\xi)t + \cdots + \mathrm{ch}_n(\xi)t^n + \cdots ,$$

where $\mathrm{ch}_j(\xi)$ is a polynomial in $c_1, \ldots, c_j$ of weight j. Since

$$e^{\gamma_j t} = \sum_{r=0}^{\infty} \frac{(\gamma_j t)^r}{r!},$$

we have

$$\sum_j e^{\gamma_j t} = \sum_j \sum_r e^{\gamma_j t} = \sum_r \Big(\frac{1}{r!} \sum_j \gamma_j^r\Big) t^r,$$

which shows that

$$\mathrm{ch}_r(c_1, \ldots, c_r) = \frac{1}{r!} \sum_j \gamma_j^r = \frac{1}{r!} s_r(\gamma_1, \ldots, \gamma_q).$$

The sums, s_r, can be computed by induction using Newton's formulae:

$$s_l - s_{l-1}c_1 + s_{l-2}c_2 + \cdots + (-1)^{l-1} s_1 c_{l-1} + (-1)^l l c_l = 0.$$

(Recall, $c_j = \sigma_j(\gamma_1, \ldots, \gamma_q)$.) We have

$$\begin{aligned}
\mathrm{ch}_1(c_1) &= c_1 \\
\mathrm{ch}_2(c_1, c_2) &= \frac{1}{2}(c_1^2 - 2c_2) \\
\mathrm{ch}_3(c_1, c_2, c_3) &= \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\
\mathrm{ch}_4(c_1, c_2, c_3, c_4) &= \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4).
\end{aligned}$$

Say

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \xi'' \longrightarrow 0$$

is an exact sequence of bundles. The Chern roots of ξ are the Chern roots of ξ' together with those of ξ''. The definition implies

$$\mathrm{ch}(\xi)(t) = \mathrm{ch}(\xi')(t) + \mathrm{ch}(\xi'')(t).$$

If ξ and η are vector bundles with Chern roots, $\gamma_1, \ldots, \gamma_q$ and $\delta_1, \ldots, \delta_r$, then $\xi \otimes \eta$ has Chern roots $\gamma_i + \delta_j$, for all i, j. By definition,

$$\mathrm{ch}(\xi \otimes \eta)(t) = \sum_{i,j} e^{(\gamma_j + \delta_j)t} = \sum_{i,j} e^{\gamma_j t} e^{\delta_j t} = \Big(\sum_i e^{\gamma_j t}\Big)\Big(\sum_j e^{\gamma_j t}\Big) = \mathrm{ch}(\xi)(t)\mathrm{ch}(\eta)(t).$$

The above facts can be summarized in the following proposition:

Proposition 3.44 *The Chern character,* $\mathrm{ch}(\xi)(t)$*, is a ring homomorphism from* $K(\mathrm{vector}(X))$ *to* $H^*(X, \mathbb{Q})$.

If ξ is a $\mathrm{U}(q)$-vector bundle over a complex analytic manifold, X, write

$$T(X,\xi)(t) = \mathrm{ch}(\xi)(t)\mathrm{td}(\xi)(t),$$

the *T-characteristic* of ξ over X.

Remark: The $T_n^{(l)}$ satisfy the *duality formula*

$$(-1)^n T_n^{(l)}(c_1,\ldots,c_n) = T_n^{(n-l)}(c_1,\ldots,c_n).$$

To compute them, we can use

$$T_n^{(l)}(c_1,\ldots,c_n) = \kappa_n(\mathrm{ch}(\bigwedge^l \xi^D)(t)\mathrm{td}(\xi)(t)),$$

where $c_1,\ldots,c_n$ are the Chern classes of the v.b., ξ, and κ_n always means the term of total degree n.

3.4 Cobordism and the Signature Theorem

Let M be a real oriented manifold. Now, if $\dim(M) \equiv 0\,(4)$, we have the Pontrjagin classes of M, say $p_1, \ldots, p_n$ (with $\dim(M) = 4n$). Say $j_1 + \cdots + j_r = n$ (a partition of n) and let $\mathcal{P}(n)$ denote all partitions of n. Write this as (j). Consider $p_{j_1} \cdots p_{j_r}$, the product of weight $j_1, \ldots, j_r$ monomials in the p's; this is in $H^{4n}(M, \mathbb{Z})$. Apply $p_{j_1} \cdots p_{j_r}$ to $[M] =$ fundamental cycle, we get an integer. Such an integer is a *Pontrjagin number* of M, there are $\#(\mathcal{P}(n))$ of them.

Since

$$(-1)^i p_i Z^i = \Bigl(\sum c_j X^j\Bigr)\Bigl(\sum c_l(-X)^l\Bigr),$$

the Pontrjagin classes are independent of the orientation. introduce $-M$, the manifold M with the opposite orientation. Then,

$$p_{j_1} \cdots p_{j_r}[-M] = -p_{j_1} \cdots p_{j_r}[M].$$

Define the *sum*, $M + N$, of two manifolds M and N as $M \amalg N$, their disjoint union, again, oriented. We have

$$H^*(M + N, \mathbb{Z}) = H^*(M, \mathbb{Z}) \prod H^*(N, \mathbb{Z})$$

and consequently, the Pontrjagin numbers of $M + N$ are the sums of the Pontrjagin numbers of M and N.

