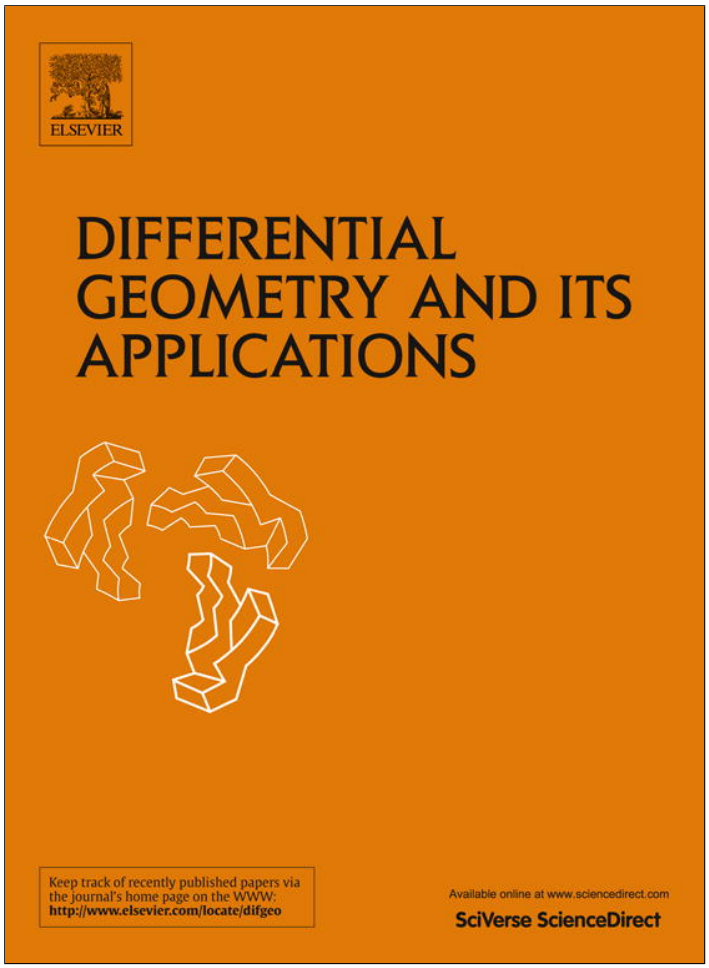

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Parametric pseudo-manifolds

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ABSTRACT

We introduce a novel and constructive definition of gluing data, and give the first rigorous proof that a universal manifold satisfying the Hausdorff condition can always be constructed from any set of gluing data. We also present a class of spaces called parametric pseudo-manifolds, which under certain conditions, are manifolds embedded in $\mathbb{R}^n$ and defined from sets of gluing data. We give a construction for building a set of gluing data from any simplicial surface in $\mathbb{R}^3$. This construction is an improvement of the construction given in Siqueira et al. (2009) [1], where the results were stated without proof. We also give a complete proof of the correctness of this construction making use of the crucial "property A." The above results enable us to develop a methodology that explicitly yields manifolds in $\mathbb{R}^n$ arising in several graphics and engineering applications.

1. Introduction

Some graphics, engineering, and artificial intelligence applications, including surface modeling [2], rendering and simulation on surfaces [3–6], spherical imagery [7,8], and manifold learning [9], deal with abstract objects that can be naturally defined as differentiable manifolds embedded in $\mathbb{R}^n$. In most cases, these objects are surfaces in $\mathbb{R}^3$ with arbitrarily large genus, but they can also be image panoramas [10], the space of bi-directional reflectance distribution functions (BRDF's) [11], or more general manifolds embedded in $\mathbb{R}^n$ and generated by well-known techniques of dimensionality reduction [12].

A common feature of all applications mentioned above is that they all need to *build* a manifold. For this purpose, the modern notion of manifold, which has been known and studied by mathematicians since the early 1900s, is not very helpful. The reason is that this notion (see Definition 2.5) is not constructive, in the sense that it does not tell us how to build a manifold. The lack of a constructive definition led many researchers and practitioners to representing manifolds by less powerful mathematical objects, making it difficult or even impossible to do differential calculus on them.

In 1995, Cindy Grimm and John Hughes [13] gave the first constructive definition of a manifold. They also provided an approach, based on their constructive definition, for building surfaces in $\mathbb{R}^3$ that approximate polygonal meshes, a classic problem in surface modeling [2]. Their approach explicitly builds an atlas for the surface by "gluing" open sets in $\mathbb{R}^2$, and by defining functions, *i.e.*, parametrizations, that take these open sets onto the surface in $\mathbb{R}^3$. The idea of gluing open sets to build manifolds can be traced back to André Weil, who introduced this idea to define abstract algebraic varieties by gluing irreducible affine sets in his book [14] published in 1946 (Chapter VII, Section 3, p. 179). However, Grimm and Hughes [13] were the first to come up with a practical approach.

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The surfaces built by the manifold-based construction in [13] are C^2. But, it was clear since then that C^k surfaces, for $k > 2$ or even $k = \infty$, could be more easily created by the approach in [13] than by all previously developed surface constructions. Furthermore, the existence of an atlas for the surface allows for a more natural and elegant way of solving differential equations on surfaces, such as the ones in [3,5,6]. The pioneering work of Grimm and Hughes caught the attention of several researchers in the field of surface modeling, and more powerful, manifold-based constructions for surfaces were developed [13,15–18,1]. Most of these constructions are based on the theoretical framework developed by Grimm and Hughes. Their work also inspired the development of manifold-based techniques for surface manipulation, such as the remeshing algorithm described in [19].

Here, we introduce a new constructive definition of manifold. In particular, we formally define *gluing data*, and prove that a "universal" manifold can always be constructed from a set of gluing data through a (constructive) gluing process. Our definition improves upon the one given by Grimm and Hughes in [13]. The main differences are two-fold. First, our definition fixes a problem with their definition. Second, we give a necessary and sufficient condition for the manifold to be Hausdorff. The condition given in [13] is only sufficient, and it excludes some Hausdorff manifolds from being constructed. We also introduce the notion of *parametric pseudo-manifolds* (PPM's for short), which under certain conditions are manifolds embedded in $\mathbb{R}^n$. We believe that PPM's are powerful representations for the manifolds arising in applications such as the ones described in [2–8,10].

Defining explicit gluing data from a simplicial surface, especially the transition functions, φ_{ji}, is harder than it appears at first glance. The main difficulty is to find smooth diffeomorphisms which satisfy the cocycle condition (condition 3c of Definition 3.1). In this paper, we provide an original solution involving domains which are the interior of disks and simple functions which are linear in polar coordinates (see Section 5). Some of our results were stated without proofs in [1]. Here, we give complete proofs for them, and we make various improvements (*e.g.*, a stronger version of condition 3c, and a cleaner and simpler presentation of the results of Section 5, using "Property A.")

The paper is organized as follows. Section 2 recalls the basic notions of charts, atlases, and manifolds. Section 3 introduces our definition of sets of gluing data, and also presents our proof that a universal manifold can always be built from such a set. Section 4 gives the definition of parametric pseudo-manifolds, and discusses some of their properties. Section 5 describes a constructive process for building sets of gluing data from simplicial surfaces. This kind of gluing data has been used in the construction of C^∞-surfaces that approximate simplicial surfaces (see [1]), but we give an improved version here. Section 6 presents our concluding remarks and some open problems, and discusses future work. Appendix A presents a proof for the correctness of the construction described in Section 5.

2. Charts, atlases, and manifolds

This section reviews standard notions related to the topic of this paper. Our presentation is based on many sources, including Warner [20], Berger and Gostiaux [21], O'Neill [22], do Carmo [23,24], and Tu [25].

Given $\mathbb{R}^n$, recall that the projection functions, $pr_i : \mathbb{R}^n \to \mathbb{R}$, for all $i \in \{1, \ldots, n\}$, are defined by

$$pr_i(x_1, \ldots, x_n) = x_i.$$

Definition 2.1. Given a topological space, M, a *chart* (or *local coordinate function*) is a pair,

$$(U, \varphi),$$

where U is an open subset of M and $\varphi : U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{R}^{n_\varphi}$ (for some $n_\varphi \geqslant 1$). For any $p \in M$, a chart, (U, φ), is a *chart at* p iff $p \in U$. If (U, φ) is a chart, then the functions $x_i = pr_i \circ \varphi$ are called *local coordinates* and for every $p \in U$, the tuple $(x_1(p), \ldots, x_n(p))$ is the set of *coordinates of* p with respect to the chart. Finally, the "inverse" chart,

$$\left(\Omega, \varphi^{-1}\right),$$

is called a *local parametrization*.

Definition 2.2. Given any two charts, (U_i, φ_i) and (U_j, φ_j), on a topological space, M, if $U_i \cap U_j \neq \emptyset$, we define the *transition maps*, $\varphi_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ and $\varphi_{ij} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$, as

$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \varphi_{ij} = \varphi_i \circ \varphi_j^{-1}.$$

Fig. 2.1 illustrates Definition 2.2.

Clearly, we have $\varphi_{ij} = \varphi_{ji}^{-1}$. We also have that φ_{ij} and φ_{ji} are maps between *open sets* of $\mathbb{R}^n$.

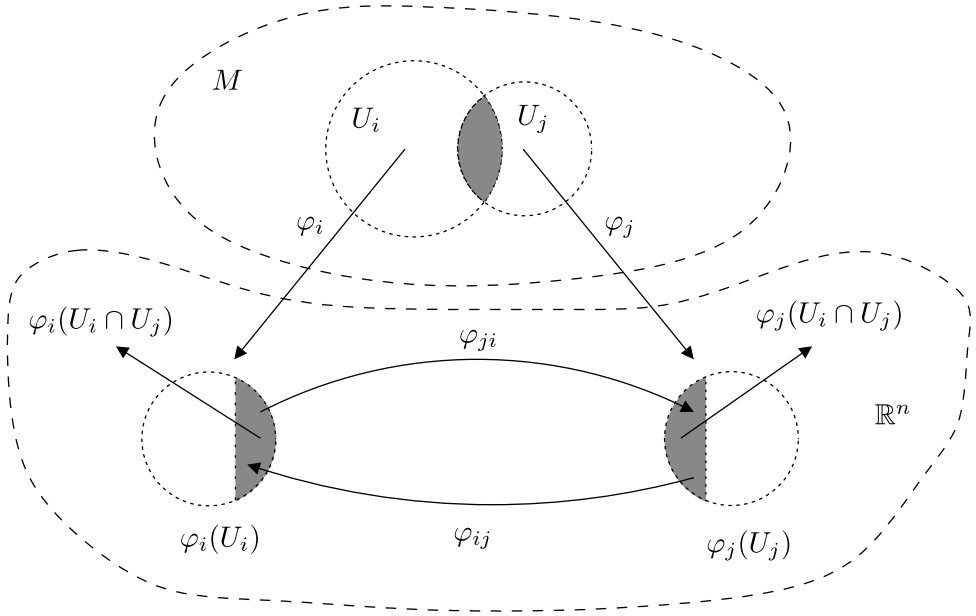

Fig. 2.1. Illustration of the definition of transition maps.

Definition 2.3. Given a topological space, M, given an integer $n \geqslant 1$, and given some k such that k is either a positive integer or $k = \infty$, a C^k *n-atlas* (or *n-atlas of class* C^k), $\mathcal{A}$, on M is a family of charts, $\{(U_i, \varphi_i)\}_I$, where I is a nonempty (and possibly uncountable) index set,[1] such that the following holds:

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i;
(2) the family $\{U_i\}_{i \in I}$ is an open cover for M, *i.e.*,

$$M = \bigcup_{i \in I} U_i;$$

and
(3) whenever $U_i \cap U_j \neq \emptyset$, the transition map φ_{ji} (resp. φ_{ij}) is a C^k diffeomorphism (when $k = \infty$, the φ_{ji} are smooth diffeomorphisms).

The existence of a C^k n-atlas on a topological space, M, is sufficient to establish that M is an n-dimensional C^k manifold, but there is still a minor subtlety in the actual definition of a manifold. This has to do with the fact that there may be many choices of atlases, but it is useful to think of a manifold as an object independent of the choice of atlas. To do so, we define the notion of atlas compatibility.

Definition 2.4. Given a C^k n-atlas, $\mathcal{A}$, on M, for any other chart, (U, φ), we say that (U, φ) is *compatible* with the atlas $\mathcal{A}$ iff every map $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ is C^k (whenever $U \cap U_i \neq \emptyset$). Two atlases, $\mathcal{A}$ and $\mathcal{A}'$, on M are said to be *compatible* iff every chart of one is compatible with the other atlas.

To say that two atlases are compatible is equivalent to saying that the union of the two atlases is still an atlas. Atlas compatibility induces an equivalence relation on C^k n-atlases on M. In particular, given an atlas, $\mathcal{A}$, for M, the collection, $\widetilde{\mathcal{A}}$, of all charts compatible with $\mathcal{A}$ is a maximal atlas in the equivalence class of charts compatible with $\mathcal{A}$. Finally, we have the definition of a manifold:

Definition 2.5. Given an integer $n \geqslant 1$ and given some k such that k is either a positive integer or $k = \infty$, a C^k *manifold of dimension n* consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases on M. Any atlas, $\mathcal{A}$, of $\overline{\mathcal{A}}$ is called a *differentiable structure of class* C^k (*and dimension n*) on M. When $k = \infty$, we say that M is a *smooth* manifold of dimension n.

For technical reasons (in particular, to ensure the existence of partitions of unity) and to avoid "esoteric" manifolds that do not arise in the practical applications we mention in this paper, from now on, all topological spaces under consideration will be assumed to be Hausdorff and second-countable.

[1] In practice, the set I is countable and even finite.

We can allow $k = 0$ in the above definitions. If this is the case, then condition 3 of Definition 2.3 is void, since a C^0 diffeomorphism is just a homeomorphism, but φ_{ji} is always a homeomorphism. When $k = 0$ we call M a *topological manifold of dimension n*. We do not require a manifold to be connected but we require all components to have the same dimension, n. Actually, on every connected component of M, it can be shown that the dimension, n_φ, of the range of every chart is the same. This is quite easy to show if $k \geqslant 1$ but for $k = 0$, this requires a deep theorem of Brouwer (the *Invariance of Domain Theorem*). We can also allow $n = 0$ in the above definitions. If this is the case, then every one-point subset of M is open. So, every subset of M is open, *i.e.*, M is any countable set (as we assumed M to be second-countable) with the discrete topology. Finally, note that every manifold is locally compact and locally connected, as $\mathbb{R}^n$ is locally compact and locally connected.

For an example of a manifold, consider the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$,

$$\mathbb{S}^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

We can regard $\mathbb{S}^n$ as a topological space by giving $\mathbb{S}^n$ the topology consisting of all subsets U of $\mathbb{S}^n$ such that, for every $p = (p_1, \ldots, p_{n+1}) \in U$, there exists a real number δ, with $\delta > 0$, such that $(\mathbb{S}^n \cap B_\delta(p, \mathbb{R}^{n+1})) \subseteq U$, where $B_\delta(p, \mathbb{R}^{n+1})$ is the open ball in $\mathbb{R}^{n+1}$ of center p and radius δ. Using the stereographic projection, we can define two charts on $\mathbb{S}^n$. In fact, denote the points $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and $(0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ by N (the north pole) and S (the south pole), respectively, and let

$$\varphi_N : \mathbb{S}^n - \{N\} \to \mathbb{R}^n \quad \text{and} \quad \varphi_S : \mathbb{S}^n - \{S\} \to \mathbb{R}^n$$

be given by

$$\varphi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{and} \quad \varphi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n),$$

which are called *stereographic projection from the north pole* and *stereographic projection from the south pole*, respectively.

The inverse stereographic projections are given by

$$\varphi_N^{-1}(x_1, \ldots, x_n) = \frac{1}{(\sum_{i=1}^n x_i^2) + 1}\left(2x_1, \ldots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

$$\varphi_S^{-1}(x_1, \ldots, x_n) = \frac{1}{(\sum_{i=1}^n x_i^2) + 1}\left(2x_1, \ldots, 2x_n, -\left(\sum_{i=1}^n x_i^2\right) + 1\right).$$

Note that φ_N and φ_S are homeomorphisms that map open sets of $\mathbb{S}^n$ to open sets of $\mathbb{R}^n$ (regarding $\mathbb{R}^n$ as a topological space equipped with the usual topology). So, (U_N, φ_N) and (U_S, φ_S) are charts. Furthermore, if we let $U_N = \mathbb{S}^n - \{N\}$ and $U_S = \mathbb{S}^n - \{S\}$, we see that (1) $\varphi_N(U_N) = \mathbb{R}^n$ and $\varphi_S(U_S) = \mathbb{R}^n$, (2) $\{U_N, U_S\}$ is an open cover for $\mathbb{S}^n$, and (3) it is easily checked that on the overlap,

$$U_N \cap U_S = \mathbb{S}^n - \{N, S\},$$

the transition maps,

$$\varphi_{SN} = \varphi_S \circ \varphi_N^{-1} \quad \text{and} \quad \varphi_{NS} = \varphi_N \circ \varphi_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

which is a smooth bijection on $\mathbb{R}^n - \{0\}$. So, (U_N, φ_N) and (U_S, φ_S) form a smooth n-atlas on $\mathbb{S}^n$.

Now, let us consider the curve $C \subset \mathbb{R}^2$ given by the zero locus of equation $y^2 = x^2 - x^3$ (*i.e.*, a *nodal cubic*):

$$C = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; y^2 = x^2 - x^3 \right\}.$$

This curve is not a manifold. The reason is that the curve has a self-intersection at the origin (see Fig. 2.2). If C were a manifold, then there would be a connected open subset, $U \subset C$, containing the origin $O = (0, 0)$, namely the intersection of a small enough open disk centered at O with C, and a local chart, (U, φ), with $\varphi : U \to \Omega$, where Ω is some connected open subset of $\mathbb{R}$ (that is, an open interval), since φ is a homeomorphism. However, $U - \{O\}$ consists of four disconnected components and $\Omega - \varphi(O)$ of two disconnected components, contradicting the fact that φ is a homeomorphism.

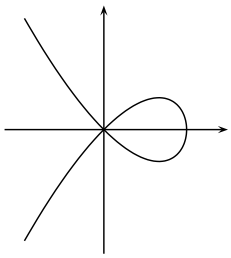

Fig. 2.2. A nodal cubic is not a manifold.

3. Sets of gluing data for manifolds

The definition of a manifold (see Definition 2.5) assumes that the topological space, M, is already known. However, there are situations of practical interest in which we only have some indirect information about the overlap of the domains, U_i, of the local charts in terms of the transition maps,

$$\varphi_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

but where the manifold M itself is not known. This is the case when trying to *build* a smooth surface to approximate a mesh in $\mathbb{R}^3$ [13,15–18,1]. If we let $\Omega_{ij} = \varphi_i(U_i \cap U_j)$ and $\Omega_{ji} = \varphi_j(U_i \cap U_j)$, then

$$\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$$

can be viewed as a "gluing map" between two open subsets, Ω_{ij} and Ω_{ji}, of Ω_i and Ω_j, respectively. Remarkably, manifolds can be constructed from what we often call "gluing data" using the "gluing process" alluded to above. It is important to note that if the Ω_{ij} arise from the charts of a manifold, then nonempty triple intersections $U_i \cap U_j \cap U_k$ of domains of charts have images $\varphi_i(U_i \cap U_j \cap U_k)$ in Ω_i, $\varphi_j(U_i \cap U_j \cap U_k)$ in Ω_j, and $\varphi_k(U_i \cap U_j \cap U_k)$ in Ω_k, and since the φ_i's are bijective maps, we get

$$\varphi_i(U_i \cap U_j \cap U_k) = \varphi_i(U_i \cap U_j \cap U_i \cap U_k) = \varphi_i(U_i \cap U_j) \cap \varphi_i(U_i \cap U_k) = \Omega_{ij} \cap \Omega_{ik},$$

and similarly

$$\varphi_j(U_i \cap U_j \cap U_k) = \Omega_{ji} \cap \Omega_{jk}, \quad \varphi_k(U_i \cap U_j \cap U_k) = \Omega_{ki} \cap \Omega_{kj},$$

and these sets are related. Indeed, we have

$$\begin{aligned}\varphi_{ji}(\Omega_{ij} \cap \Omega_{ik}) &= \varphi_j \circ \varphi_i^{-1}\big(\varphi_i(U_i \cap U_j) \cap \varphi_i(U_i \cap U_k)\big)\\ &= \varphi_j(U_i \cap U_j \cap U_k) = \Omega_{ji} \cap \Omega_{jk},\end{aligned}$$

and similar equations relating the other "triple intersections." In particular,

$$\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik},$$

which implies that

$$\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) = \varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}.$$

This is important, because $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$ is the domain of $\varphi_{kj} \circ \varphi_{ji}$ and Ω_{ik} is the domain of φ_{ki}, so the condition $\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik}$ implies that the domain of φ_{ki} is a subset of the domain of $\varphi_{kj} \circ \varphi_{ji}$. The definition of gluing data given by Grimm and Hughes [13,26] misses the above condition.

In this section, we formalize the notion of gluing data, describe the gluing process, and prove the correctness of this process in details. Our proof assumes a few mild conditions on the gluing data.

Definition 3.1. Let n be an integer with $n \geqslant 1$ and let k be either an integer with $k \geqslant 1$ or $k = \infty$. A *set of gluing data* is a triple,

$$\mathcal{G} = \big((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K}\big),$$

where I is a nonempty countable set (possibly infinite),

$$K = \big\{(i, j) \in I \times I \;\big|\; \Omega_{ij} \neq \emptyset\big\},$$

and such that the following properties hold:

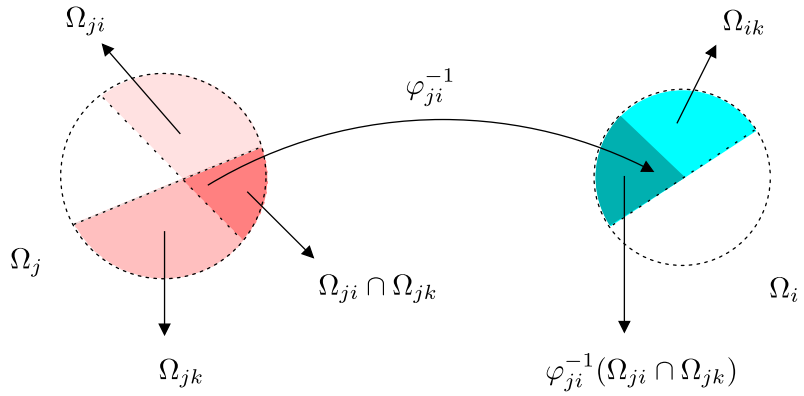

Fig. 3.1. The cocycle condition of Definition 3.1.

