CS446: Machine Learning	**Fall 2016**

October 25^{th}, 2016

- This is a closed book exam. Everything you need in order to solve the problems is supplied in the body of this exam.
- This exam booklet contains **four** problems. You need to solve all problems to get 100%.
- Please check that the exam booklet contains **14** pages, with the appendix at the end.
- The exam ends at 1:45 PM. You have 75 minutes to earn a total of 100 points.
- Answer each question in the space provided. If you need more room, write on the reverse side of the paper and indicate that you have done so.
- A list of potentially useful functions has been provided in the appendix at the end.
- **Besides having the correct answer, being concise and clear is very important. For full credit, you must show your work and explain your answers.**

Good Luck!

Name (NetID): (1 Point)

Decision Trees		/20
PAC Learning		/29
Neural Networks		/25
Short Questions		/25
Total		/100

Decision Trees [20 points]

You work in a weather forecasting company and your job as a machine learning expert is to design a decision tree which would predict whether it is going to rain today ('WillRain?' = 1) or not ('WillRain?' = 0). You are given a dataset D with the following attributes: $IsHumid \in \{0,1\}$, $IsCloudy \in \{0,1\}$, $RainedYesterday \in \{0,1\}$ and $Temp{>}20 \in \{0,1\}$.

IsHumid	IsCloudy	RainedYesterday	Temp>20	WillRain?
1	1	1	0	1
0	1	0	0	0
1	0	0	0	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
0	1	0	0	0
1	0	1	1	0

To simplify your computations please use: $\log_2(3) \approx \frac{3}{2}$.

(a) **(4 points)** What is the entropy of the label 'WillRain?'?

Entropy('WillRain?')= $-\frac{2}{8}log_2(\frac{2}{8})$ $-\frac{6}{8}log_2(\frac{6}{8}) = \frac{7}{8} = 0.875$

(b) **(4 points)** What should the proportion of the examples labeled 'WillRain?'=1 be, in order to get the maximum entropy value for the label?

Half of the examples should have label 1 and the other half have the label 0.

(c) **(4 points)** Compute the Gain(D, IsCloudy).

Entropy(D, IsCloudy=1) = -$\frac{2}{4}log_2(\frac{2}{4})$-$\frac{2}{4}log_2(\frac{2}{4})$

$\Rightarrow$ 1

Entropy(D, IsCloudy=0) = -$\frac{0}{4}log_2(\frac{0}{4})$-$\frac{4}{4}log_2(\frac{4}{4})$

$\Rightarrow$ 0

Gain(D, IsCloudy) = 0.875 - $\frac{4}{8}\times 1$ - $\frac{4}{8}\times 0$

$\Rightarrow$ 0.375

(d) **(4 points)** You are given that:

- Gain(D, IsHumid) = 0.25,
- Gain(D, RainedYesterday) = 0.11,
- Gain(D, Temp>20) = 0
- Gain(D, IsCloudy) is as computed in part c.

i. Which node should be the root node?

IsCloudy should be the root node since it gives the highest information gain.

ii. Without any additional computation, draw a decision tree that is consistent with the given dataset and uses the root chosen in (i).

```
if(IsCloudy):
    if(IsHumid):
        1
    else:
        0
else:
    0
```

(e) **(4 points)** Express the function 'WillRain?' as a simple Boolean function over the features defining the data set D. That is, define a Boolean function that returns true if an only if 'WillRain?'=1.
IsCloudy $\bigwedge$ IsHumid

PAC Learning [29 points]

We define a set of functions

$$T = \{f(x) = \mathbb{1}[x > a] : a \in \mathbb{R}\},$$

where $\mathbb{1}[x > a]$ is the indicator function returning 1 if $x > a$ and returning 0 otherwise. For input domain $\mathcal{X} = \mathbb{R}$, and a *fixed* positive number k, consider a concept class DT_k consisting of all decision trees of depth at most k where the function at each non-leaf node is an element of T. Note that if the tree has only one decision node (the root) and two leaves, then $k = 1$.

(a) **(4 points)** We want to learn a function in DT_k. Define

i. The Instance Space X

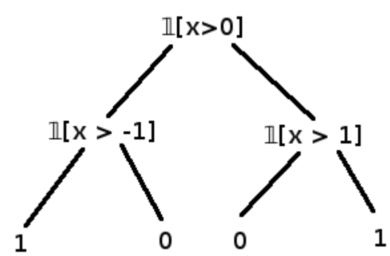
$X = \mathbb{R}$

ii. The Label Space Y

$Y = \{0, 1\}$

iii. Give an example of $f \in DT_2$.