We also define $M \prod N$, the cartesian product of M and N. By Künneth,

$$[M \prod N] = [M \otimes 1][1 \otimes N],$$

so the Pontrjagin numbers of $M \prod N$ are the products of the Pontrjagin numbers of M and N.

The Pontrjagin numbers of manifolds of dimension $n \not\equiv 0\,(4)$ are all zero.

We make an equivalence relation (*Pontrjagin equivalence*) on oriented manifolds by saying that

$$M \equiv N\;(P)$$

iff every Pontrjagin number of M is the equal to the corresponding Pontrjagin number of N. Let $\widetilde{\Omega}_n$ be the set of equivalence classes of dimension n manifolds, so that $\widetilde{\Omega}_n = (0)$ iff $n \not\equiv 0\,(4)$ and

$$\prod_{n \geq 0} \widetilde{\Omega}_n = \prod_{r \geq 0} \widetilde{\Omega}_{4r}.$$

We see that $\widetilde{\Omega}$ is a graded abelian torsion-free group. For $\widetilde{\Omega} \otimes_{\mathbb{Z}} \mathbb{Q}$, a ring of interest.

Proposition 3.45 *For a sequence, $\{M_{4k}\}_{k=0}^{\infty}$ of manifolds, the following are equivalent:*

(1) For every k, $s_k[M_{4k}] \neq 0$. Here, write $1 + p_1 X + \cdots + p_n Z^n$ as a product $\prod_{j=1}^{m \geq n}(1 + \beta_j Z)$, where equality means up to terms of degree n if $m > n$ and then

$$s_k = \beta_1^k + \cdots + \beta_m^k (m \geq k)$$

a polynomial in $p_1, \ldots, p_k$, of weight k, so it makes sense on M_{4k}.

(2) The mapping from multiplicative sequences with coefficients in B ($\supseteq \mathbb{Q}$) to $\prod_{\aleph_0} B$, via

$$\{K_j\}_{j=1}^{\infty} \mapsto (K_1[M_1], \ldots, (K_k[M_k], \ldots)$$

is a bijection. That is, given any sequence $a_1, \ldots, a_k, \ldots$ of elements of B, there is one and only one multiplicative sequence, $\{K_l\}$ (coeffs in B), so that

$$K_k(p_1, \ldots, p_k)[M_{4k}] = a_k.$$

Proof. (1) $\Longrightarrow$ (2). Choose $a_1, a_2, \ldots$ from B. Now, multiplicative sequences with coefficients in B are in one-to-one correspondence with one-units of $B[[z]]$, say $Q(z)$ is the 1-unit. If

$$1 + p_1 Z + \cdots + p_k Z^k + \cdots = \prod_j (1 + \beta_j Z),$$

then

$$1 + K_1(p_1) Z + \cdots + K_k(1, \ldots, p_k) Z^k + \cdots = \prod_j Q(\beta_j Z).$$

We must produce a unique 1-unit $1 + b_1 Z + \cdots = Q(Z)$, so that a_k is equal to the coefficient of Z^k applied to M_{4k} in $\prod_j Q(\beta_j Z) b_k +$ some polynomial in $b_1, \ldots, b_{k-1}$, of weight k. This polynomial has $\mathbb{Z}$-coefficients and depends on the M_{4k}. We need

$$a_k = s_k[M_{4k}] + \text{poly in } b_1, \ldots, b_{k-1} \tag{$\dagger$}$$

By (1), all $s_l[M_{4k}] \neq 0$; by induction we can find unique b_k's from the a_k's.

(2) $\Longrightarrow$ (1). By (2), the equations ($\dagger$) have a unique b-solution given the a's. But then, all $s_k[M_{4k}] \neq 0$, else no unique solution or worse, no solution. $\square$

Corollary 3.46 *The sequence $\{\mathbb{P}^{2k}_{\mathbb{C}}\}$ satisfies (1) and (2). Such a sequence is called a basis sequence for the n-manifolds.*

Proof. We have

$$1 + p_1 Z + \cdots + p_k Z^k = (1 + h^2 Z)^{2k+1},$$

where $h^2 \in H^4(\mathbb{P}^{2k}_{\mathbb{C}}, \mathbb{Z})$ (square of the hyperplane class). But then, $\beta_j = h^2$, for $j = 1, \ldots, 2k+1$ and

$$s_k(\mathbb{P}^{2k}_{\mathbb{C}}) = \sum_{j=1}^{2k+1} h^{2k}(\mathbb{P}^{2k}_{\mathbb{C}}) = 2k + 1 \neq 0$$

establishing the corollary. $\square$

Theorem 3.47 *Suppose $\{M_{4k}\}$ is a basis sequence for $\widetilde{\Omega} \otimes \mathbb{Q}$. Then, each $\alpha \in \widetilde{\Omega} \otimes \mathbb{Q}$ has the unique form $\sum_{(j)} \rho_{(j)} M_{(j)}$, where*

(1) $(j) = (j_1, \ldots, j_r)$; $j_1 + \cdots + j_r = k$; $M_{(j)} = M_{4j_1} \prod \cdots \prod M_{4j_r}$.

(2) $\rho_{(j)} \in \mathbb{Q}$. Secondly, given any rational numbers, $\rho_{(j)}$, there is some $\alpha \in \widetilde{\Omega} \otimes \mathbb{Q}$ so that

$$p_{(j)}(\alpha) = p_{j_1} p_{j_2} \cdots p_{j_r}(\alpha) = \rho_{(j)}.$$

(3) Given any sequence, $\{M_{4k}\}$, of manifolds suppose $\alpha = \sum_{(j)} \rho_{(j)} M_{(j)}$, then, for every $k \geq 0$, we have $s_k(\alpha) = \rho_k s_k(M_{4k})$.