(1) For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{R}^n$ called *parametrization domain* or *p-domain*, and any two distinct p-domains are pairwise disjoint (*i.e.*, $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$).
(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each nonempty subset Ω_{ij} (with $i \neq j$) is called a *gluing domain*.
(3) The map

$$\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$$

is a C^k bijection for every $(i, j) \in K$ called a *transition* (or *gluing*) map, which satisfies the following:
(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$,
(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and
(c) For all i, j, k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then $\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik}$, and $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$, for all $x \in \Omega_{ij} \cap \Omega_{ik}$.
(4) For every pair $(i, j) \in K$, with $i \neq j$, for every $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, there are open balls, V_x and V_y, centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by φ_{ji}.

There are several subtle points related to conditions 1–4 of Definition 3.1. First, we note that the index set I is assumed to be countable (finite in practical applications). For technical reasons that will become clear later, we also assume in condition 1 that any two p-domains, Ω_i and Ω_j, with $i \neq j$, are disjoint. This assumption is by no means a restriction, as I is countable. In fact, we can always map Ω_i to the unit open ball centered at $(i, \underbrace{0, \ldots, 0}_{n-1})$, for $i \in \mathbb{N}$. To do so, we can use the map

$$\beta : (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}}, \ldots, \frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)$$

which is a smooth diffeomorphism from $\mathbb{R}^n$ to the open unit ball $B_1(O, \mathbb{R}^n)$, whose inverse map is given by

$$\beta^{-1} : (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \ldots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right),$$

to map Ω_i to the unit ball centered at $O \in \mathbb{R}^n$, and then translate this ball by i units along a single axis.

In condition 3, we are only interested in the Ω_{ij}'s that are nonempty, but empty Ω_{ij}'s do arise in proofs and constructions and this is why we allow empty gluing domains in condition 2. Also, observe that $\Omega_{ij} \subseteq \Omega_i$ and $\Omega_{ji} \subseteq \Omega_j$. If $i \neq j$, then Ω_i and Ω_j are disjoint, and so are Ω_{ij} and Ω_{ji}.

Condition 3c is called the *cocycle condition*. Note that it differs from the condition given in earlier versions of this work [1,27]. It is a bit stronger, which makes proofs easier. This condition may seem overly complicated, but it is actually needed to guarantee the transitivity of the relation, $\sim$, defined in the proof of Theorem 3.1. The problem is that $\varphi_{kj} \circ \varphi_{ji}$ is a partial function whose domain, $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$, is not necessarily related to the domain, Ω_{ik}, of φ_{ki}. To ensure transitivity of $\sim$, we must assert that whenever the composition $\varphi_{kj} \circ \varphi_{ji}$ has a nonempty domain, this domain is contained in the domain, Ω_{ik}, of φ_{ki} (refer to Fig. 3.1), and that $\varphi_{kj} \circ \varphi_{ji}$ and φ_{ki} agree in $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.

Since the φ_{ji}'s are bijective, condition 3c implies conditions 3a and 3b. In fact, to get condition 3a, set $i = j = k$. Then, condition 3b follows immediately from conditions 3a and 3c by letting $k = i$.

Finally, condition 4 is a necessary and sufficient condition for the space obtained by gluing the p-domains to be Hausdorff. We shall prove this statement later on. Fig. 3.2 illustrates condition 4.

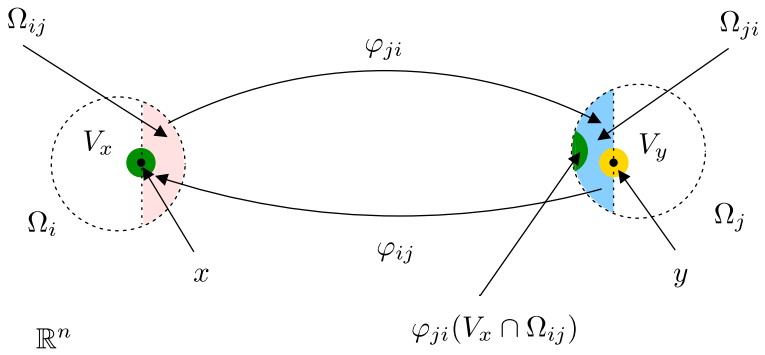

Fig. 3.2. The Hausdorff condition (condition 4) of Definition 3.1.

The idea of defining gluing data for manifolds can be traced back to André Weil, who introduced it to define abstract algebraic varieties by gluing irreducible affine sets in his book [14] published in 1946 (Chapter VII, Section 3, p. 179). The same idea is well-known in bundle theory and can be found in standard texts such as Steenrod [28]. The beauty of the idea is that it allows the reconstruction of a manifold without having prior knowledge of its topology, but by gluing open subsets of $\mathbb{R}^n$ (the Ω_i's) according to prescribed gluing instructions (namely, glue Ω_i and Ω_j by identifying Ω_{ij} and Ω_{ji} using φ_{ji}).

The gluing process clearly separates the local structure of the manifold (given by the Ω_i's) from its global structure, which is specified by the gluing functions. Furthermore, this method ensures that the resulting manifold is C^k (even for $k = \infty$) with no extra effort since the φ_{ji}'s are assumed to be C^k. Grimm and Hughes [13,26] were the first to have realized the power of the gluing process for practical applications, and also to propose a manifold-based approach for fitting smooth surfaces to meshes in $\mathbb{R}^3$. However, the cocycle condition given in the definition of a set of gluing data in [13,26] is not strong enough to ensure transitivity of the relation $\sim$. In addition, Grimm [26] uses a condition stronger than our condition 4 to ensure that the resulting space is Hausdorff. We will come back to these issues later. A correct definition of a set of gluing data, along with a necessary and sufficient Hausdorff condition (*i.e.*, condition 4 of Definition 3.1), is among the main contributions of this paper.

Let us now prove that a C^k manifold can be defined from a set of gluing data in a *natural* way:

Theorem 3.1. *For every set of gluing data,*

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big),$$

there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition maps are the φ_{ji}'s.

Proof. Define the binary relation, $\sim$, on the disjoint union, $\coprod_{i\in I}\Omega_i$, of the open sets, Ω_i, as follows: For all $x, y \in \coprod_{i\in I}\Omega_i$,

$$x \sim y \quad \text{iff} \quad \big(\exists (i,j)\in K\big)\big(x\in\Omega_{ij}, y\in\Omega_{ji}, y = \varphi_{ji}(x)\big).$$

Note that if $x \sim y$ and $x \neq y$, then $i \neq j$, as $\varphi_{ii} = \mathrm{id}$. But then, as $x \in \Omega_{ij} \subseteq \Omega_i$, $y \in \Omega_{ji} \subseteq \Omega_j$ and $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$, if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$. We claim that $\sim$ is an equivalence relation. This follows easily from the cocycle condition. Clearly, condition 3a of Definition 3.1 ensures reflexivity, while condition 3b ensures symmetry. To check transitivity, assume that $x \sim y$ and $y \sim z$. Then, there are some i, j, k such that (i) $x \in \Omega_{ij}$, $y \in \Omega_{ji} \cap \Omega_{jk}$, $z \in \Omega_{kj}$, and (ii) $y = \varphi_{ji}(x)$ and $z = \varphi_{kj}(y)$. Consequently, $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$ and $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$, so by 3c, we get $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \subseteq \Omega_{ik}$. So, $\varphi_{ki}(x)$ is defined and by 3c again, $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x) = z$, *i.e.*, $x \sim z$, as desired.

Since $\sim$ is an equivalence relation, let

$$M_{\mathcal{G}} = \Big(\coprod_{i\in I}\Omega_i\Big)/\sim$$

be the quotient set and let $p : \coprod_{i\in I}\Omega_i \to M_{\mathcal{G}}$ be the quotient map, with $p(x) = [x]$, where $[x]$ denotes the equivalence class of x (see Fig. 3.3). Also, for every $i \in I$, let $\mathrm{in}_i : \Omega_i \to \coprod_{i\in I}\Omega_i$ be the natural injection and let

$$\tau_i = p \circ \mathrm{in}_i : \Omega_i \to M_{\mathcal{G}}.$$

Since we already noted that if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$, we can conclude that every τ_i is injective. We give $M_{\mathcal{G}}$ the coarsest topology that makes the bijections, $\tau_i : \Omega_i \to \tau_i(\Omega_i)$, into homeomorphisms. Then, if we let $U_i = \tau_i(\Omega_i)$ and $\varphi_i = \tau_i^{-1}$, it is immediately verified that the (U_i, φ_i) are charts and that this collection of charts forms a C^k atlas for $M_{\mathcal{G}}$. As there are countably many charts, $M_{\mathcal{G}}$ is second-countable.

To prove that the topology is Hausdorff, we first prove the following:

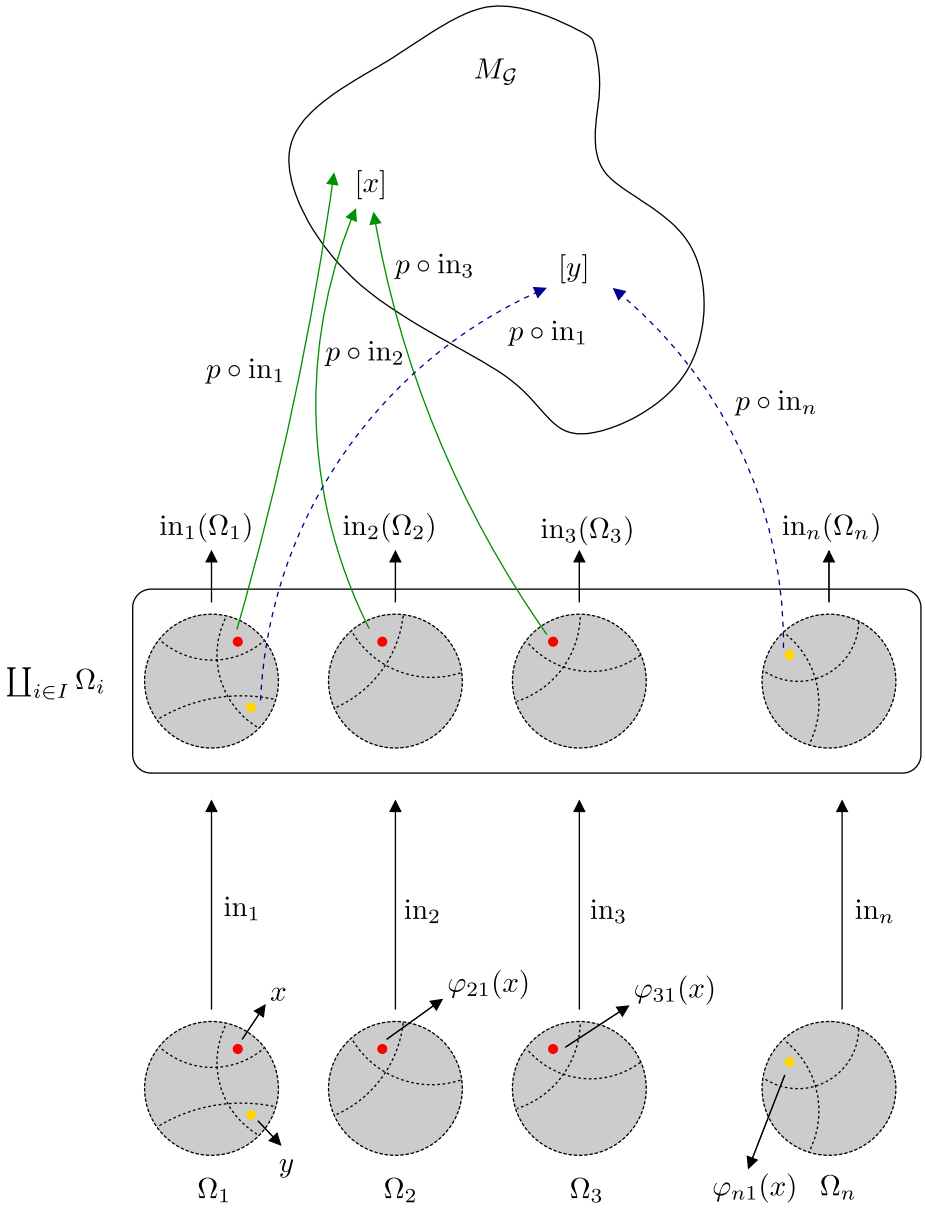

Fig. 3.3. The quotient construction.

Claim. For all $(i, j) \in I \times I$, we have $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ iff $(i, j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Assume that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ and let $[z] \in \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$. Observe that $[z] \in \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$ iff $z \sim x$ and $z \sim y$, for some $x \in \Omega_i$ and some $y \in \Omega_j$. Consequently, $x \sim y$, which implies that $(i, j) \in K$, $x \in \Omega_{ij}$ and $y \in \Omega_{ji}$. We have $[z] \in \tau_i(\Omega_{ij})$ iff $z \sim x$, for some $x \in \Omega_{ij}$. Then, either $i = j$ and $z = x$ or $i \neq j$ and $z \in \Omega_{ji}$, which shows that $[z] \in \tau_j(\Omega_{ji})$ and consequently, we get $\tau_i(\Omega_{ij}) \subseteq \tau_j(\Omega_{ji})$. Since the same argument applies by interchanging i and j, we have that $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$, for all $(i, j) \in K$. Furthermore, because $\Omega_{ij} \subseteq \Omega_i$, $\Omega_{ji} \subseteq \Omega_j$, and $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$, for all $(i, j) \in K$, we also have that $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}) \subseteq \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$, for all $(i, j) \in K$.

For the reverse inclusion, if $[z] \in \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$, then we know that there is some $x \in \Omega_{ij}$ and some $y \in \Omega_{ji}$ such that $z \sim x$ and $z \sim y$, so $[z] = [x] \in \tau_i(\Omega_{ij})$ and $[z] = [y] \in \tau_j(\Omega_{ji})$, and then we get

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \subseteq \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

This proves that if $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$, then $(i, j) \in K$ and

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Finally, assume that $(i, j) \in K$. Then, for any $x \in \Omega_{ij} \subseteq \Omega_i$, we have $y = \varphi_{ji}(x) \in \Omega_{ji} \subseteq \Omega_j$ and $x \sim y$, so that $\tau_i(x) = \tau_j(y)$, which proves that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$. So, our claim is true, and we can use it.

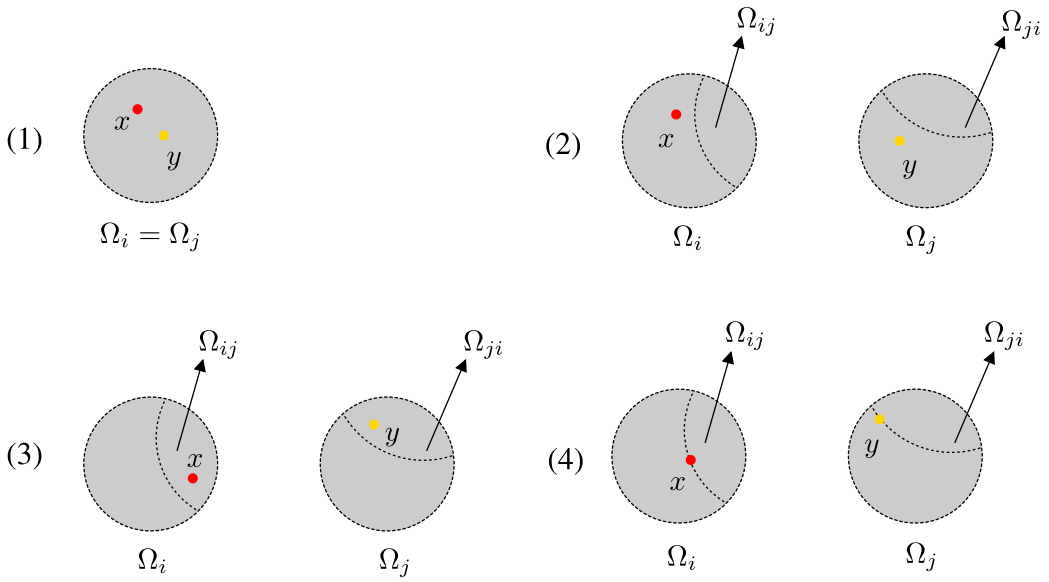

Fig. 3.4. The four cases of the proof of condition 4 of Definition 3.1.

We now prove that the topology of $M_{\mathcal{G}}$ is Hausdorff. Pick $[x], [y] \in M_{\mathcal{G}}$ with $[x] \neq [y]$, for some $x \in \Omega_i$ and some $y \in \Omega_j$. Either $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \emptyset$, in which case, as τ_i and τ_j are homeomorphisms, $[x]$ and $[y]$ belong to the two disjoint open sets $\tau_i(\Omega_i)$ and $\tau_j(\Omega_j)$. If not, then by the Claim, $(i, j) \in K$ and

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

There are several cases to consider (refer to Fig. 3.4):

(1) If $i = j$ then x and y can be separated by disjoint opens, V_x and V_y, and as τ_i is a homeomorphism, $[x]$ and $[y]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_j(V_y)$.
(2) If $i \neq j$, $x \in \Omega_i - \overline{\Omega_{ij}}$ and $y \in \Omega_j - \overline{\Omega_{ji}}$, then $\tau_i(\Omega_i - \overline{\Omega_{ij}})$ and $\tau_j(\Omega_j - \overline{\Omega_{ji}})$ are disjoint open subsets separating $[x]$ and $[y]$, where $\overline{\Omega_{ij}}$ and $\overline{\Omega_{ji}}$ are the closures of Ω_{ij} and Ω_{ji}, respectively.
(3) If $i \neq j$, $x \in \Omega_{ij}$ and $y \in \Omega_{ji}$, as $[x] \neq [y]$ and $y \sim \varphi_{ij}(y)$, then $x \neq \varphi_{ij}(y)$. We can separate x and $\varphi_{ij}(y)$ by disjoint open subsets, V_x and V_y, and $[x]$ and $[y] = [\varphi_{ij}(y)]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_i(V_{\varphi_{ij}(y)})$.
(4) If $i \neq j$, $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, then we use condition 4 of Definition 3.1. This condition yields two disjoint open subsets, V_x and V_y, with $x \in V_x$ and $y \in V_y$, such that no point of $V_x \cap \Omega_{ij}$ is equivalent to any point of $V_y \cap \Omega_{ji}$, and so $\tau_i(V_x)$ and $\tau_j(V_y)$ are disjoint open subsets separating $[x]$ and $[y]$.

Therefore, the topology of $M_{\mathcal{G}}$ is Hausdorff and $M_{\mathcal{G}}$ is indeed a manifold. Finally, it is trivial to verify that the transition maps of $M_{\mathcal{G}}$ are the original gluing functions, φ_{ij}, since $\varphi_i = \tau_i^{-1}$ and $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1}$. □

In what follows, we show that condition 4 (the Hausdorff condition) of Definition 3.1 is necessary, and we also show that the cocycle condition given by Grimm in [13,26] does not ensure the transitivity of $\sim$.

Consider the open intervals $\Omega_1 = (-3, -1)$, $\Omega_2 = (1, 3)$, $\Omega_{12} = (-3, -2)$, and $\Omega_{21} = (2, 1)$ in $\mathbb{R}$, and let $\varphi_{21}(x) = x + 4$ and $\varphi_{12}(x) = x - 4$ be the gluing functions that identify Ω_{12} and Ω_{21}. The space, C, resulting from this gluing data is a curve looking like a "fork". Note that the gluing data does not satisfy condition 4 of Definition 3.1 for $x = -2$ in $\partial(\Omega_{12}) \cap \Omega_1$ and $y = 2$ in $\partial(\Omega_{21}) \cap \Omega_2$. But, the images of -2 and 2 in C cannot be separated, as the images in C of any two open intervals $(-2 - \epsilon, -2 + \epsilon)$ and $(2 - \eta, 2 + \eta)$, with $\epsilon, \eta > 0$, always intersect since $(-2 - \min(\epsilon, \eta), -2)$ and $(2 - \min(\epsilon, \eta), 2)$ are identified. So, C is not a Hausdorff space, and thus condition 4 is indeed necessary.

Grimm [26] (p. 40) uses a condition stronger than our condition 4 to ensure that the quotient, $M_{\mathcal{G}}$, is Hausdorff, namely, that for all $(i, j) \in K$ with $i \neq j$, the quotient $(\Omega_i \amalg \Omega_j)/\sim$ should be embeddable in $\mathbb{R}^n$. This is a stronger condition, which for instance prevents us from obtaining a 2-sphere by gluing two open disks in $\mathbb{R}^2$ along an annulus (as described in [26], Appendix C2, p. 126).