There are many possible answers for this. Here is one, where we assume that the right branch is taken if the node is satisfied and the left node is taken



otherwise.

iv. Give 3 examples that are consistent with your function f and one that is not consistent with it.

For the previous tree, here are three consistent examples:
$x = 10, y = 1$
$x = -\frac{1}{2}, y = 0$
$x = -50000, y = 1$

And here is an inconsistent example: $(x = 10, y = 0)$, since the label of $x = 10$ should be 1.

(b) **(7 points)** Determine the VC dimension of DT_k, and prove that your answer is correct.

First, note that the root node of the tree partitions the input space into two intervals. Each child node then recursively divides the corresponding interval into two more intervals. Hence, a full decision tree of depth k divides the input space into at most 2^k intervals, each of which can be assigned a label of 0 or 1. We will show that the VC dimension of DT_k is 2^k.

Proof that $VCDim(DT_k) \geq 2^k$: given 2^k points, construct a decision tree such that each point lies in a separate interval. Then, one can assign labels to the leaves of the tree corresponding to any possible labeling of the points.

Proof that $VCDim(DT_k) < 2^k+1$: recall that a function from DT_k can divide the input space into at most 2^k intervals. Consequentially, the pigeonhole principle tells us that, given $2^k + 1$ points, at least one interval must contain two points no matter which function we are considering. This implies that no function in DT_k can shatter $2^k + 1$ points, since every function in the class will contain an interval with more than one point and thus cannot assign different labels to those points.

(c) **(5 points)** Now consider a concept class DT_∞ consisting of all decision trees of unbounded depth where the function at each node is an element of T. Give the VC dimension of DT_∞, and prove that your answer is correct.

We will show that $VCDim(DT_\infty) = \infty$.

Proof: For all positive integers m, given m points, we can construct a tree of height $\lceil log_2(m) \rceil$ that places each point in a separate interval, thus allowing the set of points to be shattered. Since there is no limit to how deep the tree can be constructed, we can therefore shatter any number of points.

(d) **(7 points)** Assume that you are given a set S of m examples that are consistent with a concept in DT_k. Give an efficient learning algorithm that produces a hypothesis h that is consistent with S.

Note: The hypothesis you learn, h, **does not need to be** in DT_k. You can represent it any way you want.

Let $S = \{(x_1, y_1), \ldots (x_m, y_m)\}$ be the set of the m examples after we sorted it based on the values of the x_is. That is, $x_1 \le x_2 \le \ldots x_m$.

We create a list of interval boundaries in the following way: we place a boundary between two points with a different label. We represent the set of intervals determined by the set S as a set of pairs (b_i, y_i), where b_i is the lower end of the interval and y_i is the label for that interval. Formally:

- Initialize list of interval boundaries $I = [(-\infty, y_1)]$
- for $i = 1 \ldots m - 1$:
  - if $y_i \ne y_{i+1}$:
    - add $(\frac{x_i + x_{i+1}}{2}, y_{i+1})$ to I

The set I is the hypothesis learned by our algorithm.

To classify a x, find the boundary with the largest b_i such that $b_i \le x$. and assign x the label y_i. Note that this way it is clear that the hypothesis I is consistent with the training set S.

Since the set S of examples given is known to be consistent with a function in DT_k, at most 2^k intervals will be constructed. Building the hypothesis can thus be done in $O(m \log m)$ (assuming an efficient sorting algorithm is used). Classifying new points can be accomplished in $O(k)$ time if binary search is used.

(e) **(6 points)** Is the concept class DT_k PAC learnable? Explain your answer.

Yes. Any hypothesis class with a finite VC dimension is PAC Learnable.

Neural Networks [25 points]

Consider the following set S of examples over the feature space $X = \{X_1, X_2\}$. These examples were labeled based on the XNOR (NOT XOR) function.

X_1	X_2	y^* (Label)
0	0	1
0	1	0
1	0	0
1	1	1

(a) **(4 points)** The set of 4 examples given above is not linearly separable in the $X = \{X_1, X_2\}$ space. Explain this statement in **one sentence**.