(4) If each $\alpha \in \widetilde{\Omega} \otimes \mathbb{Q}$ is a sum $\sum_{(j)} \rho_{(j)} M_{(j)}$, then the $\{M_{4k}\}$ are a basis sequence. So, the $\{M_{4k}\}$ are a basis sequence iff the monomials $M_{(j)} = M_{4j_1} \prod \cdots \prod M_{4j_r}$ (over $\mathcal{P}(k)$, all k) form a basis of $\widetilde{\Omega} \otimes \mathbb{Q}$ in the usual sense.

Proof. Note that, as abelian group, $\widetilde{\Omega}_{4k}$ has rank $\#(\mathcal{P}(k))$ (the number of Pontrjagin numbers of weight k is $\#(\mathcal{P}(k))$).

(1) Pick indeterminates $q_1, \ldots, q_l$ over $\mathbb{Q}$ and choose any integer $l \geq 0$. By the previous proposition, since $\{M_{4k}\}$ is a basis sequence there exists one and only one multiplicative sequence, call it $\{K^{(l)}_m\}_{m=1}^{\infty}$, so that

$$K^{(l)}_m[M_{4m}] = q^l_m.$$

We need only check our conclusion for $\alpha \in \widetilde{\Omega}_{4k} \otimes \mathbb{Q}$ for fixed k. Now,

$$\dim_{\mathbb{Q}} \widetilde{\Omega}_{4k} \otimes \mathbb{Q} = \#(\mathcal{P}(k))$$

and there exist exactly $\#(\mathcal{P}(k))$ elements $M_{(j)}$, so all we need to show is

$$\sum_{(j)} \rho_{(j)} M_{(j)} = 0 \quad \text{implies all } \rho_{(j)} = 0.$$

Suppose $\sum_{(j)} \rho_{(j)} M_{(j)} = 0$ and apply the multiplicative sequence $\{K_m^{(l)}\}_{m=1}^{\infty}$. We get

$$\sum_{(j)} \rho_{(j)} q_{j_1}^l \cdots q_{j_r}^l = 0 \quad \text{for all } l \geq 0. \tag{$*$}$$

Write $q_{j_1}^l \cdots q_{j_r}^l = q_{(l)}^l$. The $q_{(l)}^l$ are all pairwise distinct, so by choosing enough l, the equation $(*)$ gives a system of linear equations (unknowns the $\rho_{(j)}$) with a Vandermonde determinant. By linear algebra, all $\rho_{(j)} = 0$.

(2) This is now clear as the $M_{(j)}$ span $\widetilde{\Omega}_{4k} \otimes \mathbb{Q}$ for all k.

(3) Look at $Q(Z) = 1 + Z^k$ and make the corresponding multiplicative sequence. We have

$$1 + k_1(p_1)Z + \cdots + K_k(p_1, \ldots, p_k)Z^k + \cdots = \prod_{j \geq k} (1 + \beta_j^k Z^k).$$

Therefore, $K_l(p_1, \ldots, p_l) = 0$ if $l < k$ and $K_k(p_1, \ldots, p_k) = \beta_1^k + \beta_2^k + \cdots = s_k$. Apply this multiplicative sequence to α, we get $s_k(\alpha) = \rho_k s_k(M_{4k})$, as required.

(4) Suppose each $\alpha = \sum_{(j)} \rho_{(j)} M_{(j)}$, yet, for some k, $s_k(M_k) = 0$. By (3), we have $s_k(\alpha) = \rho_k s_k(M_{4k}) = 0$. It follows that $s_l(\alpha) = 0$, for all α. Now, let $\alpha = \mathbb{P}_{\mathbb{C}}^{2k}$. We get

$$2k + 1 = s_k(\alpha) = 0,$$

a contradiction. $\square$

Corollary 3.48 *The map $M_{4k} \mapsto Z_k$ (and $M_{(j)} \mapsto Z_{j_1} \cdots Z_{j_r}$) gives a $\mathbb{Q}$-algebra isomorphism $\widetilde{\Omega} \otimes \mathbb{Q} \cong \mathbb{Q}[Z_1, Z_2, \ldots]$, where $\deg(Z_l) = 4l$. (Here, $\{M_{4k}\}$ is a basis sequence.)*

Corollary 3.49 *The $\mathbb{Q}$-algebra maps, $\widetilde{\Omega} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$, are in one-to-one correspondence with the multiplicative sequences with coefficients in $\mathbb{Q}$ (or, what's the same, with the 1-units of $\mathbb{Q}[[Z]]$). The map is*

$$\alpha \in \widetilde{\Omega} \otimes \mathbb{Q} \mapsto K(\alpha).$$

Proof. Multiplicative sequences correspond to 1-units $1 + b_1 Z + \cdots +$ and $(\dagger)$ above shows we know the b's iff we know the value of the homomorphism on the M_{4k}, i.e., on the Z_k's and then, use Corollary 3.48. $\square$

Note that manifolds with boundary also have a notion of orientation.

An oriented n-dimensional manifold, M, *bounds* iff there is an oriented manifold, V and an orientation preserving diffeomorphism, $\partial V \cong M$.