Remark 3.2. Readers familiar with fiber bundles may wonder why the cocycle condition 3c of Definition 3.1 is more arcane than the corresponding definition found in bundle theory. The reason is that if $\pi : E \to B$ is a (smooth or C^k) fiber bundle with fiber, F, then there is some open cover, (U_α), of the base space, B, and for every index, α, there is a *local trivialization map*, namely a diffeomorphism,

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F,$$

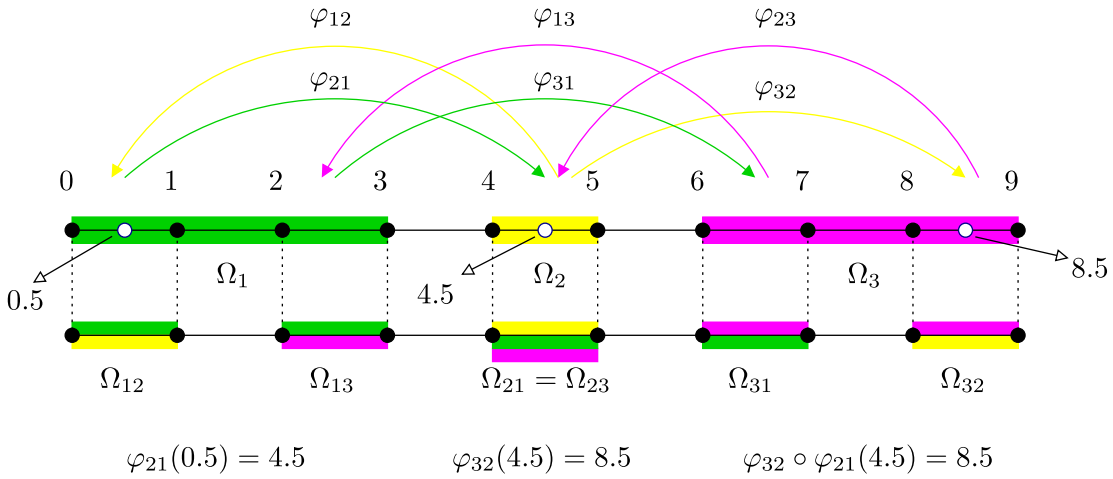

Fig. 3.5. An example to show why condition (c′) does not ensure the transitivity of $\sim$.

such that

$$\pi = p_1 \circ \varphi_\alpha,$$

where $p_1 : U_\alpha \times F \to U_\alpha$ is the projection onto U_α. Then, whenever $U_\alpha \cap U_\beta \neq \emptyset$, we have a map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F,$$

and because $\pi = p_1 \circ \varphi_\alpha$ for all α, there is a map,

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F),$$

where $\mathrm{Diff}(F)$ denotes the group of diffeomorphisms of the fiber, F, such that

$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, p) = \big(b, g_{\beta\alpha}(b)(p)\big),$$

for all $b \in U_\alpha \cap U_\beta$ and all $p \in F$. The maps, $g_{\beta\alpha}$, are the *transition maps* of the bundle. Observe that for *all* $b \in U_\alpha \cap U_\beta$, the maps, $g_{\beta\alpha}(b)$, have *the same domain* and *the same range*, F. So, whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, for all $b \in U_\alpha \cap U_\beta \cap U_\gamma$, the maps $g_{\beta\alpha}$, $g_{\gamma\beta}$ and $g_{\gamma\alpha}$ have the same domain and the same range. Consequently, in this case, the cocycle condition can be simply stated as

$$g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha},$$

without taking any precautions about the domains of these maps. However, in our situation (a manifold), the transition maps are of the form $\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$, where the Ω_{ij} are various unrelated open subsets of $\mathbb{R}^n$, and so, the composite map, $\varphi_{kj} \circ \varphi_{ji}$ only makes sense on a subset of Ω_{ij} (the domain of φ_{ji}). However, this subset need not be contained in the domain of φ_{ki}. So, in order to avoid the extra complications we saw before, the constraints in condition 3c of Definition 3.1 must be imposed.

The cocycle condition given by Grimm in [26] (p. 40) and [13] (p. 361) is stated as follows:

(c′) For all $x \in \Omega_{ij} \cap \Omega_{ik}$,

$$\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x).$$

This condition is not strong enough to imply transitivity of the relation $\sim$. In fact, consider the open intervals $\Omega_1 = (0, 3)$, $\Omega_2 = (4, 5)$, $\Omega_3 = (6, 9)$, $\Omega_{12} = (0, 1)$, $\Omega_{13} = (2, 3)$, $\Omega_{21} = \Omega_{23} = (4, 5)$, $\Omega_{32} = (8, 9)$, and $\Omega_{31} = (6, 7)$ in $\mathbb{R}$, and the gluing functions $\varphi_{21}(x) = x + 4$, $\varphi_{32}(x) = x + 4$, and $\varphi_{31}(x) = x + 4$ (refer to Fig. 3.5). Note that the pairwise gluing yields Hausdorff spaces. Clearly, $\varphi_{32} \circ \varphi_{21}(x) = x + 8$, for all $x \in \Omega_{12}$, but $\Omega_{12} \cap \Omega_{13} = \emptyset$. So, we get $0.5 \sim 4.5 \sim 8.5$. However, $0.5 \nsim 8.5$ since $\varphi_{31}(0.5)$ is undefined. The problem is that because $\Omega_{12} \cap \Omega_{13} = \emptyset$, condition (c′) is vacuous, and it is never checked that the nonempty domain of $\varphi_{32} \circ \varphi_{21}$ is a subset of the domain of φ_{31}.

Remark 3.3. In reconstructing a fiber bundle from B and the transition maps, $g_{\beta\alpha}$, we use the $g_{\beta\alpha}$ to glue the spaces $U_\alpha \times F$ and $U_\beta \times F$ along $(U_\alpha \cap U_\beta) \times F$, where two points (a, p) and (b, q) in $(U_\alpha \cap U_\beta) \times F$ are identified iff $a = b$ and $q = g_{\beta\alpha}(a)(p)$. In reconstructing a manifold from a set of gluing data, we glue the open sets Ω_i and Ω_j along Ω_{ij} and Ω_{ji}, which are identified using the maps, φ_{ji}.

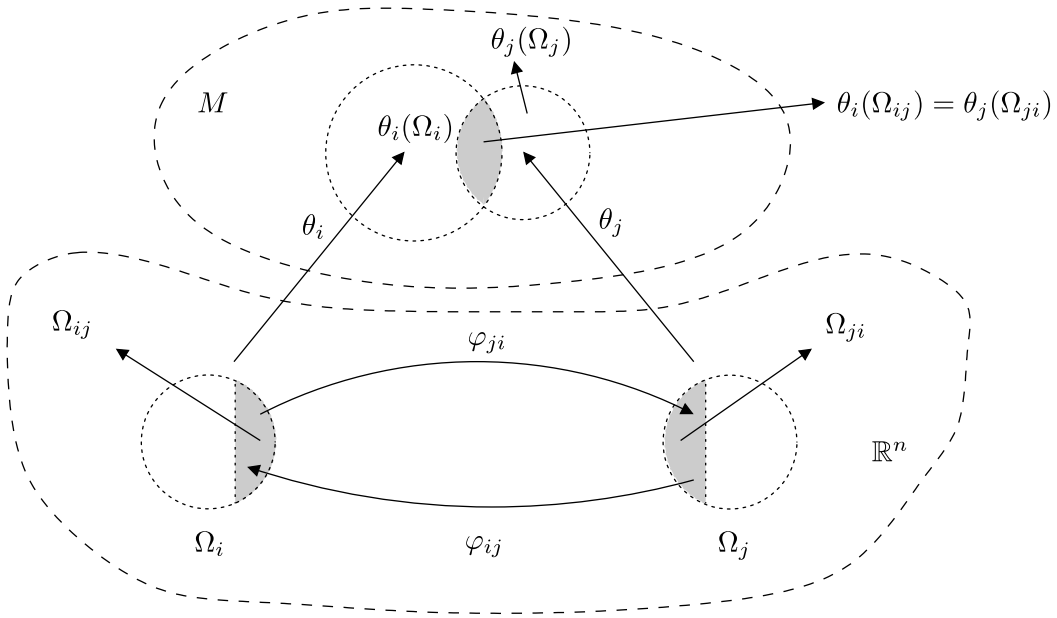

Fig. 4.1. A manifold induced from a set of gluing data.

4. Manifolds from sets of gluing data

The proof of Theorem 3.1 gives us a theoretical construction, which yields an "abstract" manifold, $M_{\mathcal{G}}$, but does not yield any information on the geometry of this manifold. In addition, the manifold $M_{\mathcal{G}}$ may not be orientable nor compact, even if we start with a finite set of p-domains. In practice, we often need a compact and orientable manifold embedded in $\mathbb{R}^n$, for some small integer n, with a prescribed geometry. In this section, we define such a "concrete" manifold from a set of gluing data.

Given a set of gluing data

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big),$$

it is natural to consider the collection of those (abstract) manifolds M that are parametrized by maps

$$\theta_i : \Omega_i \to M,$$

whose domains are the Ω_i's and whose transition maps are given by the φ_{ji}'s, that is, such that

$$\varphi_{ji} = \theta_j^{-1} \circ \theta_i.$$

We say that such manifolds are *induced* by the set of gluing data, $\mathcal{G}$. Fig. 4.1 illustrates this notion.

Recall from the proof of Theorem 3.1 that the parametrization maps τ_i's of the manifold $M_{\mathcal{G}}$ satisfy the condition

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$$

iff $(i, j) \in K$, and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Furthermore, they also satisfy the consistency condition,

$$\tau_i = \tau_j \circ \varphi_{ji} \quad \text{on } \Omega_{ij},$$

for all $(i, j) \in K$. If M is a manifold induced by the set of gluing data, $\mathcal{G}$, because the θ_i's are injective maps and $\varphi_{ji} = \theta_j^{-1} \circ \theta_i$, the two properties stated above for the τ_i's also hold for the θ_i's. In practice, however, it is often hard to ensure injectivity of the θ_i's [1]. Fortunately, we can still define a useful class of spaces from gluing data and parametrizations maps that are not necessarily injective. Roughly speaking, the gluing data specify the topology and the parametrizations define the geometry of the space. These spaces, called *parametric pseudo-manifolds* (or simply, PPMs), are not quite manifolds, but they have successfully been used as such in several applications [29,6,30,19].

4.1. Parametric pseudo-manifolds

Parametric pseudo-manifolds are topological spaces induced from gluing data which have two distinguishing properties: they are embedded in $\mathbb{R}^d$, for some positive integer d, *i.e.*, the image of each parametrization map θ_i is a subset of $\mathbb{R}^d$, and the parametrization maps themselves are not necessarily injective.

Definition 4.1. Let n, d, and k be three integers with $d > n \geqslant 1$ and $k \geqslant 1$ or $k = \infty$. A *parametric C^k pseudo-manifold of dimension n in $\mathbb{R}^d$* (for short, *parametric pseudo-manifold* or PPM) is a pair,

$$\mathcal{M} = \big(\mathcal{G}, (\theta_i)_{i\in I}\big),$$

such that

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$$

is a set of gluing data, for some finite set I, and each θ_i is a C^k function, $\theta_i : \Omega_i \to \mathbb{R}^d$, that satisfies condition C stated below:

(C) For all $(i, j) \in K$, we have

$$\theta_i(x) = (\theta_j \circ \varphi_{ji})(x) \quad \text{for all } x \in \Omega_{ij}.$$

We call θ_i a *parametrization*. The subset, $M \subset \mathbb{R}^d$, given by

$$M = \bigcup_{i\in I} \theta_i(\Omega_i)$$

is called the *image* of the parametric pseudo-manifold, $\mathcal{M}$. Whenever $n = 2$ and $d = 3$, we say that $\mathcal{M}$ is a *parametric pseudo-surface* (or PPS, for short), and that M, the image of $\mathcal{M}$, is a *pseudo-surface*.

Condition C obviously implies that

$$\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}),$$

for all $(i, j) \in K$. Consequently, θ_i and θ_j are consistent parametrizations of the overlap $\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ij})$. Thus, the set M, whatever it is, is covered by pieces, $U_i = \theta_i(\Omega_i)$, not necessarily open, such that each U_i is parametrized by θ_i, and each overlapping piece, $U_i \cap U_j$, is parametrized consistently. The local structure of M is given by the θ_i's and its global structure is given by the gluing data. More importantly, we can equip M with a manifold structure if we require the θ_i's to be injective and to satisfy

(C′) For all $(i, j) \in K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}),$$

which means that $\theta_i(\Omega_i)$ and $\theta_j(\Omega_j)$ do not intersect for any points outside Ω_{ij} or Ω_{ji}.

(C″) For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset,$$

which means that the images of disjoint parametrization domains are also disjoint.

Even if the θ_i's are not injective, properties C′ and C″ are still desirable since they ensure that $\theta_i(\Omega_i - \Omega_{ij}) = \theta_i(\Omega_i) - \theta_i(\Omega_{ij})$ and $\theta_j(\Omega_j - \Omega_{ji}) = \theta_j(\Omega_j) - \theta_j(\Omega_{ji})$ are disjoint, and thus unambiguously parametrized. Unfortunately, properties C′ and C″ seem to be difficult to enforce in practice by some automatic procedure (at least for surface constructions based on the gluing process [13,15,16,18,1]). Interestingly, regardless whether conditions C′ and C″ are satisfied, we can still show that M is the image in $\mathbb{R}^d$ of the abstract manifold, $M_{\mathcal{G}}$, as stated by Proposition 4.1 below:

Proposition 4.1. *Let $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i\in I})$ be a parametric C^k pseudo-manifold of dimension n in $\mathbb{R}^d$, where $\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K})$ is a set of gluing data, for some finite set I. Then, the parametrization maps, θ_i, induce a surjective map, $\Theta : M_{\mathcal{G}} \to M$, from the abstract manifold, $M_{\mathcal{G}}$, specified by $\mathcal{G}$ to the image, $M \subseteq \mathbb{R}^d$, of the parametric pseudo-manifold, $\mathcal{M}$, and the following property holds*:

$$\theta_i = \Theta \circ \tau_i,$$

for every Ω_i, where $\tau_i : \Omega_i \to M_{\mathcal{G}}$ are the parametrization maps of the manifold $M_{\mathcal{G}}$ (see Theorem 3.1). In particular, every manifold, $M \subset \mathbb{R}^d$, such that M is induced by $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by a map

$$\Theta : M_{\mathcal{G}} \to M.$$

Proof. Recall that

$$M_{\mathcal{G}} = \Big(\coprod_{i\in I} \Omega_i\Big)/\!\sim,$$

where $\sim$ is the equivalence relation defined so that, for all $x, y \in \coprod_{i\in I} \Omega_i$,

$$x \sim y \quad \text{iff} \quad \big(\exists (i,j) \in K\big)\big(x \in \Omega_{ij},\, y \in \Omega_{ji},\, y = \varphi_{ji}(x)\big).$$

From the proof of Theorem 3.1, we have that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ iff $(i,j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

In particular,

$$\tau_i(\Omega_i - \Omega_{ij}) \cap \tau_j(\Omega_j - \Omega_{ji}) = \emptyset$$

for all $(i,j) \in I \times I$ ($\Omega_{ij} = \Omega_{ji} = \emptyset$ when $(i,j) \notin K$). These properties with the fact that the τ_i's are injections for all $(i,j) \notin K$ suggest defining $\Theta_i : \tau_i(\Omega_i) \to \mathbb{R}^d$ and $\Theta_j : \tau_j(\Omega_j) \to \mathbb{R}^d$ in three pieces: let

$$\Theta_i\big([x]\big) = \theta_i(x), \quad x \in \Omega_i - \Omega_{ij}$$

on $\tau_i(\Omega_i - \Omega_{ij})$, and let

$$\Theta_j\big([y]\big) = \theta_i(y), \quad y \in \Omega_j - \Omega_{ji}$$

on $\tau_j(\Omega_j - \Omega_{ji})$. On the overlap $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$, we define Θ_i and Θ_j in such a way that they agree, using condition C. Recall that condition C in Definition 4.1 says that $\theta_i(x) = \theta_j(\varphi_{ji}(x))$, for all $x \in \Omega_{ij}$. Consequently, if we define the maps Θ_i on $\tau_i(\Omega_{ij})$ and Θ_j on $\tau_j(\Omega_{ji})$ by

$$\Theta_i\big([x]\big) = \theta_i(x),\ x \in \Omega_{ij} \quad \text{and} \quad \Theta_j\big([y]\big) = \theta_j(y), \quad y \in \Omega_{ji},$$

as $x \sim \varphi_{ji}(x)$, we have

$$\Theta_i\big([x]\big) = \theta_i(x) = \theta_j\big(\varphi_{ji}(x)\big) = \Theta_j\big(\big[\varphi_{ji}(x)\big]\big) = \Theta_j\big([x]\big),$$

which means that Θ_i and Θ_j agree on $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$. But then, the functions, Θ_i, agree whenever their domains overlap and consequently, they patch to yield a function, Θ, with domain $M_{\mathcal{G}}$ and image M, as desired. Finally, since $\theta_i = \Theta \circ \tau_i$ (by construction), and since any manifold in $\mathbb{R}^d$ induced by $\mathcal{G}$ is a PPM, every manifold, M, in $\mathbb{R}^d$ induced by $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by a map $\Theta : M_{\mathcal{G}} \to M$. □

4.2. Equivalence of gluing data and isomorphic manifolds

To end our discussion on the definition of manifolds from gluing data, we show that it is possible to characterize isomorphism between two manifolds *induced by the same set of gluing data* in terms of a condition on their transition maps. This characterization suggests a notion of equivalence on sets of gluing data.

This notion of equivalence stated in Definition 4.2 is directly inspired by the definition of the equivalence of fiber bundles *over the same base B and using the same open cover of B*, discussed in Hirzebruch [31] and Chern [32]. It may be possible to come up with a more general definition of equivalence involving different sets of gluing data and to obtain a generalization of Proposition 4.3. We were unable to come up with such a definition, and we leave this as a topic of further research. With our definition, any two manifolds induced by two equivalent sets of gluing data are indeed isomorphic.

Proposition 4.2. *Given any set of gluing data,*

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big),$$

for any two manifolds M and M' induced by $\mathcal{G}$ given by families of parametrizations $(\Omega_i, \theta_i)_{i\in I}$ and $(\Omega_i, \theta'_i)_{i\in I}$, respectively, if $f : M \to M'$ is a C^k isomorphism, then there are C^k bijections,

$$\rho_i : W_{ij} \to W'_{ij},$$

for some open subsets $W_{ij}, W'_{ij} \subseteq \Omega_i$, such that

$$\varphi'_{ji}(x) = \rho_j \circ \varphi_{ji} \circ \rho_i^{-1}(x), \quad \text{for all } x \in W_{ij},$$

with $\varphi_{ji} = \theta_j^{-1} \circ \theta_i$ and $\varphi'_{ji} = \theta_j'^{-1} \circ \theta'_i$. Furthermore, $\rho_i = (\theta_i'^{-1} \circ f \circ \theta_i) \mid W_{ij}$ and if $\theta_i'^{-1} \circ f \circ \theta_i$ is a bijection from Ω_i to itself and $\theta_i'^{-1} \circ f \circ \theta_i(\Omega_{ij}) = \Omega_{ij}$, for all i, j, then $W_{ij} = W'_{ij} = \Omega_i$.

Proof. The composition $\theta_i'^{-1} \circ f \circ \theta_i$ is actually a partial function with domain

$$\operatorname{dom}\big(\theta_i'^{-1} \circ f \circ \theta_i\big) = \big\{x \in \Omega_i \mid \theta_i(x) \in f^{-1} \circ \theta_i(\Omega_i)\big\}$$

and its "inverse" $\theta_i^{-1} \circ f^{-1} \circ \theta_i'$ is a partial function with domain

$$\operatorname{dom}\big(\theta_i^{-1} \circ f^{-1} \circ \theta_i'\big) = \big\{x \in \Omega_i \mid \theta_i'(x) \in f \circ \theta_i(\Omega_i)\big\}.$$

The composition $\theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ \theta_i^{-1} \circ f^{-1} \circ \theta_i'$ is also a partial function and we let

$$W_{ij} = \Omega_{ij} \cap \operatorname{dom}\big(\theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ \theta_i^{-1} \circ f^{-1} \circ \theta_i'\big), \quad \rho_i = \big(\theta_i'^{-1} \circ f \circ \theta_i\big)\big|W_{ij}, \quad \text{and} \quad W_{ij}' = \rho_i(W_{ij}).$$

Observe that $\theta_j \circ \varphi_{ji} = \theta_j \circ \theta_j^{-1} \circ \theta_i = \theta_i$; that is,

$$\theta_i = \theta_j \circ \varphi_{ji}.$$

Using this, on W_{ij}, we get

$$\begin{aligned}
\rho_j \circ \varphi_{ji} \circ \rho_i^{-1} &= \theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ \big(\theta_i'^{-1} \circ f \circ \theta_i\big)^{-1} \\
&= \theta_j'^{-1} \circ f \circ \theta_j \circ \varphi_{ji} \circ \theta_i^{-1} \circ f^{-1} \circ \theta_i' \\
&= \theta_j'^{-1} \circ f \circ \theta_i \circ \theta_i^{-1} \circ f^{-1} \circ \theta_i' \\
&= \theta_j'^{-1} \circ \theta_i' \\
&= \varphi_{ji}',
\end{aligned}$$

as claimed. The last part of the proposition is clear. □

Proposition 4.2 suggests defining the following notion of equivalence on sets of gluing data:

Definition 4.2. Two sets of gluing data,

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big) \quad \text{and} \quad \mathcal{G}' = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, \big(\varphi_{ji}'\big)_{(i,j)\in K}\big),$$

over the same p-domains and gluing domains, Ω_i's and Ω_{ij}'s, are *equivalent* iff there is a family of C^k bijections,

$$(\rho_i : \Omega_i \to \Omega_i)_{i\in I}$$

such that for all $i, j \in I$ and for all $x \in \Omega_{ij}$,

$$\rho_i(\Omega_{ij}) = \Omega_{ij} \quad \text{and} \quad \varphi_{ji}'(x) = \rho_j \circ \varphi_{ji} \circ \rho_i^{-1}(x).$$

Based on the notion of equivalence of gluing data given by Definition 4.2, we prove the converse of Proposition 4.2:

Proposition 4.3. *If two sets of gluing data,*

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big) \quad \textit{and} \quad \mathcal{G}' = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, \big(\varphi_{ji}'\big)_{(i,j)\in K}\big),$$

are equivalent, then there is a C^k isomorphism, $f : M_{\mathcal{G}} \to M_{\mathcal{G}'}$, between the manifolds induced by $\mathcal{G}$ and $\mathcal{G}'$. Furthermore,

$$f \circ \tau_i = \tau_i' \circ \rho_i,$$

for all $i \in I$.