It means that there exists no triple of real numbers (w_1, w_2, b) such that for all labeled examples (X_1, X_2, y^*) given, we have that: $y^*(w_1X_1 + w_2X_2 + b) > 0$.

(b) **(6 points)** Propose a new set of features $Z = \{Z_1, \dots Z_k\}$ such that in the Z space, this set of examples is linearly separable.

i. Define each Z_i as a function of the X_is.

There are many such mappings, and any reasonable mapping that results in linearly separable data is acceptable. One such mapping is :-

$Z_1 = \neg X_1 \wedge \neg X_2, \quad Z_2 = X_1 \wedge X_2$

ii. Write down the set of 4 examples given above in the new Z space.

Following the same order of examples as in the table above, we get:

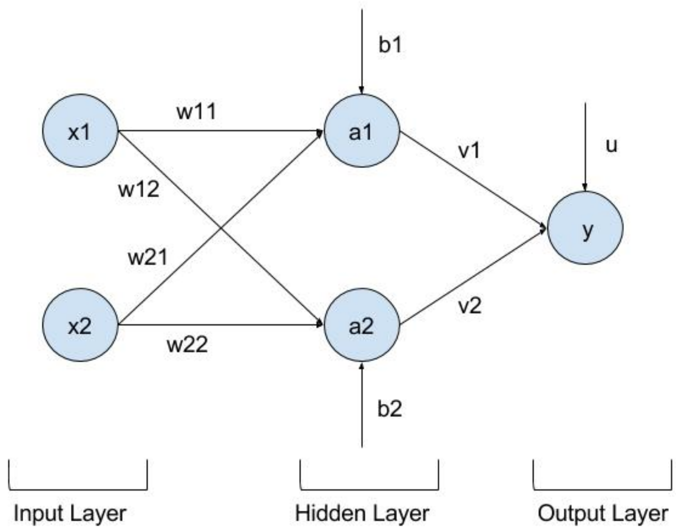
Z_1	Z_2	y^* (Label)
1	0	1
0	0	0
0	0	0
0	1	1

iii. Show that the data set is linearly separable. (**Show**, don't just say that it is separable.)

Given the definition of linear separability in (a), we need to provide a triple (w_1, w_2, b) that linearly separates the points in the Z space. We note that in the Z space the target function is a disjunction $Z_1 \vee Z_2$ and therefore one such triple is $(w_1 = 1, w_2 = 1, b = -0.5)$.

(c) **(5 points)** Now consider running the set S of examples presented above in the space X through a neural network with a single hidden layer, as shown in the figure below.

Note that numbers on the edges correspond to weights and the arrows into the units indicate the bias term. Recall that the output of a node (denoted by the terms inside the nodes in the graph e.g. a_1, a_2, y) in the neural network is given by $f(w^T x + b)$, where x is the input to the unit, w are the weights on the input, b is the bias in the unit, and f is the activation function.



For the sake of simplicity, assume that the function $sgn(x)$ ($sgn(x) = 1$ if $x \geq 0$, 0 otherwise) is used as the activation function at all the nodes of the network.

Which of the following sets of weights guarantees that the neural network above is *consistent* with all the examples in S? (That is, the 0-1 loss is 0).

The correct set of weights is ______(3)______
{option (1) | option (2) | option (3)}

Options:

Options	w_{11}	w_{21}	b_1	w_{12}	w_{22}	b_2	v_1	v_2	u
1	1	0	- 0.5	0	1	- 0.5	- 1	- 1	0.9
2	1	1	0.5	- 1	- 1	2.5	0	- 1	0.5
3	1	1	- 0.5	- 1	- 1	1.5	- 1	- 1	1.5

To show that the neural network is consistent with that dataset, one needs to run each example through the neural network, and check that the output of the neural network matches with the true label. We show this for option (3) below.

X_1	X_2	a_1	a_2	y	y^* (Label)
0	0	0	1	1	1
0	1	1	1	0	0
1	0	1	1	0	0
1	1	1	0	1	1

(d) **(10 points)** We now want to use the data set S to *learn* the neural network depicted earlier.