Definition 3.4 *(R. Thom)* Two manifolds, M and N are *cobordant* if $M + (-N)$ bounds.

Introduce *cobordism*, the equivalence relation

$$M \equiv N \ (C) \quad \text{iff } M \text{ is cobordant to } N.$$

We see immediately that if $M \equiv N \ (C)$ and $M' \equiv N' \ (C)$, then

(1) $M \amalg M' \equiv N \amalg N'$ (C)

(2) $-M \equiv -N$ (C)

(3) $M \prod M' \equiv N \prod N'$ (C).

Using this equivalence, we have the graded abelian group (under $\amalg$)

$$\Omega = \coprod_n \Omega_n,$$

where Ω_n is the set of equivalence classes of n-dimensional oriented manifolds under cobordism. We make Ω into a ring as follows: Given $\alpha \in \Omega_m$ and $\beta \in \Omega_n$, then

$$\alpha\beta = \text{class of}(\alpha \prod \beta)$$

and (use homology), $\alpha\beta = (-1)^{mn}\beta\alpha$. We call Ω the *oriented cobordism ring*.

Theorem 3.50 *(Pontrjagin) If M bounds (i.e., $M \equiv 0$ (C)) then* **all** *its Pontrjagin numbers vanish (i.e., $M \equiv 0$ (P)). Hence, there is a surjection $\Omega \longrightarrow \widetilde{\Omega}$ and hence a surjection $\Omega \otimes \mathbb{Q} \longrightarrow \widetilde{\Omega} \otimes \mathbb{Q}$.*

Proof. We have $M = \partial V$, write $i\colon M \hookrightarrow V$ for the inclusion. Let $p_1, \ldots, p_l, \ldots$ be the Pontrjagin classes of T_V; note, as $M = \partial V$,

$$i^* T_V = T_V \restriction \partial V = T_V \restriction M = T_M \amalg \mathbb{I},$$

where $\mathbb{I}$ denotes the trivial bundle. Therefore, the Pontrjagin classes of M are i^*(those of V). So, for $4k = \dim M$ and $j_1 + \cdots + j_r = k$,

$$p_{(j)}[M] = i^*((p_{j_1} \cdots p_{j_r})[M]),$$

where $[M]$ is the $4k$-cycle in $H_{4k}(V, \mathbb{Z})$. But, $[M] = 0$ in $H_{4k}(V, \mathbb{Z})$, as $M = \partial V$. Therefore, the right hand side is zero. $\square$

We will need a deep theorem of René Thom. The proof uses a lot of homotopy theory and is omitted.

Theorem 3.51 *(R. Thom, 1954, Commentari) The groups Ω_n of oriented n-manifolds are finite if $n \not\equiv 0 \pmod 4$ and Ω_{4k} = free abelian group of rank $\#(\mathcal{P}(k)) \amalg$ finite abelian group. Hence, $\Omega_n \otimes \mathbb{Q} = (0)$ if $n \not\equiv 0 \pmod 4$ and $\dim(\Omega_{4k} \otimes \mathbb{Q}) = \#(\mathcal{P}(k)) = \dim(\widetilde{\Omega}_{4k} \otimes \mathbb{Q})$. We conclude that the surjection $\Omega \otimes \mathbb{Q} \longrightarrow \widetilde{\Omega} \otimes \mathbb{Q}$ is an isomorphism. Therefore,*

$$\Omega \otimes \mathbb{Q} \cong_{\text{alg}} \mathbb{Q}[Z_1, \ldots, Z_n, \ldots].$$

We will also need another theorem of Thom. First, recall the notion of index of a manifold, from Section 2.6. The index of M, denoted $I(M)$ is by definition the signature, $\text{sgn}(Q)$, where Q is the intersection form on the middle cohomology, $H^n(M, \mathbb{C})$, when n is even. So, $I(M)$ makes sense if $\dim_{\mathbb{R}} M \equiv 0\,(4)$.

Theorem 3.52 *(R. Thom, 1952, Ann. Math. ENS) If the n-dimensional oriented manifold bounds, then $I(M) = 0$.*

In view of these two theorems we can reformulate our algebraic theorem on $\text{Hom}_{\mathbb{Q}\text{-alg}}(\widetilde{\Omega} \otimes \mathbb{Q}, \mathbb{Q})$ in terms of $\Omega \otimes \mathbb{Q}$.

Theorem 3.53 *Suppose λ is a function from oriented n-manifolds to $\mathbb{Q}$, $M \mapsto \lambda(M)$, satisfying*

(1) $\lambda(M + N) = \lambda(M) + \lambda(N)$; $\lambda(-M) = -\lambda(M)$.

(2) *If M bounds, then $\lambda(M) = 0$.*

(3) *If $\{M_{4k}\}$ is a basis sequence for Ω, then when $j_1 + \cdots + j_r = k$, we have*

$$\lambda\Big(M_{4j_1} \coprod \cdots \coprod M_{4j_r}\Big) = \lambda(M_{4j_1}) \cdots \lambda(M_{4j_r}).$$

Then, there exists a unique multiplicative sequence, $\{K_l\}$, so that for every M of dimension n,

$$\lambda(M) = K_{\frac{n}{4}}(p_1, \ldots, p_{\frac{n}{4}})[M].$$

We get the fundamental theorem:

Theorem 3.54 *(Hirzebruch Signature Theorem) For all real differentiable oriented manifolds, M, we have:*