Proof. Let $f_i : \tau_i(\Omega_i) \to \tau_i'(\Omega_i)$ be the C^k bijection given by

$$f_i = \tau_i' \circ \rho_i \circ \tau_i^{-1},$$

where the $\rho_i : \Omega_i \to \Omega_i$'s are the maps giving the equivalence of $\mathcal{G}$ and $\mathcal{G}'$. If we prove that f_i and f_j agree on the overlap, $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) \cap \tau_j(\Omega_{ji})$, then the f_i patch and yield a C^k isomorphism, $f : M_{\mathcal{G}} \to M_{\mathcal{G}'}$. Since $\mathcal{G}$ and $\mathcal{G}'$ are equivalent, we know that

$$\varphi_{ji}' \circ \rho_i = \rho_j \circ \varphi_{ji}.$$

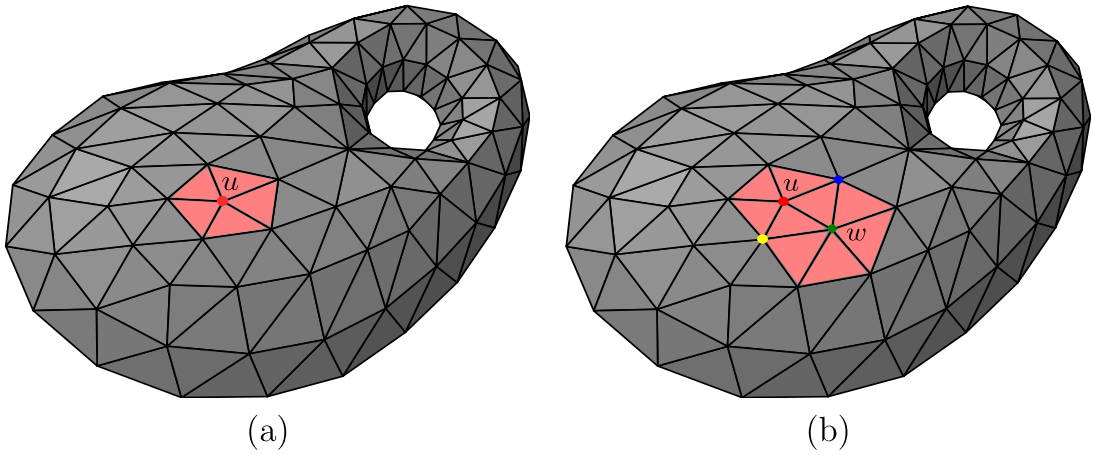

Fig. 5.1. (a) The star of a vertex of a triangle mesh. (b) The stars of two adjacent vertices.

But, we also know that

$$\tau_i' = \tau_j' \circ \varphi_{ji}'.$$

Consequently, for every $[x] \in \tau_j(\Omega_{ji}) = \tau_i(\Omega_{ij})$, with $x \in \Omega_{ij}$, we have

$$\begin{aligned} f_j([x]) &= \tau_j' \circ \rho_j \circ \tau_j^{-1}([x]) \\ &= \tau_j' \circ \rho_j \circ \tau_j^{-1}([\varphi_{ji}(x)]) \\ &= \tau_j' \circ \rho_j \circ \varphi_{ji}(x) \\ &= \tau_j' \circ \varphi_{ji}' \circ \rho_i(x) \\ &= \tau_i' \circ \rho_i(x) \\ &= \tau_i' \circ \rho_i \circ \tau_i^{-1}([x]) \\ &= f_i([x]), \end{aligned}$$

which shows that f_i and f_j agree on $\tau_i(\Omega_i) \cap \tau_j(\Omega_j)$, as claimed. □

5. Building a "concrete" set of gluing data

This section describes a construction for defining a "concrete" set of gluing data from a given simplicial surface in $\mathbb{R}^3$. Such a surface is known as a "triangle mesh" in the graphics and engineering literature. Our construction uses the connectivity information of the given triangle mesh to define the set of gluing data. In [1], we show how to define a PPS from this set of gluing data. We start with an informal description of our construction, while pointing out the problems we ran into during the process of defining the construction. We also motivate the decisions we made to overcome these difficulties.

5.1. Informal description of the method

Suppose you are flying over a triangle mesh (or mesh, for short), and refer to Fig. 5.1. Looking down at the mesh, around every vertex u, we see a star of triangles (*i.e.*, the collection of triangles incident to u along with their edges and vertices). For every pair of adjacent vertices u and w, there are two stars that overlap in a (generally nonflat) quadrangle consisting of two triangles sharing the edge $[u, w]$, as shown in Fig. 5.1b. Thus, it is natural to associate to every vertex, u, of our mesh a p-domain Ω_u, which is a flattened version of the star of u, namely the interior of a regular polygon of radius $\cos(\pi/m_u)$, where m_u is the degree of u. There is a natural piecewise affine map s_u which maps the star of u to that p-domain, so that the image of every point v on the star of u is $s_u(v)$. In particular, $s_u(u)$ is the center of Ω_u. Now, given two adjacent vertices u and w on the mesh, the overlap of the stars of u and w is a quadrangle, say $[u, v, w, z]$, and this quadrangle corresponds to two quadrangles: $[s_u(u), s_u(v), s_u(w), s_u(z)]$ in Ω_u and $[s_w(w), s_w(z), s_w(u), s_w(v)]$ in Ω_w (see Fig. 5.2).

It is then natural to let Ω_{uw} be the interior of $[s_u(u), s_u(v), s_u(w), s_u(z)]$ and Ω_{wu} be the interior of $[s_w(w), s_w(z), s_w(u), s_w(v)]$. To simplify notation, we assume that some vertex u_0 is chosen on the star of u, so that we can denote the vertices of the star by $u_0, \ldots, u_{m_u-1}$ (listed according to some cyclic traversal of the star boundary), and we denote $s_u(u_i)$ by u_i'. We also denote $s_u(u)$ by u'.

Now, the problem is to define transition maps, $\varphi_{wu} : \Omega_{uw} \to \Omega_{wu}$, which are smooth diffeomorphisms, and to find explicit formulae for the φ_{wu}'s *that satisfy the cocycle condition*. Furthermore, we would like the φ_{wu} to be easily computable.

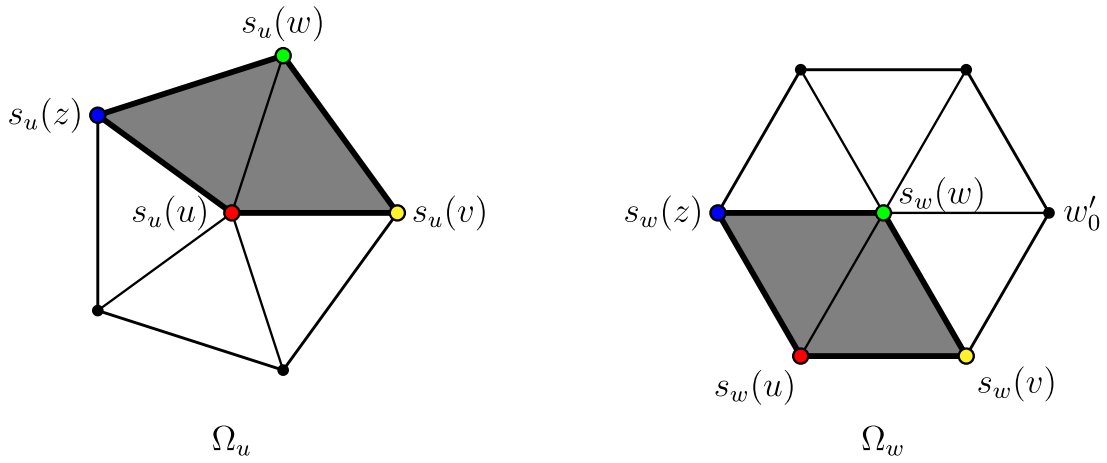

Fig. 5.2. Two p-domains corresponding to overlapping stars.

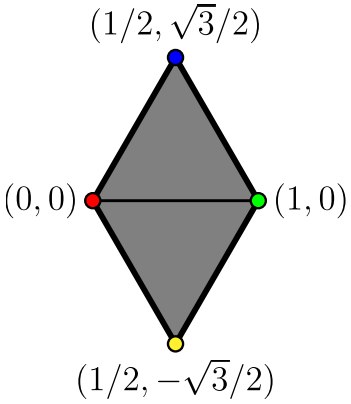

Fig. 5.3. The canonical diamond Q.

Since Ω_{uw} and Ω_{wu} are bounded open subset of the plane, at first glance, it would appear that the problem is solved by appealing to the Riemann mapping Theorem (see Ahlfors [33], Chapter 6, Section 1.1). However, there is no closed-form formula giving the Riemann mapping for quadrangles. Instead, a conformal diffeomorphism is given by the Schwarz–Christoffel formula, which involves an integral (see Ahlfors [33], Chapter 6, Section 2.2). To make things worse, we have not been able to prove that such a formula yields a diffeomorphism satisfying the cocycle condition. We believe that this is a difficult problem, and we leave it for future research.

In view of the above considerations, we will seek an approach where we modify the open subsets Ω_u and Ω_{uw} a little bit. A similar strategy was used by Ying and Zorin in [16]. Indeed, they construct transition functions whose domains are unions of concave or convex sectors. These domains are not convex in general. As we will see, our p-domains have the advantage of being always convex. While there is only one type of gluing domain in our construction, theirs has two types of gluing domains.[2] Moreover, Ying and Zorin consider quadrangle meshes and their function g_u (see Definition 5.5) rescales the polar radius r using an exponential. Their g_u function is conformal; ours is not, but our g_u has the advantage of being linear in polar coordinates, and thus less expensive to compute (and invert).

Let us go back to an informal description of our approach. First, we can simplify the problem of finding the smooth diffeomorphisms φ_{wu} by reducing it to finding a diffeomorphism between a quadrangle whose angle at the origin is $4\pi/m_u$ and the *canonical diamond* Q, consisting of two equilateral triangles, whose vertices are $(0,0)$, $(1/2, -\sqrt{3}/2)$, $(1,0)$, $(1/2, \sqrt{3}/2)$, as shown in Fig. 5.3.

Let $R_{(u,w)}$ be the rotation of center $(0,0)$ which maps the edge $[u', s_u(w)]$ onto the edge $[u', u'_0]$, and refer to Fig. 5.4. Now, rotate the quadrilateral $[s_u(u), s_u(v), s_u(w), s_u(z)]$ in Ω_u using $R_{(u,w)}$, obtaining the quadrilateral $[u', u'_{m_u-1}, u'_0, u'_1]$ that "stands upright." At this stage, assume that we are in the possession of a diffeomorphism g_u that maps the interior of $[u', u'_{m_u-1}, u'_0, u'_1]$ to the interior of the canonical diamond Q. Similarly, we can rotate the quadrilateral $[s_w(w), s_w(z), s_w(u), s_w(v)]$ in Ω_w by the rotation $R_{(w,u)}$, obtaining the quadrilateral $[w', w'_{m_w-1}, w'_0, w'_1]$, and then map this quadrilateral into the canonical diamond Q using g_w. However, observe that because under the rotation $R_{(u,w)}$ the vertex v goes to u'_{m_u-1} (resp. the vertex z goes to u'_1), and under the rotation $R_{(w,u)}$ the vertex v goes to w'_1 (resp. the

[2] This information is not explicitly given in their paper.

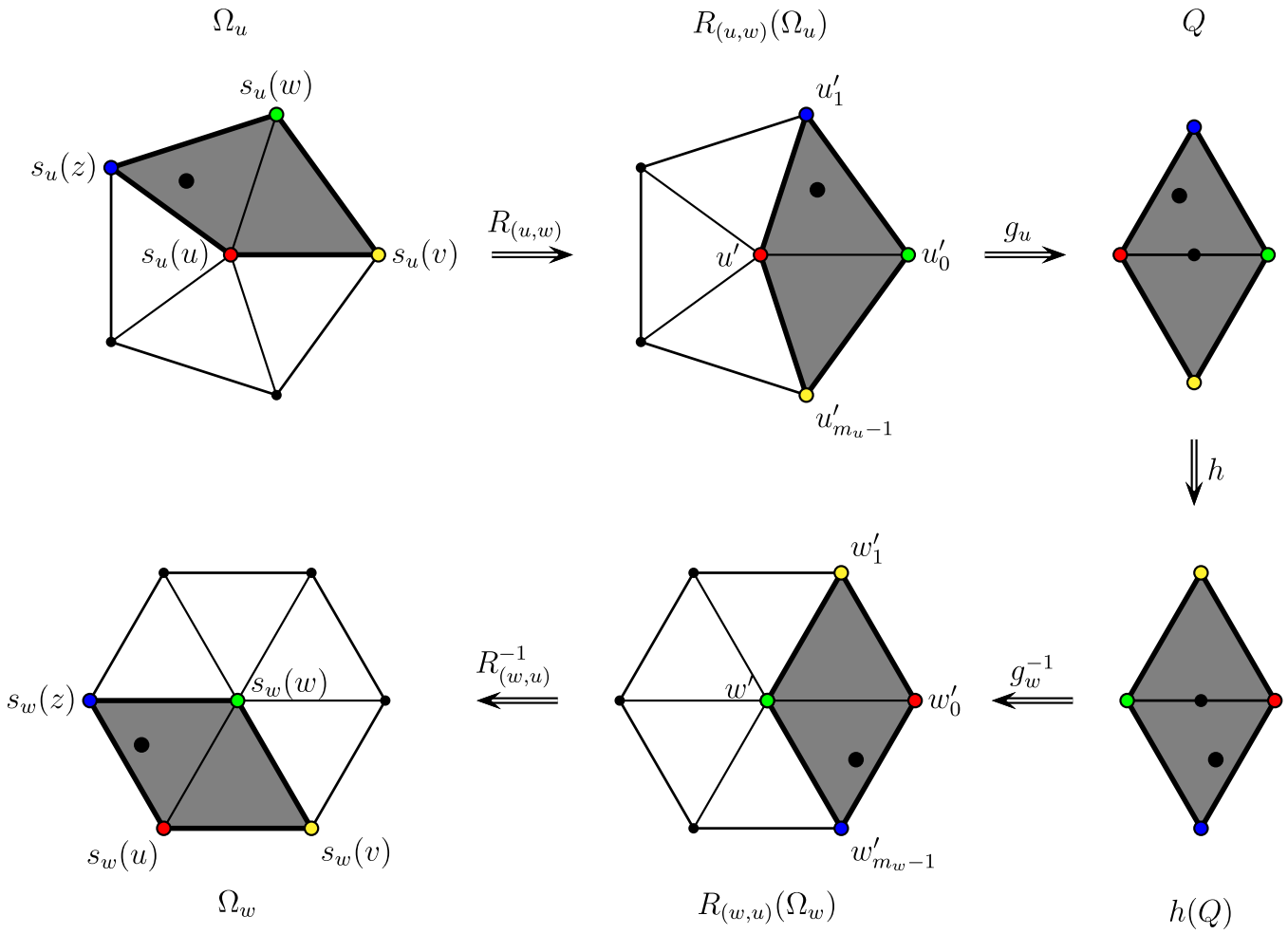

Fig. 5.4. The transition function φ_{wu}.

vertex z goes to u'_{m_z-1}), the images of v and z under the rotations $R_{(u,w)}$ and $R_{(w,u)}$ are flipped, and similarly for the images of u and w. The solution is simple: in order for $g_u \circ R_{(u,w)}$ and $g_w \circ R_{(w,u)}$ to yield the same result, rotate Q by π around the point $(1/2, 0)$. Let h denote this rotation. In summary, we can now define our diffeomorphism φ_{wu} by

$$\varphi_{wu} = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)},$$

with $\varphi_{uu} = \mathrm{id}$. So, the problem is reduced to finding diffeomorphisms g_u that satisfy the cocycle condition.

We must show that

$$\varphi_{wu} = \varphi_{wx} \circ \varphi_{xu},$$

whenever the right-hand side is defined (which turns out to imply that the left-hand side is also defined). Since

$$\varphi_{wu} = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)},$$
$$\varphi_{xu} = R^{-1}_{(x,u)} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)},$$
$$\varphi_{wx} = R^{-1}_{(w,x)} \circ g_w^{-1} \circ h \circ g_x \circ R_{(x,w)},$$

we get

$$\varphi_{wx} \circ \varphi_{xu} = R^{-1}_{(w,x)} \circ g_w^{-1} \circ h \circ g_x \circ R_{(x,w)} \circ R^{-1}_{(x,u)} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)},$$

and hence we must show that right-hand side above is equal to $\varphi_{wu} = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}$.

Without loss of generality, suppose that $s_u(x)$ follows $s_u(w)$ in a counterclockwise traversal. This means that $s_w(u)$ follows $s_w(x)$ in a counterclockwise traversal, and that $s_x(w)$ follows $s_x(u)$ in a counterclockwise traversal. First, note that

$$R_{(x,w)} \circ R^{-1}_{(x,u)} = M_{-\frac{2\pi}{m_x}},$$

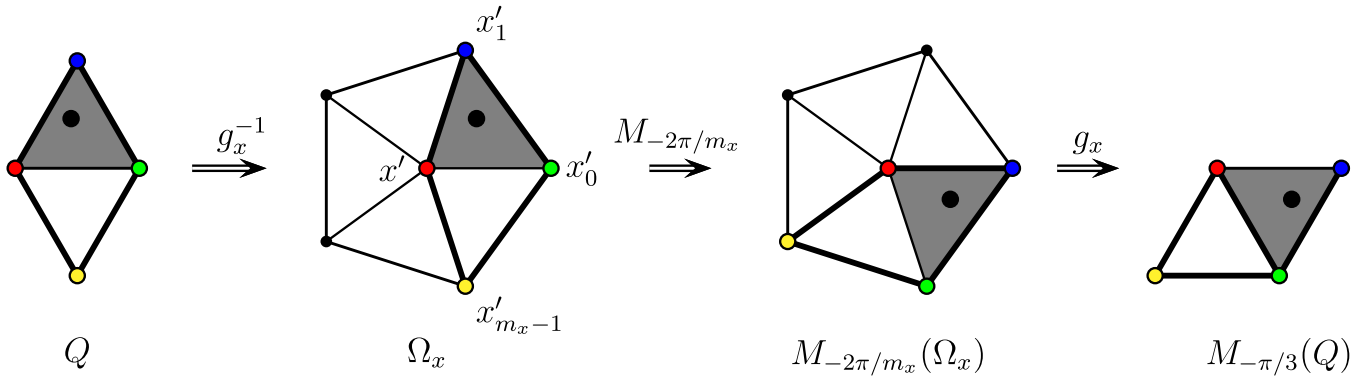

Fig. 5.5. Property A.

where $M_{-\frac{2\pi}{m_x}}$ is the rotation of center $(0,0)$ and angle $-2\pi/m_x$, as $s_x(w)$ follows $s_x(u)$ in a counterclockwise traversal, where m_x is the degree of x. We get

$$\varphi_{wx}\circ\varphi_{xu} = R^{-1}_{(w,x)}\circ g_w^{-1}\circ h\circ g_x\circ M_{-\frac{2\pi}{m_x}}\circ g_x^{-1}\circ h\circ g_u\circ R_{(u,x)}.$$

In order to make progress, it would be nice if we could simplify the middle term, $g_x\circ M_{-\frac{2\pi}{m_x}}\circ g_x^{-1}$. In fact, if we look at Fig. 5.5, we see that it would be desirable to assume that the middle term is

$$g_x\circ M_{-\frac{2\pi}{m_x}}\circ g_x^{-1} = M_{-\frac{\pi}{3}}, \tag{A}$$

where $M_{-\frac{\pi}{3}}$ is the rotation of center $(0,0)$ and angle $-\pi/3$. If we do so, we get the simplified expression

$$R^{-1}_{(w,x)}\circ g_w^{-1}\circ h\circ g_x\circ M_{-\frac{2\pi}{m_x}}\circ g_x^{-1}\circ h\circ g_u\circ R_{(u,x)} = R^{-1}_{(w,x)}\circ g_w^{-1}\circ h\circ M_{-\frac{\pi}{3}}\circ h\circ g_u\circ R_{(u,x)}.$$

We seem to be stuck, but it turns out that property A implies the following identity (named property B):

$$g_u\circ R_{(u,x)} = M_{-\frac{\pi}{3}}\circ g_u\circ R_{(u,w)}, \tag{B}$$

as shown in Proposition 5.1. Similarly, we have

$$g_w\circ R_{(w,x)} = M_{\frac{\pi}{3}}\circ g_w\circ R_{(w,u)}.$$

Using the above identities, we get

$$\begin{aligned} &R^{-1}_{(w,x)}\circ g_w^{-1}\circ h\circ M_{-\frac{\pi}{3}}\circ h\circ g_u\circ R_{(u,x)}\\ &\quad = R^{-1}_{(w,u)}\circ g_w^{-1}\circ M_{-\frac{\pi}{3}}\circ h\circ M_{-\frac{\pi}{3}}\circ h\circ M_{-\frac{\pi}{3}}\circ g_u\circ R_{(u,w)}, \end{aligned}$$

so

$$\varphi_{wx}\circ\varphi_{xu} = R^{-1}_{(w,u)}\circ g_w^{-1}\circ M_{-\frac{\pi}{3}}\circ h\circ M_{-\frac{\pi}{3}}\circ h\circ M_{-\frac{\pi}{3}}\circ g_u\circ R_{(u,w)}.$$

Luckily, the above expression can be further simplified because

$$h = M_{-\frac{\pi}{3}}\circ h\circ M_{-\frac{\pi}{3}}\circ h\circ M_{-\frac{\pi}{3}},$$

and consequently, we get

$$\varphi_{wx}\circ\varphi_{xu} = R^{-1}_{(w,u)}\circ g_w^{-1}\circ h\circ g_u\circ R_{(u,w)} = \varphi_{wu},$$

as desired (we refer the reader to Appendix A for detailed proofs). We still need to find a diffeomorphism g_u from the interior of $[u', u'_{m_u-1}, u'_0, u'_1]$ to the canonical diamond Q which satisfies property A.

