Chapter 4

Polyhedra and Polytopes

4.1 Polyhedra, $\mathcal{H}$-Polytopes and $\mathcal{V}$-Polytopes

There are two natural ways to define a convex polyhedron, A:

(1) As the convex hull of a finite set of points.

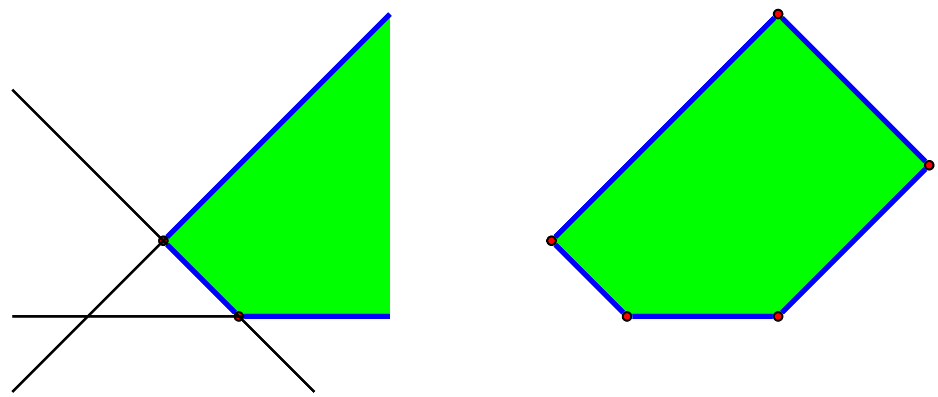
(2) As a subset of $\mathbb{E}^n$ cut out by a finite number of hyperplanes, more precisely, as the intersection of a finite number of (closed) half-spaces.

As stated, these two definitions are not equivalent because (1) implies that a polyhedron is bounded, whereas (2) allows unbounded subsets. Now, if we require in (2) that the convex set A is bounded, it is quite clear for $n = 2$ that the two definitions (1) and (2) are equivalent; for $n = 3$, it is intuitively clear that definitions (1) and (2) are still equivalent, but proving this equivalence rigorously does not appear to be that easy. What about the equivalence when $n \geq 4$?

It turns out that definitions (1) and (2) are equivalent for all n, but this is a nontrivial theorem and a rigorous proof does not come by so cheaply. Fortunately, since we have Krein and Milman's theorem at our disposal and polar duality, we can give a rather short proof. The hard direction of the equivalence consists in proving that definition (1) implies definition (2). This is where the duality induced by polarity becomes handy, especially, the fact that $A^{**} = A$! (under the right hypotheses). First, we give precise definitions (following Ziegler [45]).

Definition 4.1 Let $\mathcal{E}$ be any affine Euclidean space of finite dimension, n.[1] An *$\mathcal{H}$-polyhedron* in $\mathcal{E}$, for short, a *polyhedron*, is any subset, $P = \bigcap_{i=1}^{p} C_i$, of $\mathcal{E}$ defined as the intersection of a finite number, $p \geq 1$, of closed half-spaces, C_i; an *$\mathcal{H}$-polytope* in $\mathcal{E}$ is a bounded polyhedron and a *$\mathcal{V}$-polytope* is the convex hull, $P = \mathrm{conv}(S)$, of a finite set of points, $S \subseteq \mathcal{E}$.

[1] This means that the vector space, $\overrightarrow{\mathcal{E}}$, associated with $\mathcal{E}$ is a Euclidean space.



(a) (b)

Figure 4.1: (a) An $\mathcal{H}$-polyhedron. (b) A $\mathcal{V}$-polytope

Obviously, polyhedra and polytopes are convex and closed (in $\mathcal{E}$). Since the notions of $\mathcal{H}$-polytope and $\mathcal{V}$-polytope are equivalent (see Theorem 4.7), we often use the simpler locution polytope. Examples of an $\mathcal{H}$-polyhedron and of a $\mathcal{V}$-polytope are shown in Figure 4.1.