(1) *If $\dim_{\mathbb{R}} M \not\equiv 0 \pmod 4$, then $I(M) = 0$.*

(2) *If $\dim_{\mathbb{R}} M = 4k$, then*
$$I(M) = L_k(p_1, \ldots, p_k)[M].$$

Proof. Recall, I is a function from manifolds to $\mathbb{Z}$ and clearly satisfies (1). By Thom's second Theorem (Theorem 3.52), I satisfies (2). Take as basis sequence: $M_{4k} = \mathbb{P}^{2k}_{\mathbb{C}}$. We have

$$I(M_{4k}) = \sum_{p=0}^{2k} (-1)^p h^{p,q}(M_{4k}),$$

by the Hodge Index Theorem (Theorem 2.77). As $h^{p,p} = 1$ and $h^{p,q} = 0$ if $p \neq q$, we get

$$I(M_{4k}) = 1.$$

Now we further know the Künneth formula for the $h^{p,q}$ of a product (of two, hence any finite number of complex manifolds). Apply this and get (DX)

$$I\Big(\mathbb{P}^{j_1}_{\mathbb{C}} \coprod \cdots \coprod \mathbb{P}^{j_r}_{\mathbb{C}}\Big) = 1.$$

Therefore, (3) holds. Then, our previous theorem implies $I(M) = K(M)$ for some K, a multiplicative sequence. But, $K(\mathbb{P}^{2k}_{\mathbb{C}}) = 1$, there and we know there is one and only one multiplicative sequence $\equiv 1$ on all $\mathbb{P}^{2k}_{\mathbb{C}}$, it is L. Therefore, $I(M) = L$, as claimed. $\square$

3.5 The Hirzebruch–Riemann–Roch Theorem (HRR)

We can now state and understand the theorem:

Theorem 3.55 *(Hirzebruch–Riemann–Roch) Suppose X is a complex, smooth, projective algebraic variety of complex dimension n. If E is a rank q complex vector bundle on X, then*

$$\chi(X, \mathcal{O}_X(E)) = \kappa_n\big(\mathrm{ch}(E)(t)\mathrm{td}(X)(t)\big)[X].$$

Here,

$$\chi(X, \mathcal{O}_X(E)) = \sum_{i=0}^{n}(-1)^i \dim H^i(X, \mathcal{O}_X(E)).$$

We need to explicate the theorem.

(a) Write it using the Chern roots

$$1 + c_1(E)t + \cdots + c_q(E)t^q = \prod_{i=1}^{q}(1+\gamma_i t), \quad 1 + c_1(X)t + \cdots + c_q(X)t^n = \prod_{j=1}^{n}(1+\delta_j t),$$

and the theorem says

$$\chi(X, \mathcal{O}_X(E)) = \kappa_n\left(\sum_{i=1}^{q} e^{\gamma_i t}\prod_{j=1}^{n}\frac{\delta_j t}{1-e^{-\delta_j t}}\right)[X].$$

(b) Better explication: Use

$$\begin{aligned}\mathrm{td}(X)(t) = 1 &+ \frac{1}{2}c_1(X)t + \frac{1}{12}(c_1^2(X) + c_2(X))t^2 + \frac{1}{24}c_1(X)c_2(X)t^3\\ &+ \frac{1}{720}(-c_4(X) + c_3(X)c_1(X) + 3c_2^2(X) + 4c_2(X)c_1^2(X) - c_1^4(X))t^4 + O(t^5)\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}(E)(t) = \mathrm{rk}(E) &+ c_1(E)t + \frac{1}{2}(c_1^2(E) - 2c_2(E))t^2 + \frac{1}{6}(c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E))t^3\\ &+ \frac{1}{24}(c_1^4(E) - 4c_1^2(E)c_2(E) + 4c_1(E)c_3(E) + 2c_2^2(E) - 4c_4(E))t^4 + O(t^5).\end{aligned}$$

(A) Case $n = 1$, X = Riemann surface = complex curve; E = rank q vector bundle on X. HRR says:

$$\chi(X, \mathcal{O}_X(E)) = \left(\frac{1}{2}qc_1(X) + c_1(E)\right)[X].$$

Now, $c_1(X) = \chi(X) = \text{Euler-Poincaré}(X) = (\text{highest Chern class}) = 2-2g$ (where g is the genus of X). Also, $c_1(E) = \deg(E)$ $(= \deg \bigwedge^q E)$, so

$$\chi(X, \mathcal{O}_X(E)) = (1-g)\mathrm{rk}(E) + \deg E.$$

Now,

$$\chi(X, \mathcal{O}_X(E)) = \dim H^0(X, \mathcal{O}_X(E)) - \dim H^1(X, \mathcal{O}_X(E));$$

by Serre duality,

$$\dim H^1(X, \mathcal{O}_X(E)) = \dim H^0(X, \mathcal{O}_X(E^D \otimes \omega_X)),$$

so we get

$$\dim H^0(X, \mathcal{O}_X(E)) - \dim H^0(X, \mathcal{O}_X(E^D \otimes \omega_X)) = \deg E + (\mathrm{rk}(E))(1-g).$$

(Note: We proved this before using the Atiyah-Serre Theorem, see Theorem 3.13.)