We will use the **sigmoid** function, $sigmoid(x) = (1 + exp^{-x})^{-1}$, as the activation function in the hidden layer, and **no activation function in the output layer** (i.e. it's just a linear unit). As the loss function we will use the Hinge Loss:

$$\text{Hinge loss(w, x, b, y*)} = \begin{cases} 1 - y^*(w^T x + b), & \text{if } y^*(w^T x + b) > 1 \\ 0, & \text{otherwise} \end{cases}$$

Write down the BackPropagation update rules for the weights in the output layer ($\boldsymbol{\Delta v_i}$), and the hidden layer ($\boldsymbol{\Delta w_{ij}}$). By definition, we have that:

$$v_i^{t+1} = v_i^t + \Delta v_i$$

and

$$w_{ij}^{t+1} = w_{ij}^t + \Delta w_{ij}$$

where the updates are computed by:

$$\Delta v_i = -\eta \nabla v_i$$

and

$$\Delta w_{ij} = -\eta \nabla w_{ij}$$

We now need to compute the derivatives. First, assuming no activation function on the output layer, we get that:

$$\nabla v_i = \begin{cases} -y^* a_i, & \text{if } y^* y > 1 \\ 0, & \text{otherwise} \end{cases}$$

and:

$$\delta_i = \begin{cases} -y^* v_i, & \text{if } y^* y > 1 \\ 0, & \text{otherwise} \end{cases}$$

And, assuming the sigmoid function as the activation function in the hidden layer, we get:

$$\nabla w_{ij} = \delta_j a_j (1 - a_j) x_i$$

Short Questions [25 points]

(a) **(10 points)** In this part of the problem we consider Adaboost. Let D_t be the probability distribution in the tth round of Adaboost, h_t be the weak learning hypothesis learned in the tth round, and ϵ_t its error.

i. Denote by $D_t(i)$ the weight of the ith example under the distribution D_t. Use it to write an expression for the error ϵ_t of the AdaBoost weak learner in the tth round.

Let S be the set of all examples. We denote by $[a]$ the characteristic function that returns 1 if a is true and 0 otherwise. Then, the error is defined as the total weight, under D_t, of the examples h_t misclassifies:

$$\epsilon_t = Error_{D_t}(h_t) = \sum_{i \in S} D_t(i)[h_t(i) \neg = y(i)]$$

ii. Consider the following four statements **with respect to the hypothesis at time t,** h_t. Circle the one that is true, and provide a short explanation.

A. $\forall t, Error_{D_t}(\boldsymbol{h_t}) = Error_{D_{t+1}}(\boldsymbol{h_t})$

B. $\forall t, Error_{D_t}(\boldsymbol{h_t}) > Error_{D_{t+1}}(\boldsymbol{h_t})$

C. $\forall t, Error_{D_t}(\boldsymbol{h_t}) < Error_{D_{t+1}}(\boldsymbol{h_t})$

D. The relation between $Error_{D_t}(\boldsymbol{h_t})$ and $Error_{D_{t+1}}(\boldsymbol{h_t})$ cannot be determined in general.

Explanation: The right option is C. Consider the set of examples that are misclassified by h_t. On the left hand side we have an expression that gives the total weight of these examples under D_t. On the right hand side we have an expression that gives the total weight of the same set of examples under D_{t+1}. However, Adaboost reweighs each example e that is *misclassified* by h_t so that $D_{t+1}(e) > D_t(e)$, resulting in C being correct.

Note also that we know that the value of the left hand side is less than 1/2, by the weak learning assumption, and it can be shown (but not needed as part of the explanation) that the value of the right hand side is exactly 1/2.

(b) **(10 points)** We consider Boolean functions in the class $L_{10,20,100}$. This is the class of 10 *out of* 20 *out of* 100*, defined over* $\{x_1, x_2, \ldots x_{100}\}$.

Recall that a function in the class $L_{10,20,100}$ is defined by a set of 20 *relevant variables*. An example $x \in \{0,1\}^{100}$ is positive if and only if *at least* 10 out these 20 are **on**.