Note that Definition 4.1 allows $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes to have an empty interior, which is somewhat of an inconvenience. This is not a problem, since we may always restrict ourselves to the affine hull of P (some affine space, E, of dimension $d \leq n$, where $d = \dim(P)$, as in Definition 2.1) as we now show.

Proposition 4.1 *Let $A \subseteq \mathcal{E}$ be a $\mathcal{V}$-polytope or an $\mathcal{H}$-polyhedron, let $E = \mathrm{aff}(A)$ be the affine hull of A in $\mathcal{E}$ (with the Euclidean structure on E induced by the Euclidean structure on $\mathcal{E}$) and write $d = \dim(E)$. Then, the following assertions hold:*

(1) The set, A, is a $\mathcal{V}$-polytope in E (i.e., viewed as a subset of E) iff A is a $\mathcal{V}$-polytope in $\mathcal{E}$.

(2) The set, A, is an $\mathcal{H}$-polyhedron in E (i.e., viewed as a subset of E) iff A is an $\mathcal{H}$-polyhedron in $\mathcal{E}$.

Proof. (1) This follows immediately because E is an affine subspace of $\mathcal{E}$ and every affine subspace of $\mathcal{E}$ is closed under affine combinations and so, *a fortiori*, under convex combinations. We leave the details as an easy exercise.

(2) Assume A is an $\mathcal{H}$-polyhedron in $\mathcal{E}$ and that $d < n$. By definition, $A = \bigcap_{i=1}^{p} C_i$, where the C_i are closed half-spaces determined by some hyperplanes, $H_1, \ldots, H_p$, in $\mathcal{E}$. (Observe that the hyperplanes, H_i's, associated with the closed half-spaces, C_i, may not be distinct.

For example, we may have $C_i = (H_i)_+$ and $C_j = (H_i)_-$, for the two closed half-spaces determined by H_i.) As $A \subseteq E$, we have

$$A = A \cap E = \bigcap_{i=1}^{p} (C_i \cap E),$$

where $C_i \cap E$ is one of the closed half-spaces determined by the hyperplane, $H_i' = H_i \cap E$, in E. Thus, A is also an $\mathcal{H}$-polyhedron in E.

Conversely, assume that A is an $\mathcal{H}$-polyhedron in E and that $d < n$. As any hyperplane, H, in $\mathcal{E}$ can be written as the intersection, $H = H_- \cap H_+$, of the two closed half-spaces that it bounds, E itself can be written as the intersection,

$$E = \bigcap_{i=1}^{p} E_i = \bigcap_{i=1}^{p} (E_i)_+ \cap (E_i)_-,$$

of finitely many half-spaces in $\mathcal{E}$. Now, as A is an $\mathcal{H}$-polyhedron in E, we have

$$A = \bigcap_{j=1}^{q} C_j,$$

where the C_j are closed half-spaces in E determined by some hyperplanes, H_j, in E. However, each H_j can be extended to a hyperplane, H_j', in $\mathcal{E}$, and so, each C_j can be extended to a closed half-space, C_j', in $\mathcal{E}$ and we still have

$$A = \bigcap_{j=1}^{q} C_j'.$$

Consequently, we get

$$A = A \cap E = \bigcap_{i=1}^{p} ((E_i)_+ \cap (E_i)_-) \cap \bigcap_{j=1}^{q} C_j',$$

which proves that A is also an $\mathcal{H}$-polyhedron in $\mathcal{E}$. $\square$

The following simple proposition shows that we may assume that $\mathcal{E} = \mathbb{E}^n$:

Proposition 4.2 *Given any two affine Euclidean spaces, E and F, if $h\colon E \to F$ is any affine map then:*

(1) If A is any $\mathcal{V}$-polytope in E, then $h(E)$ is a $\mathcal{V}$-polytope in F.

(2) If h is bijective and A is any $\mathcal{H}$-polyhedron in E, then $h(E)$ is an $\mathcal{H}$-polyhedron in F.

Proof. (1) As any affine map preserves affine combinations it also preserves convex combination. Thus, $h(\mathrm{conv}(S)) = \mathrm{conv}(h(S))$, for any $S \subseteq E$.