(i) $E = \mathcal{O}_X =$ trivial bundle, then $\deg E = 0$ and $\mathrm{rk}\, E = 1$. We get

$$\dim H^0(X, \mathcal{O}_X) - \dim H^0(X, \Omega_X^1) = 1 - g.$$

Now, X connected implies $\dim H^0(X, \mathcal{O}_X) = h^{0,1} = 1$, so

$$g = \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega_X^1) = h^{1,0}.$$

(ii) $E = \omega_X = \Omega_X^1$, $\mathrm{rk}\, E = 1$ and HRR says

$$\dim H^0(X, \Omega_X^1) - \dim H^0(X, \mathcal{O}_X) = \deg \Omega_X^1 + 1 - g.$$

The left hand side is g and $\dim H^0(X, \mathcal{O}_X) = 1$, so

$$\deg \Omega_X^1 = 2g - 2.$$

(iii) $E = T_X = \Omega_X^{1,D}$. Then, $\mathrm{rk}\, E = 1$, $\deg E = 2 - 2g$ and HRR says

$$\dim H^0(X, T_X) - \dim H^1(X, T_X) = 2 - 2g + 1 - g.$$

Assume $g \geq 2$, then $\deg T_X = 2 - 2g < 0$. Therefore, $H^0(X, T_X) = (0)$ and so,

$$-\dim H^1(X, T_X) = 3 - 3g,$$

so that

$$\dim H^1(X, T_X) = 3g - 3.$$

Remark: The group $H^1(X, T_X)$ is the space of infinitesimal analytic deformations of X. Therefore, $3g-3$ is the dimension of the complex space of infinitesimal deformations of X as complex manifold. suppose we know that there was a “classifying” variety of the genus g Riemann surfaces, say $\mathfrak{M}_g$. Then, if X (our Riemann surface of genus g) corresponds to a smooth point of $\mathfrak{M}_g$, then

$$T_{\mathfrak{M}_g, X} = H^1(X, T_X).$$

Therefore, $\dim_{\mathbb{C}} \mathfrak{M}_g = 3g - 3$ (Riemann's computation).

(B) The case $n = 2$, an algebraic surface. Here, HRR says

$$\chi(X, \mathcal{O}_X(E)) = \left(\frac{1}{12}(c_1^2(X) + c_2(X))\mathrm{rk}(E) + \frac{1}{2}c_1(X)c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E))\right)[X].$$

The left hand side is

$$\dim H^0(X, \mathcal{O}_X(E)) - \dim H^1(X, \mathcal{O}_X(E)) + \dim H^0(X, \mathcal{O}_X(E^D \otimes \omega_X)).$$

Take $E =$ trivial bundle, $\mathrm{rk}\, E = 1$, $c_1(E) = c_2(E) = 0$, and we get

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X))[X] = \frac{1}{12}(\mathcal{K}_X^2 + \chi(X))[X],$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X. We proved that this holds iff $I(X) = \frac{1}{3}p_1(X) = L_1(p_1)[X]$ (see Section 2.6, just after Theorem 2.82). By the Hirzebruch signature theorem, our formula is OK.

Observe, if we take ω_X, not $\mathcal{O}_X$, then the left hand side, $\chi(X, \mathcal{O}_X)$, is

$$\dim H^0(X, \omega_X) - \dim H^1(X, \omega_X) + \dim H^2(X, \omega_X) = \dim H^2(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) + \dim H^0(X, \omega_X)$$

(by Serre duality) and the left hand side stays the same.

Take $E = T_X$; $\mathrm{rk}\, E = 2$, $c_1(E) = c_1(X)$, $c_2(E) = c_2(X)$ and the right hand side of HRR is

$$\begin{aligned}\left(\frac{2}{12}(c_1^2(X) + c_2(X)) + \frac{1}{2}c_1^2(X) + \frac{1}{2}c_1^2(E) - c_2(X)\right)[X] &= \left(\frac{7}{6}c_1^2(X) - \frac{5}{6}c_2(X)\right)[X] \\ &= \left(\frac{7}{6}\mathcal{K}_X^2 - \frac{5}{6}\chi(X)\right)[X].\end{aligned}$$

The left hand side of HRR is

$$\dim H^0(X, T_X) - \dim H^1(X, T_X) + \dim H^0(X, T_X^D \otimes \omega_X).$$

Now,

$$T_X^D \otimes T_X^D \longrightarrow T_X^D \wedge T_X^D = \omega_X$$

gives by duality

$$\begin{aligned} T_X^D &\cong \mathrm{Hom}(T_X^D, \omega_X) \\ &\cong \mathrm{Hom}(T_X^D \otimes \omega_X^D, \mathcal{O}_X) \\ &\cong T_X \otimes \omega_X, \end{aligned}$$

so the left hand side is

dim(global holo vector fields on X) − dim(infinitesimal deformations of X)
+ dim(global section of $T_X \otimes \omega_X^{\otimes 2}$).