In the following discussion, for the sake of simplicity, whenever we consider a member in $L_{10,20,100}$, we will consider the function f in which the *first* 20 coordinates are the relevant coordinates.

i. Show that the perceptron algorithm can be used to learn functions in the class $L_{10,20,100}$. In order to do so,

A. Show a linear threshold function h that behaves just like $f \in L_{10,20,100}$ on $\{0,1\}^{100}$.

f is determined by a bias terms $b = -10$, and a weight vectors $w \in R^{100}$ so that $w_i = 1$, for $i = 1, 2, \ldots 20$, and $w_i = 0$ otherwise.
It is easy to see that $w \cdot x + b > 0$ iff $f(x) = 1$

B. Write h as a weight vector that goes through the origin and has size (as measured by the L_2 norm) equal to 1.

To represent h as a weight vector that goes through the origin we represent the example now as $x' = (x, 1) \in \{0,1\}^{101}$ and the weight vector as $w' = (w, b) \in R^{101}$. We note that $w \cdot x + b = w' \cdot x'$. To make sure that h has L_2 norm that is equal to 1, we normalize it by dividing (w, b) by $||(w, b)||_2 = \sqrt{20 + 100}$.

ii. Let R be the set of 20 variables defining the target function. We consider the following two data sets, both of which have examples with 50 **on** bits.

$\mathbf{D_1}$: In all the negative examples exactly 9 of the variables in R are **on**; in all the positive examples exactly 11 of the variables in R are **on**.

$\mathbf{D_2}$: In all the negative examples exactly 5 of the variables in R are **on**; in all the positive examples exactly 15 of the variables in R are **on**.

Consider running perceptron on D_1 and on D_2. On which of these data sets do you expect Perceptron to make less mistakes?

Perceptron will make less mistakes on the data set

____
$\{D_1 \mid D_2\}$

Perceptron will make less mistakes on D_2 since the margin of this data set is larger than that of D_1.

iii. Define the *margin* of a data set D with respect to weight vector w.

$$\gamma = min_{||w||_2=1}|yw \cdot x|$$

Explain your answer to (ii) using the notion of the *margin*.
Formally we can appeal to Novikoff's bound, assuming that, w.l.o.g., R will be the same, since the mistake bound of perceptron is lower bounded by R^2/γ^2.

(c) **(5 points)** Let f be a concept that is defined on examples drawn from a distribution D. The "true" error of the hypothesis h is defined as

$$Error_D(h) = Pr_{x \in D}\left(h(x) \neq f(x)\right).$$

In the class, we saw that the true error of a classifier is bounded above by two terms that relate to the training data and the hypothesis space. That is

$$Error_D(h) < \quad A \quad + \quad B$$

What are A and B? (If you do not remember the exact functional forms of these terms, it is sufficient to **briefly** describe what they mean.)

A is the training error of h.

B is a term that bounds how much will the true error of h deviate from the observed (training) error of h. This term scales proportionally with the VC dimension of the hypothesis space H, and is inversely proportional to the number of training examples.

Appendix

(a) $Entropy(S) = -p_+ log_2(p_+) - p_- log_2(p_-)$

(b) $Gain(S, a) = Entropy(S) - \sum_{v \in values(a)} \frac{|S_v|}{|S|} Entropy(S_v)$

(c) $\text{sgn(x)} = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$

(d) $\text{sigmoid(x)} = \frac{1}{1 + exp^{-x}}$

(e) $\frac{\partial}{\partial x} sigmoid(x) = sigmoid(x)\Big(1 - sigmoid(x)\Big)$

(f) $\text{ReLU(x)} = max(0, x)$

(g) $\frac{\partial}{\partial x} ReLU(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$

(h) $\text{tanh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(i) $\frac{\partial}{\partial x} tanh(x) = 1 - tanh^2(x)$

(j) $\text{Zero-One loss}(y,\, y^*) = \begin{cases} 1, & \text{if } y \neq y^* \\ 0, & \text{if } y = y^* \end{cases}$

(k) $\text{Hinge loss(w, x, b, y*)} = \begin{cases} 1 - y^*(w^T x + b), & \text{if } y^*(w^T x + b) > 1 \\ 0, & \text{otherwise} \end{cases}$

(l) $\frac{\partial}{\partial w}\ \text{Hinge loss(w, x, b, y*)} = \begin{cases} -y^*(x), & \text{if } y^*(w^T x + b) > 1 \\ 0, & \text{otherwise} \end{cases}$

(m) $\text{Squared loss}(w,\, x,\, y^*) = \frac{1}{2}(w^T x - y^*)^2$

(n) $\frac{\partial}{\partial w}\ \text{Squared loss}(w, x, y^*) = x(w^T x - y^*)$