It is natural to look for an expression of g_u in polar coordinates, and it turns out that the function

$$g_u(\theta, r) = \left(\frac{m_u}{6}\cdot\theta, \frac{\cos(\pi/6)}{\cos(\pi/m_u)}\cdot r\right)$$

does the job, in the sense that property A holds, and thus the corresponding φ_{wu} satisfies the cocycle condition. However, we have overlooked an important point, namely that the map g_u *does not* map the quadrilateral $[u', u'_{m_u-1}, u'_0, u'_1]$ onto Q!

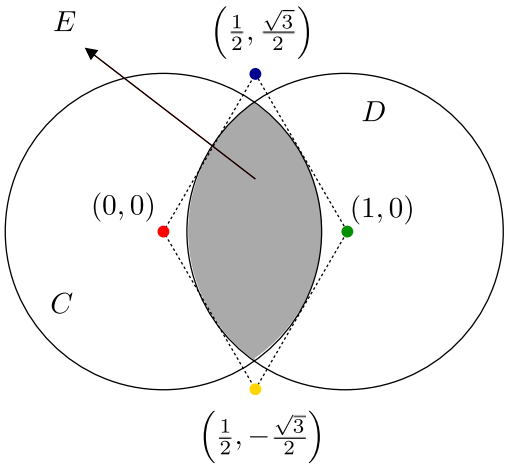

Fig. 5.6. The circles C, D, and the canonical lens E.

Indeed, the image of the straight edges $[u'_0, u'_1]$ and $[u'_0, u'_{m_u-1}]$ are *curved*, so the image of $[u', u'_{m_u-1}, u'_0, u'_1]$ under g_u is a diamond with curved edges; it is concave if $m_u \geqslant 6$, and convex otherwise. In order to fix this problem, we need to modify Ω_u or Q, so that g_u is a bijection onto Q. In particular, there are at least two ways of fixing the problem:

(1) Keep the canonical domain Q and define Ω_u as the union of m_u copies (obtained by rotations of $2k\pi/m_u$ around the origin, for $k = 1, \ldots, m_u - 1$) of the inverse image of the upper triangle of Q by g_u.
(2) Define Ω_u as the interior of the *inscribed circle* of radius $\cos(\pi/m_u)$ in the regular polygon corresponding to the old version of Ω_u, and use the *canonical lens* E instead of Q, where E is the intersection of the two open disks C and D, where C is the circle of center $(0, 0)$ and radius $\cos(\pi/6)$ and D is the circle of center $(1, 0)$ and radius $\cos(\pi/6)$, as shown in Fig. 5.6.

If we choose option (1), we obtain nonconvex p-domains that consist of concave or convex sectors (unless $n_u = 6$). We prefer the second option because the p-domains are simpler and convex, which is also a crucial property for constructing parametrizations over these domains using splines or other machinery (subdivision surfaces, etc.) Therefore, in the rest of this paper, we will adopt the second option for the p-domains, and we will prove rigorously that the properties of gluing data are satisfied. The first option is closely related to the method described in [16] for constructing transition functions, but quadrangle meshes are used to define the gluing data rather than triangle meshes.

5.2. Gluing data

In this section, we build the collections $(\Omega_i)_{i\in I}$, $(\Omega_{ij})_{(i,j)\in I\times I}$, and $(\varphi_{ji})_{(i,j)\in K}$ of the set of gluing data, $\mathcal{G}$, based on the idea informally presented in Section 5.1. Before presenting the details of our construction, we establish the notation that we will use for dealing with some basic notions of piecewise-linear topology. This subject can be found in standard textbooks such as the one written by Bloch [34].

The *simplex* σ spanned by the points $v_0, \ldots, v_d$ is denoted by

$$[v_0, \ldots, v_d].$$

The *dimension*, $\dim(\sigma)$, of σ is d, and σ is called a d-simplex. The *combinatorial boundary of* σ is denoted by $\mathrm{bd}(\sigma)$, and the *combinatorial interior of* σ is denoted by $\mathrm{int}(\sigma)$. The *underlying space* of a simplicial complex $\mathcal{K}$ is denoted by $|\mathcal{K}|$. Given a simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$, for any simplex σ in $\mathcal{K}$, we denote the *star* of σ by $\mathrm{st}(\sigma, \mathcal{K})$, and the *link* of σ by $\mathrm{lk}(\sigma, \mathcal{K})$. For each integer i, with $0 \leqslant i \leqslant \dim(\mathcal{K})$, we define $\mathcal{K}^{(i)}$ to be the simplicial complex consisting of all j-simplices of $\mathcal{K}$ with $0 \leqslant j \leqslant i$.

Now, let $\mathcal{K}$ be any given simplicial surface in $\mathbb{R}^3$. Our goal is to define a set of gluing data from $\mathcal{K}$, say

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big).$$

As we said before, each p-domain, Ω_i, in $(\Omega_i)_{i\in I}$ is the interior of a circle in $\mathbb{R}^2$, while each gluing domain, Ω_{ij}, in $(\Omega_{ij})_{(i,j)\in I\times I}$ is defined by means of two abstractions, namely, a P-polygon and its canonical triangulation, together with a composition of bijective maps. From now on, we assume that the *degree* of every vertex v in $\mathcal{K}$ (*i.e.*, the number of edges of $\mathcal{K}$ having v as a 0-face) is at least 3.

Let

$$I = \{v \mid v \text{ is a vertex of } \mathcal{K}\}.$$

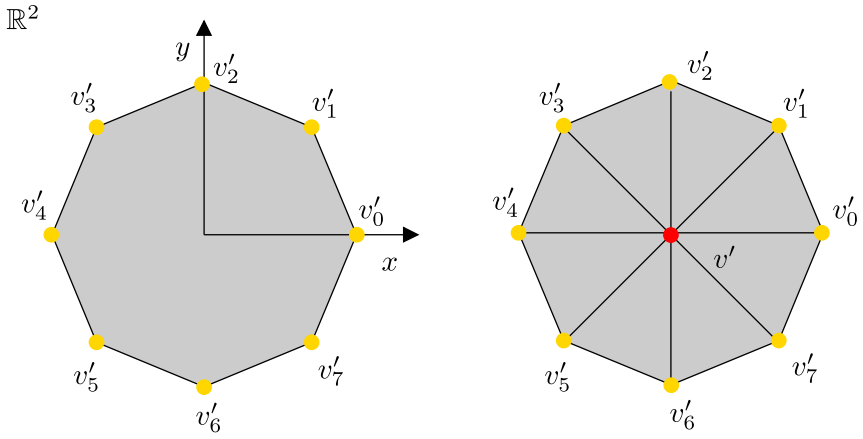

Fig. 5.7. A P-polygon (left) and its canonical triangulation (right).

Definition 5.1. For every $v \in I$, the *p-domain* Ω_v is the set

$$\Omega_v = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \left(\cos\left(\frac{\pi}{m_v} \right) \right)^2 \right\},$$

where m_v is the degree of vertex v.

Note that Ω_v is simply the interior of a circle of radius $\cos(\pi/m_v)$ centered at the origin of $\mathbb{R}^2$.

For any two $u, w \in I$, *we assume that* Ω_u *and* Ω_w *belong to distinct "copies" of* $\mathbb{R}^2$. This assumption ensures that $\Omega_u \cap \Omega_w = \emptyset$, so that condition 1 of Definition 3.1 holds. To build gluing domains and transition maps, we need the notions of P-polygon and canonical triangulation. We also need to define the transition function φ_{wu}, which is the composition of two rotations around the origin, an analytic map, a Polar to Cartesian coordinate conversion map (and its inverse), and a rotation of angle π (a double reflection), as the reader can probably guess from the discussion in Section 5.1.

Definition 5.2. For each vertex v of $\mathcal{K}$, the *P-polygon*, P_v, *associated with* v is the regular polygon in $\mathbb{R}^2$ with vertices $v'_0, \ldots, v'_{m_v-1}$, where m_v is the degree of v in $\mathcal{K}$, and such that the coordinates of v'_i are

$$\left(\cos\left(\frac{2\pi \cdot i}{m_v} \right), \sin\left(\frac{2\pi \cdot i}{m_v} \right) \right), \quad \text{for each } i \in \{0, \ldots, m_v - 1\}.$$

Moreover, we can define the *canonical triangulation,* T_v, *of* P_v as the triangulation of P_v obtained by adding the vertex $v' = (0, 0)$ to it, as well as m_v diagonals, each of which connects v' to a vertex v'_i of P_v.

Fig. 5.7 illustrates Definition 5.2.

We assume that P_v *resides in the copy of* $\mathbb{R}^2$ *that contains the p-domain* Ω_v. As a result, the p-domain Ω_v is the interior, $\text{int}(C_v)$, of the circle, C_v, inscribed in the P-polygon, P_v, *i.e.*, $\Omega_v = \text{int}(C_v)$.

Let v be a vertex in $\mathcal{K}$ of degree m_v. Since $\mathcal{K}$ is a simplicial surface, the link, $\text{lk}(v, \mathcal{K})$, of v in $\mathcal{K}$ is homeomorphic to $\mathbb{S}^1$. So, $\text{lk}(v, \mathcal{K})$ is a simple, closed polygonal chain in $\mathbb{R}^3$. Let $v_0, \ldots, v_{m_v-1}$ be any enumeration of the vertices of $\text{lk}(v, \mathcal{K})$ such that $[v_i, v_{i+1}]$ is an edge of $\text{lk}(v, \mathcal{K})$, for each $i \in \{0, \ldots, m_v - 1\}$, where the index i is considered congruent modulo m_v (unless stated otherwise).

Definition 5.3. Given $\text{st}(v, \mathcal{K})$ and the canonical triangulation, T_v, of P_v, we define the map

$$s_v : \text{st}(v, \mathcal{K})^{(0)} \to T_v^{(0)}$$

such that

$$s_v(v) = v' \quad \text{and} \quad s_v(v_i) = v'_i,$$

for every $i \in \{0, \ldots, m_v - 1\}$. So, for any $x, y, z \in \text{st}(v, \mathcal{K})$, we have that $[s_v(x), s_v(y)]$ is an edge of T_v if and only if $[x, y]$ is an edge of $\text{st}(v, \mathcal{K})$, and $[s_v(x), s_v(y), s_v(z)]$ is a triangle of T_v if and only if $[x, y, z]$ is a triangle of $\text{st}(v, \mathcal{K})$. Thus, the map s_v is a simplicial isomorphism, and $\text{st}(v, \mathcal{K})$ and T_v are isomorphic. We can extend the bijection s_v to mapping triangles in $\text{st}(v, \mathcal{K})$ onto triangles in T_v. In particular, if $\sigma = [v, v_i, v_{i+1}]$ is in $\text{st}(v, \mathcal{K})$ then $s_v(\sigma) = [v', s_v(v_i), s_v(v_{i+1})]$ is its "image" in T_v.

The conversion from Cartesian to polar coordinates and back is defined as follows:

Definition 5.4. Let

$$\Pi : \mathbb{R}^2 - \{(0,0)\} \to (-\pi, \pi] \times \mathbb{R}_+$$

be the map that converts Cartesian to polar coordinates which is given by

$$\Pi(p) = \Pi\big((x,y)\big) = (\theta, r),$$

for every $p \in \mathbb{R} - \{(0,0)\}$, where $\theta \in (-\pi, \pi]$ is the *angle* uniquely determined by

$$\cos\left(\frac{x}{r}\right) \quad \text{and} \quad \sin\left(\frac{y}{r}\right),$$

and $r \in \mathbb{R}_+$ is the *length*, with

$$r = \sqrt{x^2 + y^2}.$$

Function Π is bijective and its inverse,

$$\Pi^{-1} : (-\pi, \pi] \times \mathbb{R}_+ \to \mathbb{R}^2 - \{(0,0)\},$$

is given by

$$\Pi^{-1}\big((\theta, r)\big) = \big(r \cdot \cos(\theta), r \cdot \sin(\theta)\big).$$

Both Π and Π^{-1} are C^∞ functions. We use Π and Π^{-1} to define a map associated with each vertex of $\mathcal{K}$:

Function g_v maps Ω_u onto the circle of center $(0,0)$ and radius $\cos(\pi/6)$ and is the key to our construction, as we have already said in Section 5.1. Below, we give the formal definition of this function:

Definition 5.5. For each vertex v in I, let

$$g_v : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}$$

be given by

$$g_v(p) = \Pi^{-1} \circ f_v \circ \Pi(p)$$

for every $p \in \mathbb{R}^2 - \{(0,0)\}$, where $f_v : (-\pi, \pi] \times \mathbb{R}_+ \to (-\pi, \pi] \times \mathbb{R}_+$ is given by

$$f_v\big((\theta, r)\big) = \left(\frac{m_v}{6} \cdot \theta, \frac{\cos(\pi/6)}{\cos(\pi/m_v)} \cdot r\right),$$

(θ, r) are the polar coordinates of p and m_v is the degree of vertex v in $\mathcal{K}$.

In the context of our construction, function g_v has the following interpretation (refer to Fig. 5.8): it maps the circular sector, A, of the circle C_v inscribed in P_v, onto the circular sector, B, of the circle of radius $\cos(\pi/6)$ and centers at $(0,0)$, where A consists of $(0,0)$ and all points with polar coordinates $(\theta, r) \in [-2\pi/m_v, 2\pi/m_v] \times (0, \cos(\pi/m_v)]$, and B consists of $(0,0)$ and all points with polar coordinates $(\beta, s) \in [-\pi/3, \pi/3] \times (0, \cos(\pi/6)]$. Note that A is contained in the quadrilateral given by the vertices v', $s_v(v_{m_v-1})$, $s_v(v_0)$, and $s_v(v_1)$ of T_v. We call B the *canonical sector*.

The function g_v is bijective and its inverse,

$$g_v^{-1} : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\},$$

is given by

$$g_v^{-1}(q) = \Pi^{-1} \circ f_v^{-1} \circ \Pi(q)$$

for every $q \in \mathbb{R}^2 - \{(0,0)\}$, where $f_v^{-1} : (-\pi, \pi] \times \mathbb{R}_+ \to (-\pi, \pi] \times \mathbb{R}_+$ is given by

$$f_v^{-1}\big((\beta, s)\big) = \left(\frac{6}{m_v} \cdot \beta, \frac{\cos(\pi/m_v)}{\cos(\pi/6)} \cdot s\right),$$

(β, s) are the polar coordinates of q and m_v is the degree of vertex v in $\mathcal{K}$. Since f_v is clearly C^∞, so is g_v.

We also need the rotation h introduced in Section 5.1.

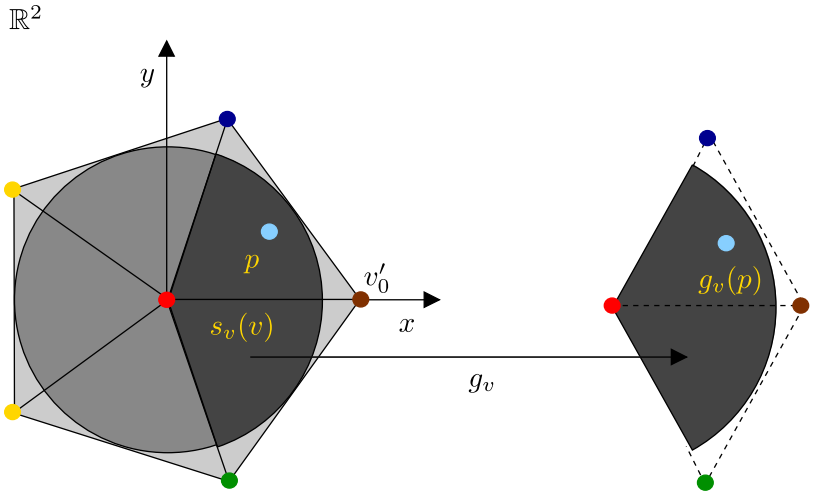

Fig. 5.8. The action of g_v upon a point $p \in C_v$.

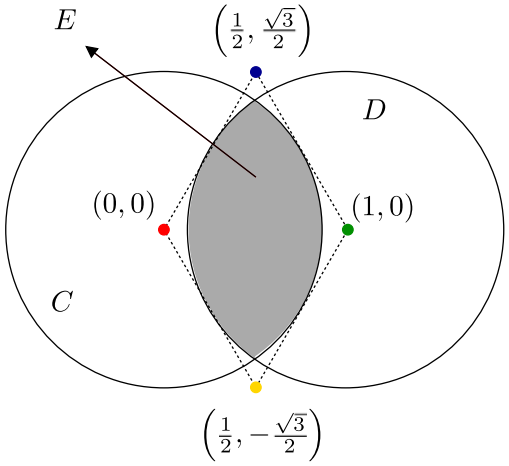

Fig. 5.9. The circles C and D, the canonical lens E, and the quadrilateral Q (drawn with dotted line).

Definition 5.6. Let

$$h : \mathbb{R}^2 \to \mathbb{R}^2$$

be the function given by

$$h(p) = h\big((x, y)\big) = (1 - x, -y),$$

for every point $p \in \mathbb{R}^2$ with Cartesian coordinates (x, y).

Function h is the rotation of center $(1/2, 0)$ and angle π. This function is a "double" reflection: $p = (x, y)$ is reflected over the line $x = 1/2$ and then over the line $y = 0$. Our transition maps are composite functions involving Π, g_v, h, rotations, and their inverses. The domains and ranges of our transition maps are more easily defined through an abstraction named the *canonical lens* (see Fig. 5.9).

More specifically, let u and w be any two vertices of $\mathcal{K}$ such that $[u, w]$ is an edge of $\mathcal{K}$, and as in Section 5.1, let $R_{(u,w)}$ denote the rotation around $(0, 0)$ that takes the edge $[s_u(u) = u', s_u(w)]$ onto the edge $[u', u_0']$ of T_u. Now, observe that $g_u \circ R_{(u,w)}$ maps $\Omega_u - \{(0, 0)\}$ onto the set $\text{int}(C) - \{(0, 0)\}$, where C is the circle of radius $\cos(\pi/6)$ and center $(0, 0)$, as illustrated by Fig. 5.9. In turn, function h maps $\text{int}(C) - \{(0, 0)\}$ onto the set $\text{int}(D) - \{(1, 0)\}$, where D is the circle of radius $\cos(\pi/6)$ and center $(1, 0)$. Finally, the composite function $R_{(w,u)}^{-1} \circ g_w^{-1}$ maps $\text{int}(C) - \{(0, 0)\}$ onto $\Omega_w - \{(0, 0)\}$.

From the above remarks, we can conclude that only the points in the set

$$\big(\text{int}(C) - \big\{(0, 0)\big\}\big) \cap \big(\text{int}(D) - \big\{(1, 0)\big\}\big) = \text{int}(C) \cap \text{int}(D)$$

are mapped by $R_{(w,u)}^{-1} \circ g_w^{-1}$ to

$$\Omega_w - \big\{(0, 0)\big\}.$$

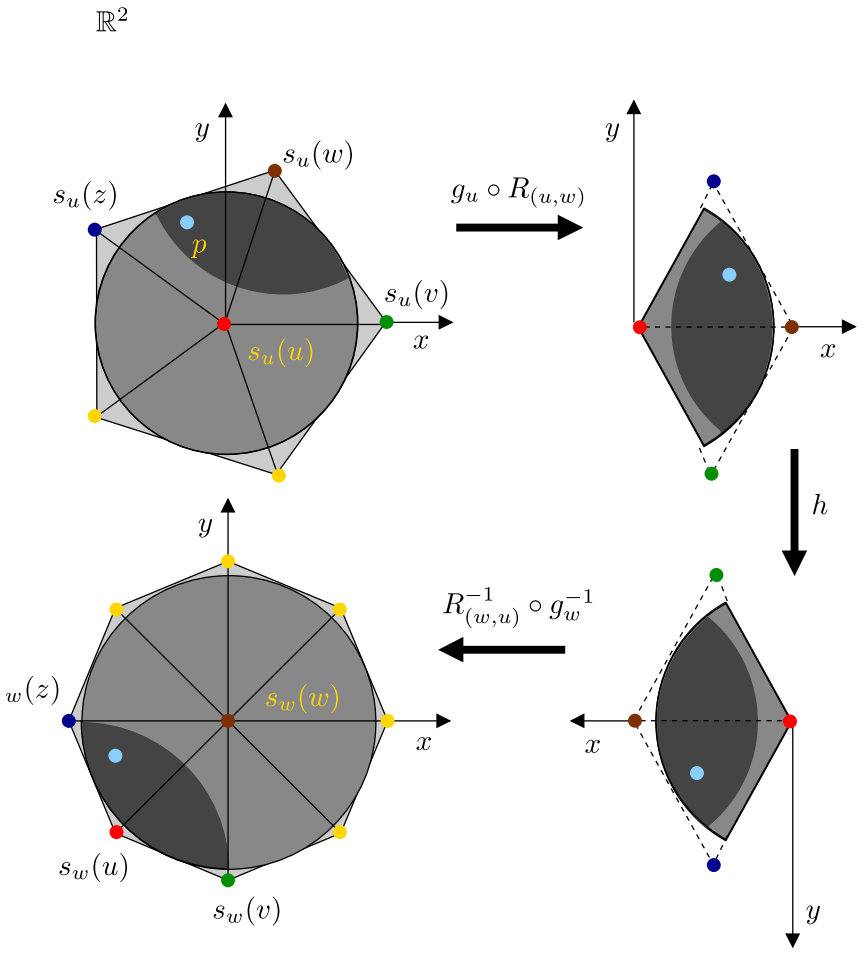

Fig. 5.10. The action of $g_{(u,w)}$ upon a point $p \in G_{uw}$, with $G_{uw} = R^{-1}_{(u,w)} \circ g_u^{-1}(E)$.