(2) Say $A = \bigcap_{i=1}^{p} C_i$ in E. Consider any half-space, C, in E and assume that

$$C = \{x \in E \mid \varphi(x) \leq 0\},$$

for some affine form, φ, defining the hyperplane, $H = \{x \in E \mid \varphi(x) = 0\}$. Then, as h is bijective, we get

$$\begin{aligned} h(C) &= \{h(x) \in F \mid \varphi(x) \leq 0\} \\ &= \{y \in F \mid \varphi(h^{-1}(y)) \leq 0\} \\ &= \{y \in F \mid (\varphi \circ h^{-1})(y) \leq 0\}. \end{aligned}$$

This shows that $h(C)$ is one of the closed half-spaces in F determined by the hyperplane, $H' = \{y \in F \mid (\varphi \circ h^{-1})(y) = 0\}$. Furthermore, as h is bijective, it preserves intersections so

$$h(A) = h\left(\bigcap_{i=1}^{p} C_i\right) = \bigcap_{i=1}^{p} h(C_i),$$

a finite intersection of closed half-spaces. Therefore, $h(A)$ is an $\mathcal{H}$-polyhedron in F. $\square$

By Proposition 4.2 we may assume that $\mathcal{E} = \mathbb{E}^d$ and by Proposition 4.1 we may assume that $\dim(A) = d$. These propositions justify the type of argument beginning with: "We may assume that $A \subseteq \mathbb{E}^d$ has dimension d, that is, that A has nonempty interior". This kind of reasonning will occur many times.

Since the boundary of a closed half-space, C_i, is a hyperplane, H_i, and since hyperplanes are defined by affine forms, a closed half-space is defined by the locus of points satisfying a "linear" inequality of the form $a_i \cdot x \leq b_i$ or $a_i \cdot x \geq b_i$, for some vector $a_i \in \mathbb{R}^n$ and some $b_i \in \mathbb{R}$. Since $a_i \cdot x \geq b_i$ is equivalent to $(-a_i) \cdot x \leq -b_i$, we may restrict our attention to inequalities with a $\leq$ sign. Thus, if A is the $p \times n$ matrix whose i^{th} row is a_i, we see that the $\mathcal{H}$-polyhedron, P, is defined by the system of linear inequalities, $Ax \leq b$, where $b = (b_1, \ldots, b_p) \in \mathbb{R}^p$. We write

$$P = P(A, b), \quad \textit{with} \quad P(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

An equation, $a_i \cdot x = b_i$, may be handled as the conjunction of the two inequalities $a_i \cdot x \leq b_i$ and $(-a_i) \cdot x \leq -b_i$. Also, if $0 \in P$, observe that we must have $b_i \geq 0$ for $i = 1, \ldots, p$. In this case, every inequality for which $b_i > 0$ can be normalized by dividing both sides by b_i, so we may assume that $b_i = 1$ or $b_i = 0$. This observation will be useful to show that the polar dual of an $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.

Remark: Some authors call "convex" polyhedra and "convex" polytopes what we have simply called polyhedra and polytopes. Since Definition 4.1 implies that these objects are

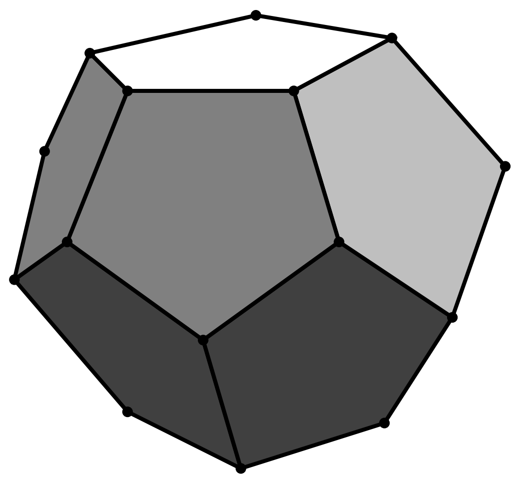


Figure 4.2: Example of a polytope (a dodecahedron)

convex and since we are not going to consider non-convex polyhedra in this chapter, we stick to the simpler terminology.