Take $E = \Omega_X^1 = T_X^D$, $\mathrm{rk}\, E = 2$, $c_1(E) = c_1(\omega_X) = -c_1(T_X) = -c_1(X)$, $c_2(E) = c_2(X)$. The right hand side of HRR is

$$\frac{2}{12}(c_1^2(X) + c_2(X)) - \frac{1}{2}c_1^2(X) + \frac{1}{2}c_1^2(X) - c_2(X) = \frac{1}{6}c_1^2(X) - \frac{5}{6}c_2(X).$$

The left hand side of HRR is

$$\dim H^0(X, \Omega_X^1) - \dim H^1(X, \Omega_X^1) + \dim H^2(X, \Omega_X^1) = h^{1,0} - h^{1,1} + h^{1,2} = h^{1,0} - h^{1,1} + h^{1,0} = b_1(X) - h^{1,1}.$$

It follows that

$$b_1(X) - h^{1,1} = \left(\frac{7}{6}\mathcal{K}_X^2 - \frac{5}{6}\chi(X)\right)[X],$$

so

$$b_1(X) - \left(\frac{7}{6}\mathcal{K}_X^2 - \frac{5}{6}\chi(X)\right)[X] = h^{1,1}.$$

Also,

$$\begin{aligned} H^0(X, \Omega_X^1) &= H^2(X, \omega_X \otimes T_X)^D \\ H^1(X, \Omega_X^1) &= H^1(X, \omega_X \otimes T_X)^D \\ H^2(X, \Omega_X^1) &= H^0(X, \omega_X \otimes T_X)^D \end{aligned}$$

and we get no new information.

When we know something about X, we can say more. For example, say X is a hypersurface of degree d in $\mathbb{P}^3_{\mathbb{C}}$. Then, write
$$H \cdot X = h = i^* H,$$
where $i\colon X \to \mathbb{P}^3_{\mathbb{C}}$. We know
$$\mathcal{N}_{X \hookrightarrow \mathbb{P}^3} = \mathcal{O}_X(d \cdot h),$$
so
$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^3} \restriction X \longrightarrow \mathcal{O}_X(dh) \longrightarrow 0 \quad \text{is exact.}$$
We have
$$(1 + c_1(X)t + c_2(X)t^2)(1 + dht) = (1 + Ht)^4 \restriction X = (1 + ht)^4,$$
so
$$1 + c_1(X)t + c_2(X)t^2 = (1 + 4ht + 6h^2t^2)(1 - dht + d^2h^2t^2) = 1 + (4 - d)ht + (6 - 4d + d^2)h^2t^2.$$
So $c_1(X) = (4 - d)h$ and $c_2(X) = (6 - 4d + d^2)h^2$. Now,
$$h^2[X] = i^*(H \cdot X)i^*(H \cdot X) = H \cdot H \cdot X = \deg X = d.$$
Consequently,
$$\chi(X, \mathcal{O}_X(E)) = \frac{1}{12}\mathrm{rk}(E)((4 - d)^2 d + (6 - 4d + d^2)d) + \frac{1}{2}c_1(E)(4 - d)h[X] + \frac{1}{2}(c_1^2(E) - 2c_2(E))[X].$$
Take eH and set $E =$ line bundle $eh = eH \cdot X = eH \restriction X = \mathcal{O}_X(e)$. In this case, $\mathrm{rk}(E) = 1$, $c_2(E) = 0$ and $c_1(E) = eh$. We get
$$\chi(X, \mathcal{O}_X(e)) = \frac{1}{6}(11 - 6d + d^2)d + \frac{1}{2}e(4 - d)d + \frac{1}{2}e^2 d,$$
i.e.,
$$\chi(X, \mathcal{O}_X(e)) = \left(\frac{1}{6}(11 - 6d + d^2) + \frac{1}{2}(e^2 - ed + 4e)\right) d.$$

(C) $X =$ abelian variety $=$ projective group variety.

As X is a group, T_X is the trivial bundle, so $c_1(X) = c_2(X) = 0$. When X is an abelian surface we get
$$\chi(X, \mathcal{O}_X(E)) = \frac{1}{2}(c_1^2(E) - 2c_2(E))[X].$$
When X is an abelian curve $=$ elliptic curve ($g = 1$), we get
$$\chi(X, \mathcal{O}_X(E)) = c_1(E) = \deg E.$$
Say the abelian surface is a hypersurface in $\mathbb{P}^3_{\mathbb{C}}$. We know $c_1(X) = 0$ and $c_2(X) = (4 - d)h$. This implies $d = 4$, but $c_2(X) = 6h^2 \neq 0$, a contradiction! Therefore, no abelian surface in $\mathbb{P}^3_{\mathbb{C}}$ is a hypersurface.

Now, assume $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$, where $N > 3$ and X is an abelian surface. Set $E = \mathcal{O}_X(h)$ and compute $\chi(X, \mathcal{O}_X(h))$, where $h = H \cdot X$. We have $c_1(\mathcal{O}_X(h)) = h$ and $c_2(\mathcal{O}_X(h)) = 0$. Then,
$$c_1^2(E)[X] = h^2[X] = H \cdot H \cdot X = \deg X$$
as subvariety of $\mathbb{P}^N_{\mathbb{C}}$. HRR for abelian surfaces embedded in $\mathbb{P}^N_{\mathbb{C}}$ with $N > 3$ yields
$$\chi(X, \mathcal{O}_X(1)) = \frac{1}{2}\deg X.$$

As the left hand side is an integer, we deduce that $\deg X$ must be even.