The set

$$E = \text{int}(C) \cap \text{int}(D)$$

is the so-called canonical lens. Set E (*i.e.*, the canonical lens) is contained in the quadrilateral, Q, given by the vertices with coordinates $(0,0)$, $(1/2, -\sqrt{3}/2)$, $(1,0)$, and $(1/2, \sqrt{3}/2)$, as can be seen in Fig. 5.9.

It is worth noticing that the set $\Omega_w - (0,0)$ is the image of the set $\text{int}(C) - \{(0,0)\}$ by $R^{-1}_{(w,u)} \circ g_w^{-1}$ rather than the image of the set $\text{int}(D) - \{(0,0)\}$. Using the notion of canonical lens, we define another key function of our construction, which is also a fundamental component of our transition maps.

Definition 5.7. For any two vertices u, w of I such that $[u, w]$ is an edge of $\mathcal{K}$, we define the function

$$g_{(u,w)} : G_{uw} \to G_{wu}$$

as

$$g_{(u,w)}(p) = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(p)$$

for every point $p \in G_{uw}$, where $G_{uw} = R^{-1}_{(u,w)} \circ g_u^{-1}(E)$, $G_{wu} = R^{-1}_{(w,u)} \circ g_w^{-1}(E)$, and E is the canonical lens.

Fig. 5.10 shows the action of $g_{(u,w)}$ upon a point $p \in G_{uw}$, with $G_{uw} = R^{-1}_{(u,w)} \circ g_u^{-1}(E)$.

Suppose that $[u, w, v]$ and $[u, w, z]$ are the two triangles of $\mathcal{K}$ sharing the edge $[u, w]$, where v and z are vertices of $\mathcal{K}$, with $v \neq z$. Let Q_u be the quadrilateral given by the vertices $s_u(u) = u'$, $s_u(v)$, $s_u(w)$, and $s_u(z)$. Then, the composite function $g_u \circ R_{(u,w)}$ maps the intersection $Q_u \cap (\Omega_u - \{(0,0)\})$ onto the intersection set $Q \cap (\text{int}(C) - \{(0,0)\})$. In turn, function h maps $Q \cap (\text{int}(C) - \{(0,0)\})$ onto $Q \cap (\text{int}(D) - \{(0,0)\})$. From the definition of h, the points in the upper (resp. lower) half of Q are mapped to the lower (resp. upper) half of Q. Next, the composite function $R^{-1}_{(w,u)} \circ g_w^{-1}$ maps the set $Q \cap (\text{int}(C) - \{(0,0)\})$ onto the set $Q_w \cap (\Omega_w - \{(0,0)\})$, where Q_w is the quadrilateral given by the vertices $s_w(w) = w'$,

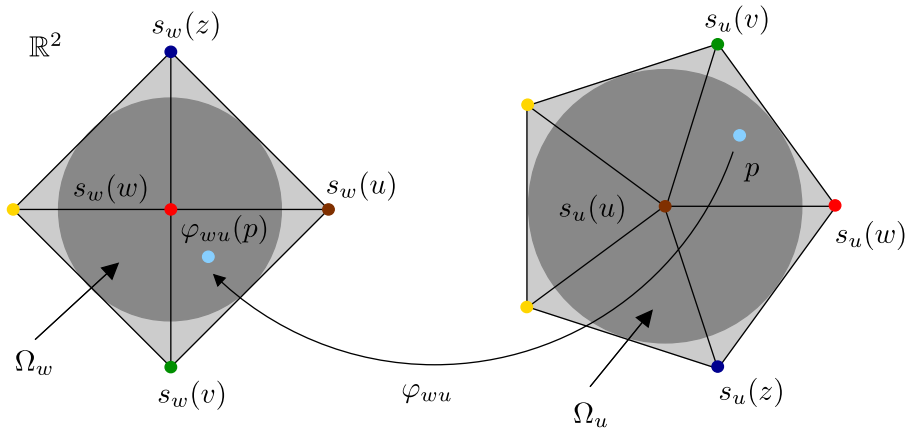

Fig. 5.11. Illustration of Definition 5.9.

$s_w(z)$, $s_w(u)$, and $s_w(v)$. However, since only the points of $Q \cap (\mathrm{int}(C) - \{(0,0)\})$ belonging to the canonical lens E are mapped to $Q_w \cap (\Omega_w - \{(0,0)\})$ by $R^{-1}_{(w,u)} \circ g_w^{-1}$, and since $E \subseteq (Q \cap \mathrm{int}(C) - \{(0,0)\})$, only the points of $Q_u \cap (\Omega_u - \{(0,0)\})$ in the subset $G_{uw} = R^{-1}_{(u,w)} \circ g_u^{-1}(E)$ of $Q_u \cap (\Omega_u - \{(0,0)\})$ are mapped to $Q_w \cap (\Omega_w - \{(0,0)\})$ by function $g_{(u,w)}$.

Observe that $g_{(u,w)}$ is bijective. Its inverse, $g^{-1}_{(u,w)} : G_{wu} \to G_{uw}$, is given by

$$g^{-1}_{(u,w)}(q) = R^{-1}_{(u,w)} \circ g_u^{-1} \circ h \circ g_w \circ R_{(w,u)}(q),$$

for every $q \in G_{wu}$. As we shall prove, $g_{(u,w)}$ is C^∞, $g_{(u,w)}(G_{uw})$ is nonempty and open in $\mathbb{R}^2$, and $g^{-1}_{(u,w)}(p) = g_{(w,u)}(p)$, for every p in $g_{(u,w)}(G_{uw})$. Finally, $g_{(u,w)}$ plays a crucial role in the following two definitions:

Definition 5.8. For any two vertices $u, w \in I$, the *gluing domain* Ω_{uw} is defined as

$$\Omega_{uw} = \begin{cases} \Omega_u & \text{if } u = w, \\ G_{uw} & \text{if } [u,w] \text{ is an edge of } \mathcal{K}, \text{ where } G_{uw} = R^{-1}_{(u,w)} \circ g_u^{-1}(E), \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

As we shall see in Appendix A, Definition 5.8 satisfies condition 2 of Definition 3.1. In addition, observe that the requirement $\Omega_{uu} = \Omega_u$, for all $u \in I$, is true by definition. So, we are left to prove that the set Ω_{uw} is open in $\mathbb{R}^2$ and $\Omega_{uw} \neq \emptyset$ if and only if $\Omega_{wu} \neq \emptyset$, for every $(u,w) \in I \times I$, with $u \neq w$.

Transition maps are bijective functions between nonempty gluing domains defined as follows:

Definition 5.9. Let K be the index set,

$$K = \{(u,w) \in I \times I \mid \Omega_{uw} \neq 0\}.$$

Then, for any pair $(u,w) \in K$, the *transition map*,

$$\varphi_{wu} : \Omega_{uw} \to \Omega_{wu},$$

is such that, for every $p \in \Omega_{uw}$, we let

$$\varphi_{wu}(p) = \begin{cases} p & \text{if } u = w, \\ g_{(u,w)}(p) & \text{otherwise.} \end{cases}$$

Fig. 5.11 illustrates Definition 5.9.

Function $g_{(u,w)}$ possesses the properties A and B described in Section 5.1, which play a crucial role in proving that the transitions maps introduced in Definition 5.9 satisfy conditions 3 and 4 of Definition 3.1. The details are given in Appendix A but we state and prove the two properties right now.

Proposition 5.1. *The maps g_u satisfy the following properties*:

(A) *For all $q \in g_u(\Omega_u)$, we have*

$$\left(g_u \circ M_{-\frac{2\pi}{m_u}} \circ g_u^{-1}\right)(q) = M_{-\frac{\pi}{3}}(q) \quad \text{and} \quad \left(g_u \circ M_{\frac{2\pi}{m_u}} \circ g_u^{-1}\right)(q) = M_{\frac{\pi}{3}}(q),$$

where $M_{-\frac{\pi}{3}}$ *(resp.* $M_{\frac{\pi}{3}}$*) is a rotation by* $-\frac{\pi}{3}$ *(resp.* $\frac{\pi}{3}$*) around the origin, and* $M_{\frac{2\pi}{m_u}}$ *(resp.* $M_{-\frac{2\pi}{m_u}}$*) is a rotation by* $\frac{2\pi}{m_u}$ *(resp.* $\frac{2\pi}{m_u}$*) around the origin, with* m_u *the degree of vertex* u *in* $\mathcal{K}$*.*

(B) *If* $s_u(w)$ *precedes* $s_u(v)$ *in a counterclockwise enumeration of the vertices of* $\mathcal{K}$*, then*

$$(g_u \circ R_{(u,w)})(p) = (M_{\frac{\pi}{3}} \circ g_u \circ R_{(u,v)})(p),$$

for all $p \in \Omega_{uw}$*.*

Proof. (A) If (α, s) and (β, t) are the polar coordinates of q and $g_u \circ M_{-\frac{2\pi}{m_u}} \circ g_u^{-1}(q)$, respectively, then the definition of g_u tells us that

$$\beta = \frac{m_u}{6} \cdot \left(-\frac{2\pi}{m_u} + \frac{6}{m_u} \cdot \alpha \right) = -\frac{\pi}{3} + \alpha$$

and

$$t = \frac{\cos(\pi/6)}{\cos(\pi/m_u)} \cdot \frac{\cos(\pi/m_u)}{\cos(\pi/6)} \cdot s = s,$$

so

$$\left(g_u \circ M_{-\frac{2\pi}{m_u}} \circ g_u^{-1}\right)(q) = M_{-\frac{\pi}{3}}(q),$$

as claimed. A similar argument with $-2\pi/m_u$ replaced by $2\pi/m_u$ proves the second identity.

(B) Since $s_u(w)$ precedes $s_u(v)$ in a counterclockwise enumeration, we have

$$R_{(u,w)} \circ R_{(u,v)}^{-1} = M_{\frac{2\pi}{m_u}}.$$

Consequently

$$(g_u \circ R_{(u,w)})(p) = (g_u \circ M_{\frac{2\pi}{m_u}} \circ R_{(u,v)})(p),$$

but using A, we have

$$g_u \circ M_{\frac{2\pi}{m_u}} = M_{\frac{\pi}{3}} \circ g_u$$

on Ω_u, so we get

$$(g_u \circ R_{(u,w)})(p) = (M_{\frac{\pi}{3}} \circ g_u \circ R_{(u,v)})(p),$$

for all $p \in \Omega_{uw}$, as claimed. □

Note that the proof of B shows that B actually follows from A, and the fundamental fact that $R_{(u,w)} \circ R_{(u,v)}^{-1} = M_{\frac{2\pi}{m_u}}$ (when $s_u(w)$ precedes $s_u(v)$ in a counterclockwise enumeration of the vertices of P_u). Also, observe that condition 3a, $\varphi_{uu} = \mathrm{id}_{\Omega_u}$, for all $u \in I$, is true by definition. So, we are left to prove conditions 3b and 3c (the *cocycle condition*). Condition 3b is proved using Proposition 5.1. Once condition 3b has been proved, it is easy to prove that condition 3c holds using Proposition 5.1 (see Lemma A.8). We also show in Appendix A that the *Hausdorff condition* (condition 4 of Definition 3.1) holds.

We developed a computer program for building sets of gluing data from simplicial surfaces.[3] Our program is based on the construction described above, and it also allows us to define a Parametric Pseudo-Surface (PPS) from the gluing data. The computational aspects of this program are detailed in [1,27]. Figs. 5.12–5.14 show approximations to the images of three PPSs generated by the program.

6. Conclusions

We gave a novel and constructive definition of gluing data, and proved that a universal manifold can always be built from a set of gluing data. Our definition fixed a flaw in the definition of the pioneering work of Grimm and Hughes [13], and provided a necessary and sufficient condition for building Hausdorff spaces from sets of gluing data. To demonstrate the applicability of our definition, we showed how to construct sets of gluing data from simplicial surfaces, and then proved the correctness of our construction. Although this construction is limited to simplicial surfaces, our definition of sets of gluing data is not. In principle, sets of gluing data can be built from other objects.

[3] http://www.dimap.ufrn.br/~mfsiqueira/Marcelo_Siqueiras_Web_Spot/Software.html.

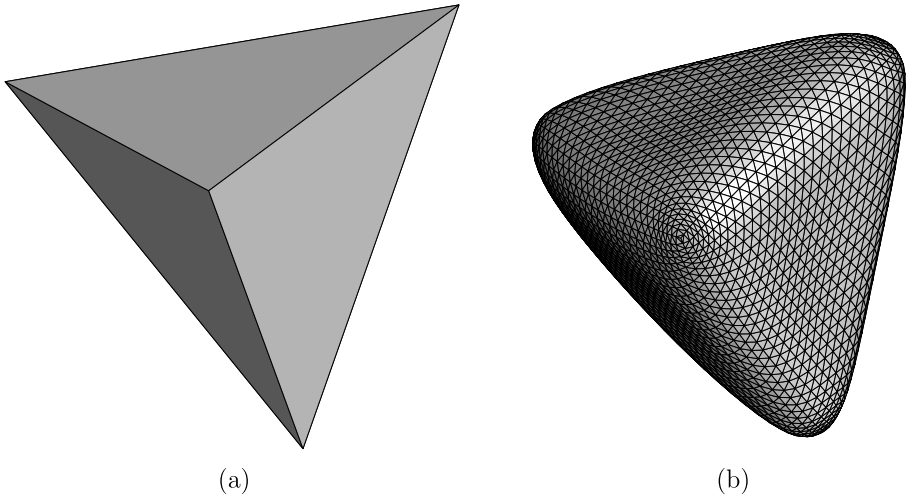

Fig. 5.12. (a) The boundary of a tetrahedron. (b) Approximation to a PPS defined from (a).

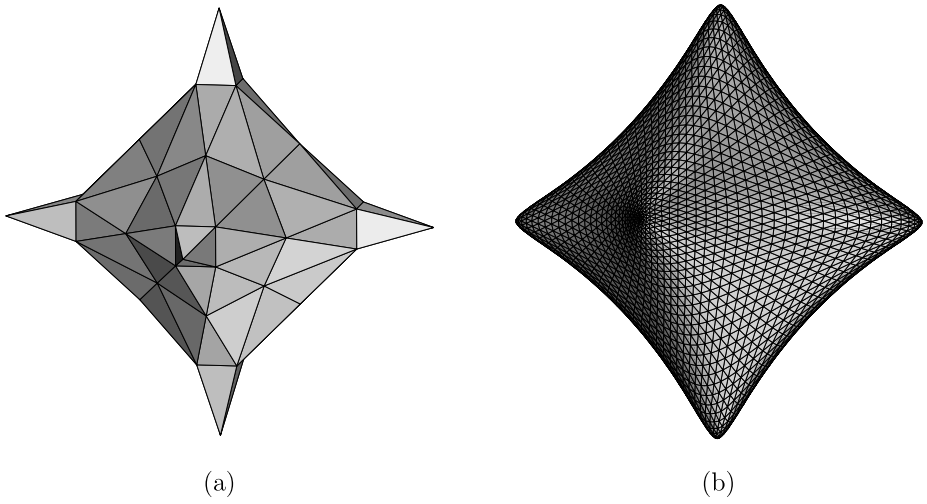

Fig. 5.13. (a) A star-shaped simplicial surface. (b) Approximation to a PPS defined from (a).

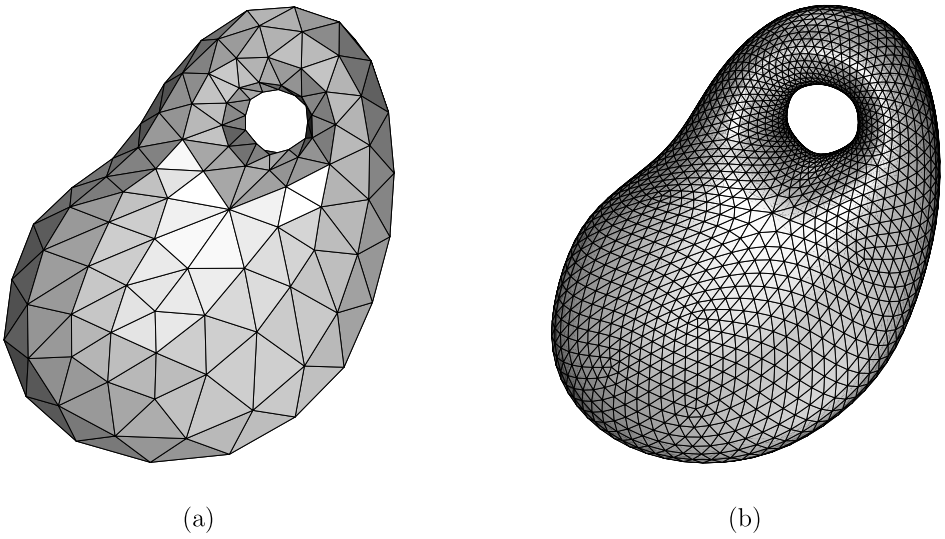

Fig. 5.14. (a) A simplicial surface. (b) Approximation to a PPS defined from (a).

We also introduced a class of spaces called parametric pseudo-manifolds (or PPM's for short), which under certain conditions are manifolds embedded in $\mathbb{R}^n$, for some positive integer n. PPM's can be naturally defined from sets of gluing data, and they are powerful representations for manifolds arising in several graphics and engineering applications [2–8, 10]. We have already used PPM's for building C^∞-surfaces in $\mathbb{R}^3$ that approximate simplicial surfaces [1]. Unlike other approaches for constructing surfaces, such as the stitching paradigm [2] or subdivision surfaces [35], sets of gluing data and PPM's allowed us to build C^k surfaces, for a large integer k or even $k = \infty$, with ease.

As future work is concerned, we intend to create an analogous definition of sets of gluing data for building manifolds with boundary. This definition would definitely increase the range of applications for manifold-based constructions. A manifold-based construction for surfaces with boundary in $\mathbb{R}^3$ already exists [36], but it is not based on any definition of sets of gluing data for manifolds with boundary. In particular, the issue of the cocycle condition is not addressed, and we know that it is a delicate point (similarly for Grimm and Hughes's approach). We are also developing a construction for building gluing data from point sets in $\mathbb{R}^n$, and using our construction for mesh optimization [19]. This sort of construction benefits from existing techniques for manifold learning [12] and point set surfaces [37].

The main obstacle in developing constructions for gluing data is the cocycle condition (see Definition 3.1). The reason is that it does not seem easy to find transition maps that satisfy the cocycle condition and are still simple to invert and evaluate. For instance, we were unable to find polynomial transition maps for the construction described in Section 5. The reader may also wonder why the p-domains are not the interior of the P-polygons (but the interior of the circle inscribed in the P-polygon). The reason is that we were also unable to find diffeomorphisms defined on the interior of the P-polygons. In this respect, we showed that if a map satisfies property A (see Proposition 5.1), then it satisfies the cocycle condition. However, we do not know whether condition A is a necessary condition. We also do not know whether the diffeomorphism given by the Schwarz–Christoffel formula satisfies the cocycle condition. We leave these as open problems to be further investigated. It is interesting to remark that affine maps between triangles of the canonical triangulations of the P-polygons satisfy the cocycle condition, but do not yield C^k-functions, for $k \geqslant 1$, along common edges of adjacent triangles. In turn, projective maps between quadrilaterals formed from two adjacent triangles are C^∞, but do not satisfy the cocycle condition (see [27] for details).

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Appendix A. Construction correctness

We now prove that the construction described in Section 5.2 is correct. This amounts to showing that the triple $\mathcal{G}$ defined in Section 5.2 is a set of gluing data in the sense of Definition 3.1. Formally, we have:

Theorem A.1. *Given any given simplicial surface, $\mathcal{K}$, in $\mathbb{R}^3$, the triple*

$$\mathcal{G} = \big((\Omega_v)_{v\in I}, (\Omega_{uw})_{(u,w)\in I\times I}, (\varphi_{uw})_{(u,w)\in K}\big),$$

where

- *$(\Omega_v)_{v\in I}$ is any set of p-domains for $\mathcal{K}$,*
- *$(\Omega_{uw})_{(u,w)\in I\times I}$ is the set of gluing domains for $\mathcal{K}$ with respect to $(\Omega_v)_{v\in I}$,*
- *$(\varphi_{uw})_{(u,w)\in K}$ is the set of transition maps defined by Definition 5.9, and*
- *$K = \{(u, w) \in I \times I \mid \Omega_{uw} \neq \emptyset\}$,*

is a set of gluing data according to Definition 3.1.

Our proof of Theorem A.1 relies on several straightforward claims, which are stated and proved in the remainder of this appendix. We start by showing that Definition 5.8 satisfies condition 2 of Definition 3.1.