(D) $X = \mathbb{P}^n_{\mathbb{C}}$. From

$$1 + c_1(X)t + \cdots + c_n(X)t^n = (1 + Ht)^{n+1}$$

we deduce

$$\delta_1 = \cdots = \delta_{n+1} = H.$$

Take

$$1 + c_1(X)t + \cdots + c_n(X)t^n = \prod_j (1 + \gamma_j t)$$

and look at $E \otimes H^{\otimes r} = E(r)$. We have

$$\begin{aligned}
\chi(\mathbb{P}^n, \mathcal{O}_X(E(r))) &= \kappa_n \left(\sum_{i=1}^{q} e^{(\gamma_i + r)t} \frac{(Ht)^n}{(1 - e^{-Ht})^n} \right) [X] \\
&= \sum_{l=1}^{q} \frac{1}{2\pi i} \int_C \frac{e^{(\gamma_l + r)Ht}}{(1 - e^{-Ht})^{n+1}}\, d(Ht) \\
&= \sum_{l=1}^{q} \frac{1}{2\pi i} \int_C \frac{e^{(\gamma_l + r)z}}{(1 - e^{-z})^{n+1}}\, d(z),
\end{aligned}$$

where C is a small circle. Let $u = 1 - e^{-z}$, then $du = e^{-z}dz = (1-u)dz$, so

$$dz = \frac{du}{1-u}.$$

We also have $e^{(\gamma_l + r)z} = (e^{-z})^{-(\gamma_l + r)} = (1-u)^{-(\gamma_l + r)}$. Consequently, the integral is

$$\sum_{l=1}^{q} \frac{1}{2\pi i} \int_{u=0}^{2\pi} \frac{du}{(1-u)^{\gamma_l + r + 1} u^{n+1}}$$

where is the path of integration is a segment of the line $z = \epsilon + iu$. It turns out that

$$\frac{1}{2\pi i} \int_{u=0}^{2\pi} \frac{du}{(1-u)^{\gamma_l + r + 1} u^{n+1}} = \beta(\gamma_l, n) = \binom{n + \gamma_l + r}{n}$$

so HRR implies

$$\chi(\mathbb{P}^n, \mathcal{O}_X(E(r))) = \sum_{l=1}^{q} \binom{n + \gamma_l + r}{n} \in \mathbb{Q}.$$

But, the right hand side has denominator $n!$ and the left hand side is an integer. We deduce that for all $r \in \mathbb{Z}$, for all $n \geq$ and all $q \geq 1$,

$$\sum_{l=1}^{q} \binom{n + \gamma_l + r}{n} \in \mathbb{Z}.$$

(Here, $1 + c_1(E)Ht + \cdots + c_q(E)(Ht)^q = \prod_{j=1}^{q} (1 + \gamma_j Ht)$.)

Take $r = 0, q = 2$. We get

$$\binom{n + \gamma_1}{n} + \binom{n + \gamma_2}{n} \in \mathbb{Z}.$$

For $n = 2$, we must have

$$(2 + \gamma_1)(1 + \gamma_1) + (2 + \gamma_2)(1 + \gamma_2) \equiv 0 \ (2),$$

i.e.,

$$2 + 3\gamma_1 + \gamma_1^2 + 2 + 3\gamma_2 + \gamma_2^2 \equiv 0 \ (2),$$

which is equivalent to

$$3c_1 + c_1^2 - 2c_2 \equiv 0 \ (2).$$

Thus, we need $c_1(3 + c_1) \equiv 0 \ (2)$, which always holds.

Now, take $n = 3$. We have

$$\binom{3+\gamma_1}{3} + \binom{3+\gamma_2}{3} \in \mathbb{Z},$$

i.e.,

$$(3 + \gamma_1)(2 + \gamma_1)(1 + \gamma_1) + (3 + \gamma_2)(2 + \gamma_2)(1 + \gamma_2) \equiv 0 \ (6).$$

This amounts to

$$(6 + 5\gamma_1 + \gamma_1^2)(1 + \gamma_1) + (6 + 5\gamma_2 + \gamma_2^2)(1 + \gamma_2) \equiv 0 \ (6)$$

which is equivalent to

$$\gamma_1(5 + \gamma_1)(1 + \gamma_1) + \gamma_2(5 + \gamma_2)(1 + \gamma_2) \equiv 0 \ (6),$$

i.e.,

$$\gamma_1(5 + 6\gamma_1 + \gamma_1^2) + \gamma_2(5 + 6\gamma_2 + \gamma_2^2) \equiv 0 \ (6)$$

which can be written in terms of the Chern classes as

$$5c_1 + 6(c_1^2 - 2c_2) + c_1^3 - 3c_1c_2 \equiv 0 \ (6),$$

i.e.,

$$c_1(c_1^2 - 3c_2 + 5) \equiv 0 \ (6).$$

Observe that

$$c_1^3 + 5c_1 \equiv 0 \ (6)$$

always, so we conclude that c_1c_2 must be *even*.

Say $i\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ is an embedding of $\mathbb{P}^2_{\mathbb{C}}$ into $\mathbb{P}^3_{\mathbb{C}}$.

Question: Does there exist a rank 2 bundle on $\mathbb{P}^3_{\mathbb{C}}$, say E, so that $i^*(E) = T_{\mathbb{P}^2_{\mathbb{C}}}$.?

If so, E has Chern classes c_1 and c_2 and

$$c_1(T_{\mathbb{P}^2_{\mathbb{C}}}) = i^*(c_1), \quad c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = i^*(c_2).$$

This implies

$$c_1c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = i^*(c_1c_2(E)),$$

which is even (case $n = 3$). But,

$$c_1(T_{\mathbb{P}^2_{\mathbb{C}}}) = 3H_{\mathbb{P}^2}, \quad c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = 3H_{\mathbb{P}^2},$$

so

$$c_1c_2(T_{\mathbb{P}^2_{\mathbb{C}}}) = 9H^2,$$

which is **not** even! Therefore, the answer is no.

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