Proposition A.2. *Let Ω_u and Ω_w be any two p-domains of $(\Omega_v)_{v\in I}$. Then, $\Omega_{uw} \neq \emptyset$ if and only if $\Omega_{wu} \neq \emptyset$.*

Proof. If $u = w$, our claim is trivially true. So, let us assume that $u \neq w$. Now, suppose that $\Omega_{uw} \neq \emptyset$. So, from Definition 5.8, we must have that $[u, w]$ is an edge of $\mathcal{K}$. Otherwise, Ω_{uw} would be empty. By Definition 5.8 again, the fact that $[u, w]$ is an edge of $\mathcal{K}$ implies that Ω_{wu} is equal to G_{wu}, which in turn is equal to $R^{-1}_{(w,u)} \circ g_w^{-1}(E)$. Since both $R^{-1}_{(w,u)}$ and g_w^{-1} are bijective, and since $E \neq \emptyset$, we get that $\Omega_{wu} \neq \emptyset$. Conversely, if $\Omega_{wu} \neq \emptyset$ then we can use the same argument to conclude that $\Omega_{uw} \neq \emptyset$. □

Proposition A.3. *Let Ω_u and Ω_w be any two p-domains of $(\Omega_v)_{v\in I}$. Then, the gluing domain Ω_{uw} is an open set of $\mathbb{R}^2$.*

Proof. If $u = w$ then our claim is trivially true, as $\Omega_{uu} = \Omega_u$ and Ω_u is open in $\mathbb{R}^2$ (by definition). So, assume that $u \neq w$. If $\Omega_{uw} = \emptyset$ then our claim is trivially true. So, assume that $\Omega_{uw} \neq \emptyset$. From Definition 5.8, if $\Omega_{uw} \neq \emptyset$ then $[u, w]$ is an edge of $\mathcal{K}$, which means that $\Omega_{uw} = G_{uw} = R^{-1}_{(u,w)} \circ g_u^{-1}(E)$. So, Ω_{uw} is the inverse image, under the function $g_u \circ R_{(u,w)}$, of an open set, E, of $\mathbb{R}^2$. But, since $g_u \circ R_{(u,w)}$ is continuous and E is open in $\mathbb{R}^2$, we must have that Ω_{uw} is also open in $\mathbb{R}^2$. □

The following propositions state several useful properties of $g_{(u,w)}$:

Proposition A.4. *For any two $u, w \in I$ such that $[u, w]$ is an edge of $\mathcal{K}$, function $g_{(u,w)}$ is C^∞.*

Proof. By definition,

$$g_{(u,w)}(p) = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(p),$$

for every $p \in \Omega_u - \{(0, 0)\}$. Since $R^{-1}_{(w,u)}$, g_w^{-1}, h, g_u, and $R_{(u,w)}$ are all C^∞ functions, so is $g_{(u,w)}$. □

Proposition A.5. *For any two vertices, u and w, of $\mathcal{K}$ such that $[u, w]$ is an edge of $\mathcal{K}$, and for every p in G_{wu},*

$$g^{-1}_{(u,w)}(p) = g_{(w,u)}(p).$$

Proof. By definition, we have that $g_{(u,w)}(p) = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(p)$, for every $p \in G_{(u,w)}$. But, the composite function $g_u \circ R_{(u,w)}$ maps $G_{(u,w)}$ onto the set E, where E is the canonical lens. In turn, $h(E) = E$, and the composite function $R^{-1}_{(w,u)} \circ g_w^{-1}$ maps E onto $G_{w,u}$. So, $g_{(u,w)}$ maps G_{uw} onto G_{wu}. Using the same argument, we can conclude that $g_{(w,u)}$ maps G_{wu} onto G_{uw}. Furthermore, we get

$$g^{-1}_{(u,w)}(p) = \left(R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}\right)^{-1}(p) = R^{-1}_{(u,w)} \circ g_u^{-1} \circ h \circ g_w \circ R_{(w,u)}(p) = g_{(w,u)}(p),$$

for every $p \in G_{wu}$. □

We now show that the transition maps (see Definition 5.9) satisfy conditions 3 and 4 of Definition 3.1. Although conditions 3a and 3b follow from condition 3c, the exposition of our proof of condition 3c assumes that conditions 3a and 3b are true, so we first show that conditions 3a and 3b hold.

Proposition A.6. *For any $(u, w) \in K$, we have that $\varphi_{wu}(p) = \varphi^{-1}_{uw}(p)$, for all $p \in \Omega_{uw}$.*

Proof. From Definition 5.9, if $u = w$ then $\varphi_{wu} = \varphi_{uw} = \mathrm{id}_{\Omega_u}$. Otherwise, we have $\varphi_{wu} = g_{(u,w)}$ and $\varphi_{uw} = g_{(w,u)}$. In the former case, our claim is trivially true. In the latter case, Proposition A.5 states that $g^{-1}_{(u,w)}(p) = g_{(w,u)}(p)$, for every $p \in \Omega_{uw}$. Since $\varphi_{uw}(p) = g_{(w,u)}(p) = g^{-1}_{(u,w)}(p) = \varphi^{-1}_{uw}(p)$, our claim follows.

Our proof of condition 3c relies on a property of function g_u, called *rotational symmetry*, as follows:

Proposition A.7. *Let $[u, w, z]$ be any triangle of $\mathcal{K}$. If $s_u(z)$ precedes $s_u(w)$ in a counterclockwise traversal of the vertices of P_u, then*

$$M_{-\pi/3} \circ g_u \circ R_{(u,w)}(\Omega_{uw}) = g_u \circ R_{(u,w)}(\Omega_{uz}) \quad \textit{and} \quad M_{\pi/3} \circ g_u \circ R_{(u,z)}(\Omega_{uz}) = g_u \circ R_{(u,z)}(\Omega_{uw}).$$

Furthermore,

$$\Omega_{uz} = M_{-\frac{2\pi}{m_u}}(\Omega_{uw}) \quad \textit{and} \quad \Omega_{uw} = M_{\frac{2\pi}{m_u}}(\Omega_{uz}).$$

Proof. From Definition 5.8, we have that

$$\Omega_{uw} = G_{uw} \quad \text{and} \quad \Omega_{uz} = G_{uz}.$$

Since $g_u \circ R_{(u,w)}$ and $g_u \circ R_{(u,z)}$ are bijective, we also have that

$$g_u \circ R_{(u,w)}(G_{uw} \cap G_{uz}) = g_u \circ R_{(u,w)}(G_{uw}) \cap g_u \circ R_{(u,w)}(G_{uz})$$

and

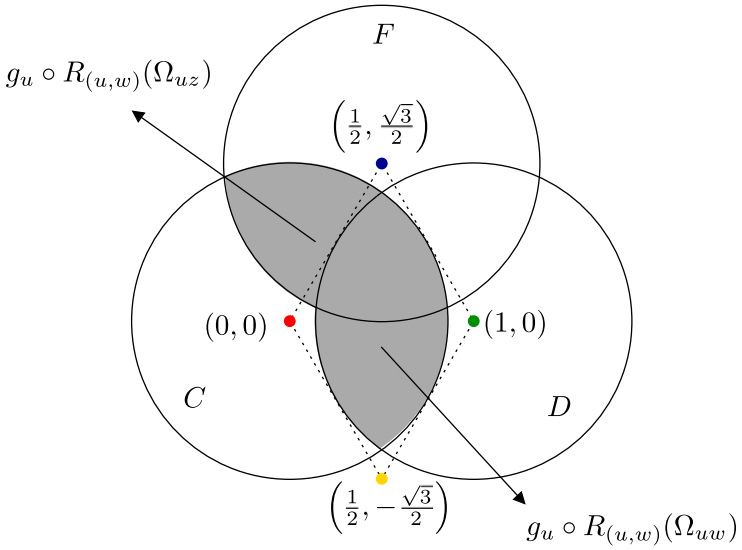

Fig. A.1. The sets $g_u \circ R_{(u,w)}(\Omega_{uw})$ and $g_u \circ R_{(u,w)}(\Omega_{uz})$.

$$g_u \circ R_{(u,z)}(G_{uw} \cap G_{uz}) = g_u \circ R_{(u,z)}(G_{uw}) \cap g_u \circ R_{(u,z)}(G_{uz}).$$

But,

$$g_u \circ R_{(u,w)}(G_{uw}) = E = g_u \circ R_{(u,z)}(G_{uz}),$$

with $E = \text{int}(C) \cap \text{int}(D)$, where C and D are the circles of radius $\cos(\pi/6)$ and centers $(0,0)$ and $(1,0)$, respectively. Furthermore, we also have that

$$\begin{aligned} g_u \circ R_{(u,w)}(G_{uz}) &= g_u \circ R_{(u,w)}\big(g_{(z,u)}(G_{zu})\big) \\ &= g_u \circ R_{(u,w)} \circ R_{(u,z)}^{-1} \circ g_u^{-1} \circ h \circ g_z \circ R_{(z,u)}(G_{zu}) \\ &= g_u \circ M_{-\frac{2\pi}{m_u}} \circ g_u^{-1} \circ h \circ g_z \circ R_{(z,u)}(G_{zu}) \\ &= M_{-\frac{\pi}{3}} \circ h \circ g_z \circ R_{(z,u)}(G_{zu}) \\ &= M_{-\frac{\pi}{3}}(E) \\ &= \text{int}(C) \cap \text{int}(F), \end{aligned}$$

where we used the property that $R_{(u,w)} \circ R_{(u,z)}^{-1} = M_{-\frac{2\pi}{m_u}}$ and property A from Proposition 5.1 to claim that $g_u \circ M_{-\frac{2\pi}{m_u}} \circ g_u^{-1} = M_{-\frac{\pi}{3}}$, where F is the circle of radius $\cos(\pi/6)$ and center $(1/2, \sqrt{3}/2)$ (see Fig. A.1).

So,

$$g_u \circ R_{(u,w)}(G_{uw}) = E = \text{int}(C) \cap \text{int}(D)$$

and

$$g_u \circ R_{(u,w)}(G_{uz}) = \text{int}(C) \cap \text{int}(F).$$

But, since

$$M_{-\frac{\pi}{3}}\big(\text{int}(D)\big) = \text{int}(F),$$

we get

$$M_{-\pi/3} \circ g_u \circ R_{(u,w)}(G_{uw}) = g_u \circ R_{(u,w)}(G_{uz}).$$

To show that $M_{\pi/3} \circ g_u \circ R_{(u,z)}(\Omega_{uz}) = g_u \circ R_{(u,z)}(\Omega_{uw})$, we can proceed as before, but noting that

$$R_{(u,z)} \circ R_{(u,w)}^{-1} = M_{\frac{2\pi}{m_u}} \qquad \text{and} \qquad g_u \circ M_{\frac{2\pi}{m_u}} \circ g_u^{-1} = M_{\frac{\pi}{3}}.$$

To prove the second claim, note that

$$\begin{aligned} M_{-\frac{2\pi}{m_u}}(\Omega_{uw}) &= M_{-\frac{2\pi}{m_u}}(G_{uw}) \\ &= M_{-\frac{2\pi}{m_u}}\big(g_{(w,u)}(G_{wu})\big) \\ &= M_{-\frac{2\pi}{m_u}} \circ R^{-1}_{(u,w)} \circ g_u^{-1} \circ h \circ g_w \circ R_{(w,u)}(G_{wu}) \\ &= R^{-1}_{(u,z)} \circ g_u^{-1}(E) \\ &= \Omega_{uz}. \end{aligned}$$

To show that $M_{\frac{2\pi}{m_u}}(\Omega_{uz}) = \Omega_{uw}$ holds, we can proceed as before, but noting that $M_{\frac{2\pi}{m_u}} \circ R^{-1}_{(u,z)} = R^{-1}_{(u,w)}$. □

We can now prove the first implication of condition 3c of Definition 3.1.

Lemma A.8. *Let Ω_u, Ω_w, and Ω_x be any three p-domains in $(\Omega_v)_{v\in I}$. If the intersection $\Omega_{xu} \cap \Omega_{xw}$ is nonempty, then*

$$\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{uw}.$$

Proof. We distinguish three cases: (a) $u = w = x$, (b) $u = w$ and $u \neq x$, or $u = x$ and $u \neq w$, or $w = x$ and $u \neq w$, and (c) $u \neq w$, $u \neq x$, and $w \neq x$. Case (a) is trivial, as $\Omega_{xu} \cap \Omega_{xw} = \Omega_x$, and thus $\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \mathrm{id}_{\Omega_x}(\Omega_x) = \Omega_x = \Omega_{uw}$. Case (b) is also trivial. If $u = w$ and $u \neq x$ then $\Omega_{xu} \cap \Omega_{xw} = \Omega_{xu}$, and thus $\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \varphi_{xu}^{-1}(\Omega_{xu}) = \Omega_{ux}$. In turn, if $u = x$ and $u \neq w$ then $\Omega_{xu} \cap \Omega_{xw} = \Omega_{xx} \cap \Omega_{xw} = \Omega_x \cap \Omega_{xw} = \Omega_{xw}$, and thus $\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \mathrm{id}^{-1}_{\Omega_x}(\Omega_{xw}) = \Omega_{xw} = \Omega_{uw}$. Finally, if $w = x$ and $u \neq w$ then $\Omega_{xu} \cap \Omega_{xw} = \Omega_{xu} \cap \Omega_{xx} = \Omega_{xu} \cap \Omega_x = \Omega_{xu}$, and thus $\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \varphi_{xu}^{-1}(\Omega_{xu}) = \Omega_{ux} = \Omega_{uw}$. So, consider case (c): $u \neq w$, $u \neq x$, and $w \neq x$.

Assume that the edges $[u, w]$, $[u, x]$, and $[w, x]$ of $\mathcal{K}$ are shared by the triangles $[u, w, x]$ and $[u, w, z]$, $[u, w, x]$ and $[u, x, y]$, and $[u, w, x]$ and $[v, x, w]$ of $\mathcal{K}$, respectively. We will first show the following:

$$g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{uw}.$$

In fact, since $g^{-1}_{(u,x)}$ is bijective,

$$g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw}) = g^{-1}_{(u,x)}(\Omega_{xu}) \cap g^{-1}_{(u,x)}(\Omega_{xw}) = g_{(x,u)}(\Omega_{xu}) \cap g_{(x,u)}(\Omega_{xw}) = \Omega_{ux} \cap g_{(x,u)}(\Omega_{xw}).$$

By definition,

$$g_{(x,u)}(\Omega_{xw}) = R^{-1}_{(u,x)} \circ g_u^{-1} \circ h \circ g_x \circ R_{(x,u)}(\Omega_{xw}).$$

From Proposition A.7, we have that

$$R^{-1}_{(u,x)} \circ g_u^{-1} \circ h \circ g_x \circ R_{(x,u)}(\Omega_{xw}) = R^{-1}_{(u,x)} \circ g_u^{-1} \circ h \circ M_{\frac{\pi}{3}} \circ g_x \circ R_{(x,u)}(\Omega_{xu}),$$

where $M_{\frac{\pi}{3}}$ is a rotation by $\frac{\pi}{3}$ around the origin. By construction, the composite function $g_x \circ R_{(x,u)}$ maps Ω_{xu} onto the canonical lens, $E = \mathrm{int}(C) \cap \mathrm{int}(D)$, where C is the circle of radius $\cos(\pi/6)$ and center $(0, 0)$ and D is the circle of radius $\cos(\pi/6)$ and center $(1, 0)$. So, we get that

$$h \circ M_{\frac{\pi}{3}} \circ g_x \circ R_{(x,u)}(\Omega_{xu})$$

is the set

$$\mathrm{int}(D) \cap \mathrm{int}(G),$$

where G is the circle of radius $\cos(\pi/6)$ and center $(1/2, -\sqrt{3}/2)$. But, only the points of the above set that are also in $\mathrm{int}(C) - \{(0, 0)\}$ are mapped by $R^{-1}_{(u,x)} \circ g_u^{-1}$ to Ω_u. So, $g_{(x,u)}(\Omega_{xw}) \cap \Omega_u$ is the image of

$$\mathrm{int}(C) \cap \mathrm{int}(D) \cap \mathrm{int}(G)$$

under $R^{-1}_{(u,x)} \circ g_u^{-1}$ (see Fig. A.2).

Now, we claim that the image of $\Omega_{ux} \cap \Omega_{uw}$ under $g_u \circ R_{(u,x)}$ is also equal to

$$\mathrm{int}(C) \cap \mathrm{int}(D) \cap \mathrm{int}(G).$$

In fact,

$$g_u \circ R_{(u,x)}(\Omega_{ux} \cap \Omega_{uw}) = g_u \circ R_{(u,x)}(\Omega_{ux}) \cap g_u \circ R_{(u,x)}(\Omega_{uw}).$$

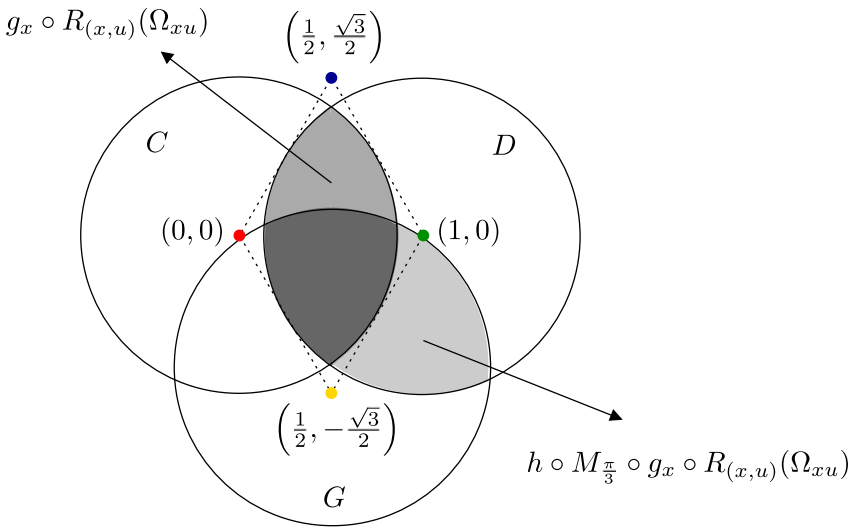

Fig. A.2. The sets $g_x \circ R_{(x,u)}(\Omega_{xu})$ and $h \circ M_{\frac{\pi}{3}} \circ g_x \circ R_{(x,u)}(\Omega_{xu})$.

By definition,

$$g_u \circ R_{(u,x)}(\Omega_{ux}) = E = \text{int}(C) \cap \text{int}(D).$$

In turn, from Proposition A.7, we know that $g_u \circ R_{(u,x)}(\Omega_{uw}) = M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,x)}(\Omega_{uw})$. So,

$$g_u \circ R_{(u,x)}(\Omega_{uw}) = M_{-\frac{\pi}{3}}(E) = \text{int}(C) \cap \text{int}(G),$$

and hence

$$g_u \circ R_{(u,x)}(\Omega_{ux} \cap \Omega_{uw}) = \text{int}(C) \cap \text{int}(D) \cap \text{int}(G).$$

So,

$$\begin{aligned}
\Omega_{ux} \cap \Omega_{uw} &= g_{(x,u)}(\Omega_{xw}) \cap \Omega_u \\
&= g_{(x,u)}(\Omega_{xw}) \cap \Omega_{ux} \\
&= g_{(x,u)}(\Omega_{xw}) \cap g_{(x,u)}(\Omega_{xu}) \\
&= g_{(x,u)}(\Omega_{xw} \cap \Omega_{xu}) \\
&= g^{-1}_{(u,x)}(\Omega_{xw} \cap \Omega_{xu}).
\end{aligned}$$

Since $\varphi^{-1}_{xu}(p) = g^{-1}_{(u,x)}(p)$, for every $p \in \Omega_{xu}$, we get $\varphi^{-1}_{xu}(\Omega_{xw} \cap \Omega_{xu}) = \Omega_{ux} \cap \Omega_{uw}$, and hence our claim is true. □

The second and last implication of condition 3c of Definition 3.1 also holds:

Lemma A.9. *Let Ω_u, Ω_w, and Ω_x be any three p-domains in $(\Omega_v)_{v \in I}$. If $\Omega_{xu} \cap \Omega_{xw} \neq \emptyset$, then*

$$\varphi_{wu}(p) = \varphi_{wx} \circ \varphi_{xu}(p),$$

for all $p \in \varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{uw}$.

Proof. From Lemma A.8, we know that φ_{wu} is well-defined for all points in $\varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{uw}$. So, we are left to show that $\varphi_{wu} = \varphi_{wx} \circ \varphi_{xu}$. Assume that u, w, and x are all distinct; otherwise, if two of them are equal or all of them are the same, our claim would be reduced to condition 3b of Definition 3.1, which has been proved. Since the indices u, w, and x are assumed to be pairwise distinct, Definition 5.9 tells us that $\varphi_{wu} = g_{(u,w)}$, $\varphi_{wx} = g_{(x,w)}$, and $\varphi_{xu} = g_{(u,x)}$. So, we need to prove that

$$g_{(u,w)}(p) = g_{(x,w)} \circ g_{(u,x)}(p),$$

for all $p \in g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{uw}$.

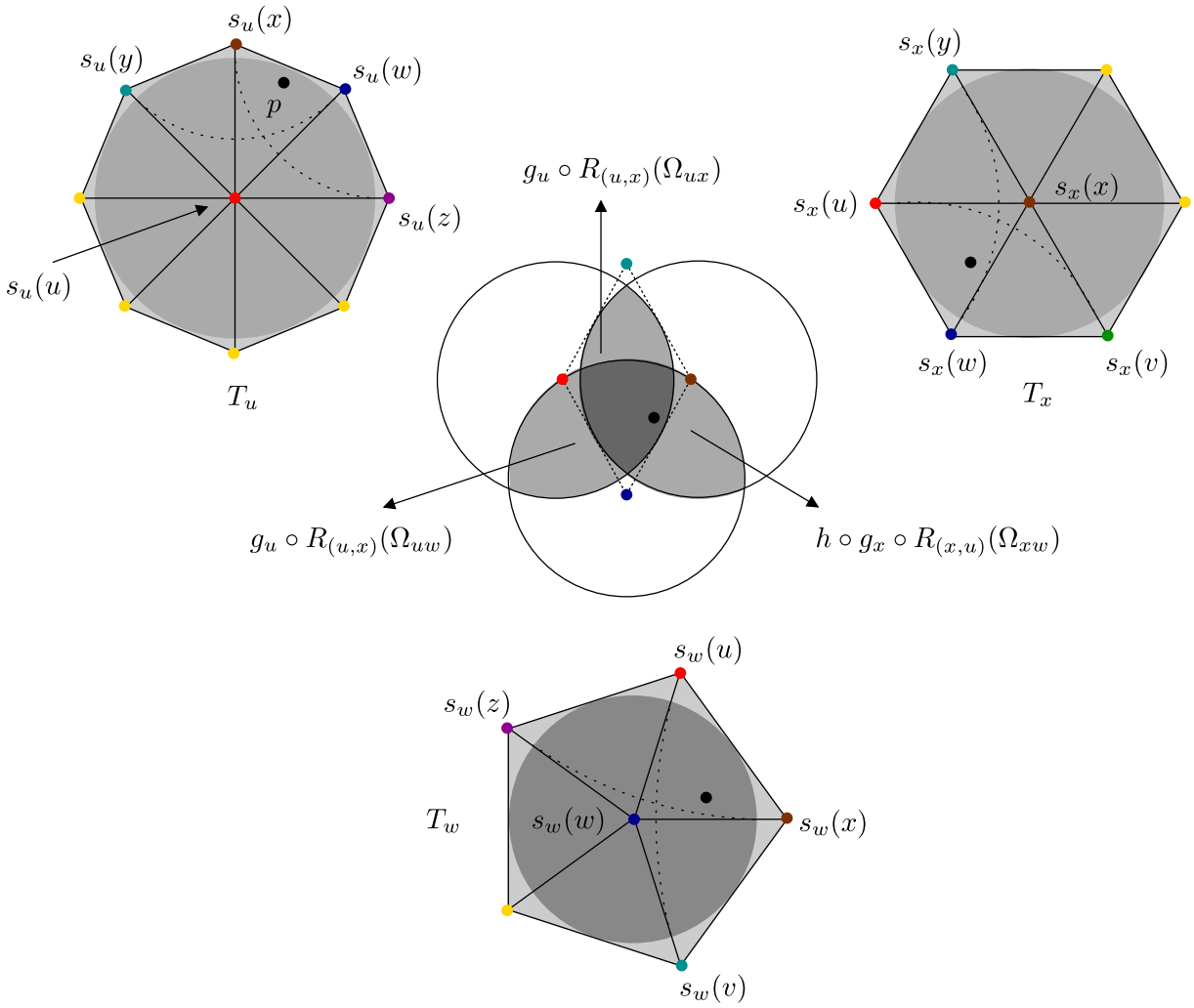

Fig. A.3. Illustration of the cocycle condition.

From Definition 5.7, we know that

$$g_{(u,w)} = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}, \tag{1}$$

$$g_{(u,x)} = R^{-1}_{(x,u)} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)}, \tag{2}$$

and

$$g_{(x,w)} = R^{-1}_{(w,x)} \circ g_w^{-1} \circ h \circ g_x \circ R_{(x,w)}. \tag{3}$$

So,

$$g_{(x,w)} \circ g_{(u,x)} = R^{-1}_{(w,x)} \circ g_w^{-1} \circ h \circ g_x \circ R_{(x,w)} \circ R^{-1}_{(x,u)} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)}. \tag{4}$$

To show that the right side of Eq. (4) is equal to the right side of Eq. (1), we invoke Proposition A.7. Consider the triangles $[s_u(u), s_u(w), s_u(x)]$, $[s_w(u), s_w(w), s_w(x)]$, and $[s_x(u), s_x(w), s_x(x)]$ of T_u, T_w, and T_x, respectively (see Fig. A.3). Without loss of generality, suppose that $s_u(x)$ follows $s_u(w)$ in a counterclockwise traversal of the vertices of P_u. This means that $s_w(u)$ follows $s_w(x)$ in a counterclockwise traversal of the vertices of P_w, and that $s_x(w)$ follows $s_x(u)$ in a counterclockwise traversal of the vertices of P_x. Now, let p be a point in $g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw})$. From Lemma A.8, we know that

$$g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{uw}.$$

We now show how to simplify the expression

$$g_{(x,w)} \circ g_{(u,x)} = R^{-1}_{(w,x)} \circ g_w^{-1} \circ h \circ g_x \circ R_{(x,w)} \circ R^{-1}_{(x,u)} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)}.$$

First, note that

$$R_{(x,w)} \circ R_{(x,u)}^{-1} = M_{-\frac{2\pi}{m_x}},$$

as $s_x(w)$ follows $s_x(u)$ in a counterclockwise traversal of P_x, where m_x is the degree of x. We get

$$g_{(x,w)} \circ g_{(u,x)} = R_{(w,x)}^{-1} \circ g_w^{-1} \circ h \circ g_x \circ M_{-\frac{2\pi}{m_x}} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)}.$$

By property A of Proposition 5.1, we have

$$g_x \circ M_{-\frac{2\pi}{m_x}} \circ g_x^{-1} = M_{-\frac{\pi}{3}}$$

on $g_x(\Omega_x)$, so we get

$$R_{(w,x)}^{-1} \circ g_w^{-1} \circ h \circ g_x \circ M_{-\frac{2\pi}{m_x}} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)} = R_{(w,x)}^{-1} \circ g_w^{-1} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ g_u \circ R_{(u,x)}. \tag{5}$$

Next, from Proposition A.7, we know that

$$g_u \circ R_{(u,x)}(\Omega_{uw}) = M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}(\Omega_{uw}).$$

Since $p \in g_{(u,x)}^{-1}(\Omega_{xu} \cap \Omega_{xw})$, we can conclude from property B of Proposition 5.1 that

$$g_u \circ R_{(u,x)}(p) = M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}(p). \tag{6}$$

For the same reason, we also know that

$$g_w \circ R_{(w,x)}(q) = M_{\frac{\pi}{3}} \circ g_w \circ R_{(w,u)}(q),$$

for every $q \in g_{(w,x)}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{wx}$. So, by property B of Proposition 5.1,

$$R_{(w,x)}^{-1} \circ g_w^{-1}(t) = R_{(w,u)}^{-1} \circ g_w^{-1} \circ M_{-\frac{\pi}{3}}(t), \tag{7}$$

for every t, where $t = g_w \circ R_{(w,x)}(q)$ for some $q \in g_{(w,x)}^{-1}(\Omega_{xu} \cap \Omega_{xw})$.

Substituting the right-hand side of the identities in Eq. (6) and Eq. (7) with their left side into Eq. (5), we get

$$\begin{aligned} &R_{(w,x)}^{-1} \circ g_w^{-1} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ g_u \circ R_{(u,x)} \\ &\quad = R_{(w,u)}^{-1} \circ g_w^{-1} \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}. \end{aligned} \tag{8}$$

This means that

$$g_{(x,w)} \circ g_{(u,x)} = R_{(w,u)}^{-1} \circ g_w^{-1} \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}. \tag{9}$$

Now, the above expression can be further simplified because

$$h = M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}}.$$

This is because it is easily checked that $h \circ M_{-\frac{\pi}{3}}$ is the rotation of center $(\sqrt{3}/4, 1/2)$ and angle $\pi - \pi/3 = 2\pi/3$, which implies that

$$h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} = \mathrm{id}_{\mathbb{R}^2},$$

and since $h \circ h = \mathrm{id}_{\mathbb{R}^2}$, we have $h = M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}}$, as claimed. So, we have that

$$g_{(x,w)} \circ g_{(u,x)}(p) = R_{(w,u)}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(p) = g_{(u,w)}(p), \tag{10}$$

for every $p \in g_{(u,x)}^{-1}(\Omega_{xu} \cap \Omega_{xw})$. □

Finally, we show that condition 4 of Definition 3.1 also holds:

Lemma A.10. *Let (u, w) be any pair in K, with $u \neq w$. Then, for every $x \in \partial(\Omega_{uw}) \cap \Omega_u$ and every $y \in \partial(\Omega_{wu}) \cap \Omega_w$, there are open balls, V_x and V_y, centered at x and y, such that no point of $V_y \cap \Omega_{wu}$ is the image of any point $V_x \cap \Omega_{uw}$ under φ_{wu}.*

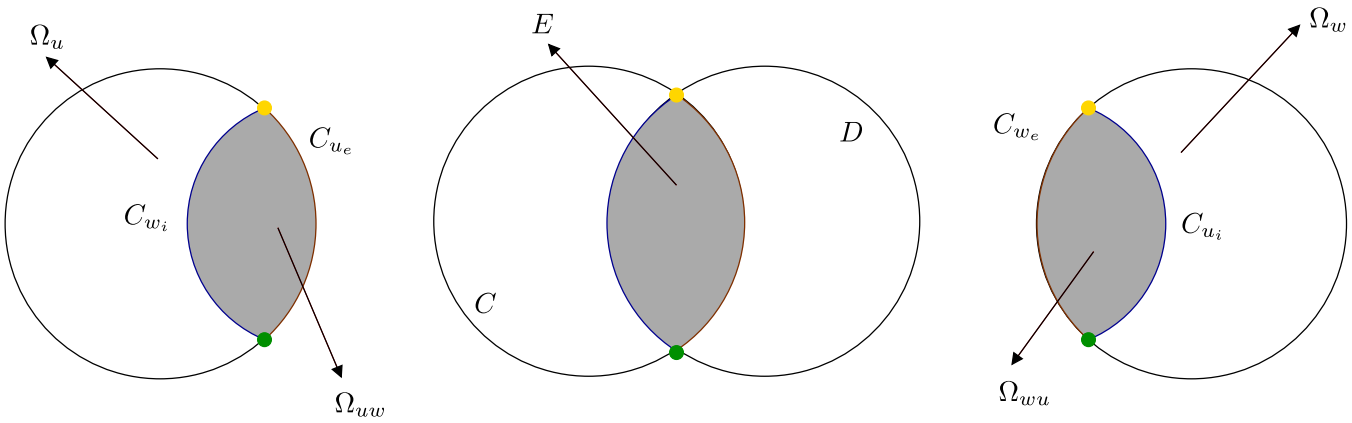

Fig. A.4. The image sets of the canonical lens, E, under $R^{-1}_{(u,w)} \circ g_u^{-1}$ and $R^{-1}_{(w,u)} \circ g_w^{-1}$.

Proof. By definition, each gluing domain, Ω_{uw}, is the image by $R^{-1}_{(u,w)} \circ g_u^{-1}$ of the canonical lens, E, given by

$$\operatorname{int}(C) \cap \operatorname{int}(D),$$

where C and D are the circles of radius $\cos(\pi/6)$ and centers $(0,0)$ and $(1,0)$, respectively. Furthermore, the gluing domain Ω_{uw} is also a lens-shaped set whose boundary, $\partial(\Omega_{uw})$, is the image by $R^{-1}_{(u,w)} \circ g_u^{-1}$ of the boundary, $\partial(E)$, of E. We can view $\partial(\Omega_{uw})$ as the union of two open and simple curve segments, C_{u_e} and C_{u_i}, such that C_{u_e} belongs to $\partial(\Omega_{uw})$ and the interior, $\operatorname{int}(C_{u_i})$, of C_{u_i} belongs to the interior of Ω_u (see Fig. A.4). In addition, the pairs of endpoints of both curves, C_{u_e} and C_{u_i}, are the same, and each pair is the image by $R^{-1}_{(u,w)} \circ g_u^{-1}$ of the two intersection points of the boundaries, $\partial(C)$ and $\partial(D)$, of C and D. Similarly, the boundary, $\partial(\Omega_{wu})$, of the gluing domain, Ω_{wu}, can be viewed as the union of two curves, C_{w_e} and C_{w_i}, such that C_{w_e} belongs to $\partial(\Omega_{wu})$ and the interior, $\operatorname{int}(C_{w_i})$, of C_{w_i} belongs to the interior of Ω_w. In addition, the pairs of endpoints of both curves, C_{w_e} and C_{w_i}, are the same, and each pair is the image by $R^{-1}_{(w,u)} \circ g_w^{-1}$ of the two intersection points of the boundaries, $\partial(C)$ and $\partial(D)$, of C and D, as also shown in Fig. A.4.

Note that

$$\operatorname{int}(C_{u_i}) = \partial(\Omega_{uw}) \cap \Omega_u \quad \text{and} \quad \operatorname{int}(C_{w_i}) = \partial(\Omega_{wu}) \cap \Omega_w.$$

Note also that

$$g_{(u,w)}(C_{u_i}) = C_{w_e} \quad \text{and} \quad g_{(w,u)}(C_{w_i}) = C_{u_e}.$$

Indeed,

$$g_{(u,w)}(C_{u_i}) = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(C_{u_i}).$$

By construction, we know that $g_u \circ R_{(u,w)}(C_{u_i}) \in \partial(C)$, which implies that $h \circ g_u \circ R_{(u,w)}(C_{u_i}) \in \partial(D)$. So,

$$R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(C_{u_i}) = C_{w_e}.$$

Finally, let x be any point in $\partial(\Omega_{uw}) \cap \Omega_u$. Since $\operatorname{int}(C_{u_i}) = \partial(\Omega_{uw}) \cap \Omega_u$, we have that $x \in \operatorname{int}(C_{u_i})$. From our discussion above, we also have that if $p = g_{(u,w)}(x)$ then $p \in \operatorname{int}(C_{w_e})$. Since $\operatorname{int}(C_{w_e}) \cap \operatorname{int}(C_{w_i}) = \emptyset$, there exists an open ball, V_p, centered at p such that $V_p \cap \operatorname{int}(C_{w_i}) = \emptyset$, which follows from the fact that $\mathbb{R}^2$ is a Hausdorff space. Since $\operatorname{int}(C_{w_i}) = \partial(\Omega_{wu}) \cap \Omega_w$, we have that

$$V_p \cap \big(\partial(\Omega_{wu}) \cap \Omega_w\big) = \emptyset.$$

In turn, for any point $y \in \partial(\Omega_{wu}) \cap \Omega_w$, there exists an open ball, V_y, such that $V_y \cap V_p = \emptyset$ (see Fig. A.5). This also follows from the fact that $\mathbb{R}^2$ is a Hausdorff space. So, define V_x to be any open ball centered at x such that $V_x \subseteq g^{-1}_{(u,w)}(V_p)$. By construction, we know that $g_{(u,w)}(V_x) \cap V_y = \emptyset$. To conclude that our claim is true, it suffices to notice that $g_{(u,w)}(V_x \cap \Omega_{uw}) \subset \Omega_w$ and that $\varphi_{wu} = g_{(u,w)}$ for every point in Ω_{uw}, which implies that $\varphi_{wu}(V_x \cap \Omega_{uw}) \cap (V_y \cap \Omega_{wu}) = \emptyset$. So, our claim follows. □

We can now prove Theorem A.1:

Proof. Our claim follows immediately from the facts that our construction yields p-domains, gluing domains, and transition functions that satisfy conditions 1 through 4 of Definition 5.9. Indeed, the p-domains are open sets in $\mathbb{R}^2$. Proposition A.2 and Proposition A.3 ensure that the gluing domains satisfy condition 2 of Definition 5.9, while Proposition A.6, Lemma A.8, and Lemma A.9 ensure that the transition functions satisfy condition 3. Finally, Lemma A.10 states that condition 4 holds. □

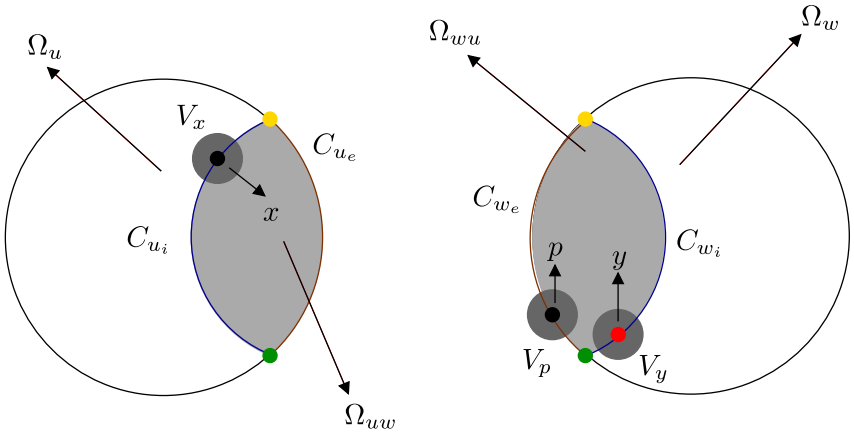

Fig. A.5. The open balls V_x, V_y, and V_p.

References

[1] Marcelo Siqueira, Dianna Xu, Jean Gallier, Luis Gustavo Nonato, Dimas Martínez Morera, Luiz Velho, A new construction of smooth surfaces from triangle meshes using parametric pseudo-manifolds, Computer & Graphics 33 (3) (2009) 331–340.
[2] Jean Gallier, Curves and Surfaces in Geometric Modeling: Theory and Algorithms, Morgan-Kaufmann, 1999.
[3] Greg Turk, Generating textures on arbitrary surfaces using reaction–diffusion, in: Proceedings of the 18th Annual Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'91), July 28–August 2, 1991, pp. 289–298.
[4] Doug L. James, Dinesh K. Pai, Artdefo: Accurate real time deformable objects, in: Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'99), August 8–13, 1999, pp. 65–72.
[5] Jos Stam, Flows on surfaces of arbitrary topology, ACM Transactions on Graphics 22 (3) (2003) 724–731.
[6] Lexing Ying, George Biros, Denis Zorin, A high-order 3d boundary integral equation solver for elliptic pdes in smooth domains, Journal of Computational Physics 219 (1) (2006) 247–275.
[7] Wolfgang Heidrich, Hans-Peter Seidel, View-independent environment maps, in: Proceedings of the ACM SIGGRAPH/EUROGRAPHICS Workshop on Graphics Hardware, HWWS'98, ACM, New York, NY, USA, 1998, pp. 39–46.
[8] Michael Kazhdan, Hughes Hoppe, Metric-aware processing of spherical imagery, ACM Transactions on Graphics 29 (December 2010) 1–10.
[9] Matthew Brand, Charting a manifold, in: S. Thrun, S. Becker, K. Obermayer (Eds.), Advances in Neural Information Processing Systems, vol. 15, MIT Press, Cambridge, Massachusetts, USA, 2003, pp. 961–968.
[10] Shmuel Peleg, Joshua Herman, Panoramic mosaics by manifold projection, in: Proceedings of the 1997 Conference on Computer Vision and Pattern Recognition (CVPR'97), CVPR'97, IEEE Computer Society, Washington, DC, USA, 1997, pp. 338–343.
[11] Wojciech Matusik, Hanspeter Pfister, Matt Brand, Leonard McMillan, A data-driven reflectance model, ACM Transactions on Graphics 22 (July 2003) 759–769.
[12] Martin H.C. Law, Anil K. Jain, Incremental nonlinear dimensionality reduction by manifold learning, IEEE Transactions on Pattern Analysis and Machine Intelligence 28 (March 2006) 377–391.
[13] Cindy M. Grimm, John F. Hughes, Modeling surfaces of arbitrary topology using manifolds, in: Proceedings of the 22nd ACM Annual Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'95), August 6–11, 1995, pp. 359–368.
[14] André Weil, Foundations of Algebraic Topology, second edition, Colloquium Publications, vol. XXIX, AMS, 1946.
[15] Josep Cotrina Navau, Núria Pla Garcia, Modeling surfaces from meshes of arbitrary topology, Computer Aided Geometric Design 7 (1) (2000) 643–671.
[16] L. Ying, Denis Zorin, A simple manifold-based construction of surfaces of arbitrary smoothness, ACM Transactions on Graphics 23 (3) (2004) 271–275.
[17] Xianfeng Gu, Ying He, Hong Qin, Manifold splines, Graphical Models 68 (3) (2006) 237–254.
[18] G.D. Vecchia, B. Jüttler, M.-S. Kim, A construction of rational manifold surfaces of arbitrary topology and smoothness from triangular meshes, Computer Aided Geometric Design 25 (9) (2008) 801–815.
[19] Igor Guskov, Manifold-based approach to semi-regular remeshing, Graphical Models 69 (1) (January 2007) 1–18.
[20] Frank Warner, Foundations of Differentiable Manifolds and Lie Groups, first edition, GTM, vol. 94, Springer-Verlag, 1983.
[21] Marcel Berger, Bernard Gostiaux, Differential Geometry, Manifolds, Curves, and Surfaces, GTM, vol. 115, Springer-Verlag, 2006.
[22] Barrett O'Neill, Semi-Riemannian Geometry with Applications to Relativity, first edition, Pure and Applied Mathematics, vol. 103, Academic Press, 1983.
[23] Manfredo Perdigão do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, 1983.
[24] Manfredo Perdigão do Carmo, Riemannian Geometry, second edition, Birkhäuser, 1992.
[25] Loring W. Tu, An Introduction to Manifolds, first edition, Universitext, Springer-Verlag, 2008.
[26] Cindy M. Grimm, Modeling surfaces of arbitrary topology using manifolds, PhD thesis, Department of Computer Science, Brown University, Providence, RI, USA, 1996.
[27] Marcelo Siqueira, Dianna Xu, Jean Gallier, Construction of c^∞ surfaces from triangular meshes using parametric pseudo-manifolds, Technical Report MS-CIS-08-14, Department of Computer and Information Science, University of Pennsylvania, Philadelphia, PA, USA, 2008.
[28] Norman Steenrod, The Topology of Fibre Bundles, Princeton Mathematics Series, vol. 14, Princeton University Press, 1956.
[29] Cindy M. Grimm, Denis Zorin, Surface modeling and parameterization with manifolds, in: ACM SIGGRAPH 2006 Courses (SIGGRAPH'06), ACM Press, New York, NY, USA, 2006, pp. 1–81.
[30] Georgeta Elizabeth Marai, David Laidlaw, Cagatay Demiralap, Stuart Andrews, Cindy Grimm, Joseph Crisco, Estimating joint contact areas and ligament lengths from bone kinematics and surfaces, IEEE Transactions on Biomechanical Engineering 51 (5) (May 2004) 790–799.
[31] Friedrich Hirzebruch, Topological Methods in Algebraic Geometry, second edition, Springer Classics in Mathematics, Springer-Verlag, 1978.
[32] Shiing-shen Chern, Complex Manifolds Without Potential Theory, second edition, Universitext, Springer-Verlag, 1995.
[33] Lars V. Ahlfors, Complex Analysis, third edition, McGraw-Hill, 1979.
[34] Ethan D. Bloch, A First Course in Geometric Topology and Differential Geometry, Birkhäuser, 1997.
[35] Jörg Peters, Ulrich Reif, Subdivision Surfaces, Geometry and Computing, vol. 3, Springer-Verlag, 2008.
[36] Elif Tosun, Denis Zorin, Manifold-based surfaces with boundaries, Computer Aided Geometric Design 28 (January 2011) 1–22.
[37] Marc Alexa, Johannes Behr, Daniel Cohen-Or, Shachar Fleishman, David Levin, Claudio T. Silva, Point set surfaces, in: Proceedings of the Conference on Visualization, VIS'01, IEEE Computer Society, Washington, DC, USA, 2001, pp. 21–28.