One should consult Ziegler [45], Berger [6], Grunbaum [24] and especially Cromwell [14], for pictures of polyhedra and polytopes. Figure 4.2 shows the picture a polytope whose faces are all pentagons. This polytope is called a *dodecahedron*. The dodecahedron has 12 faces, 30 edges and 20 vertices.

Even better and a lot more entertaining, take a look at the spectacular web sites of George Hart,

Virtual Polyedra: http://www.georgehart.com/virtual-polyhedra/vp.html,

George Hart's web site: http://www.georgehart.com/

and also

Zvi Har'El's web site: http://www.math.technion.ac.il/ rl/

The *Uniform Polyhedra* web site: http://www.mathconsult.ch/showroom/unipoly/

Paper Models of Polyhedra: http://www.korthalsaltes.com/

Bulatov's *Polyhedra Collection*: http://www.physics.orst.edu/ bulatov/polyhedra/

Paul Getty's *Polyhedral Solids*: http://home.teleport.com/ tpgettys/poly.shtml

Jill Britton's *Polyhedra Pastimes*: http://ccins.camosun.bc.ca/ jbritton/jbpolyhedra.htm

and many other web sites dealing with polyhedra in one way or another by searching for "polyhedra" on *Google*!

Obviously, an n-simplex is a $\mathcal{V}$-polytope. The *standard n-cube* is the set

$$\{(x_1, \ldots, x_n) \in \mathbb{E}^n \mid |x_i| \leq 1, \quad 1 \leq i \leq n\}.$$

The standard cube is a $\mathcal{V}$-polytope. The *standard n-cross-polytope* (or *n-co-cube*) is the set

$$\{(x_1, \ldots, x_n) \in \mathbb{E}^n \mid \sum_{i=1}^{n} |x_i| \leq 1\}.$$

It is also a $\mathcal{V}$-polytope.

What happens if we take the dual of a $\mathcal{V}$-polytope (resp. an $\mathcal{H}$-polytope)? The following proposition, although very simple, is an important step in answering the above question:

Proposition 4.3 *Let $S = \{a_i\}_{i=1}^{p}$ be a finite set of points in $\mathbb{E}^n$ and let $A = \mathrm{conv}(S)$ be its convex hull. If $S \neq \{O\}$, then, the dual, A^*, of A w.r.t. the center O is the $\mathcal{H}$-polyhedron given by*

$$A^* = \bigcap_{i=1}^{p} (a_i^\dagger)_-.$$

Furthermore, if $O \in \overset{\circ}{A}$, then A^* is an $\mathcal{H}$-polytope, i.e., the dual of a $\mathcal{V}$-polytope with nonempty interior is an $\mathcal{H}$-polytope. If $A = S = \{O\}$, then $A^* = \mathbb{E}^d$.

Proof. By definition, we have

$$A^* = \{b \in \mathbb{E}^n \mid \mathbf{Ob} \cdot (\sum_{j=1}^{p} \lambda_j \mathbf{Oa_j}) \leq 1, \quad \lambda_j \geq 0, \sum_{j=1}^{p} \lambda_j = 1\},$$

and the right hand side is clearly equal to $\bigcap_{i=1}^{p} \{b \in \mathbb{E}^n \mid \mathbf{Ob} \cdot \mathbf{Oa_i} \leq 1\} = \bigcap_{i=1}^{p} (a_i^\dagger)_-$, which is a polyhedron. (Recall that $(a_i^\dagger)_- = \mathbb{E}^n$ if $a_i = O$.) If $O \in \overset{\circ}{A}$, then A^* is bounded (by Proposition 3.21) and so, A^* is an $\mathcal{H}$-polytope. $\square$

Thus, the dual of the convex hull of a finite set of points, $\{a_1, \ldots, a_p\}$, is the intersection of the half-spaces containing O determined by the polar hyperplanes of the points a_i.

It is convenient to restate Proposition 4.3 using matrices. First, observe that the proof of Proposition 4.3 shows that

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = \mathrm{conv}(\{a_1, \ldots, a_p\} \cup \{O\})^*.$$

Therefore, we may assume that not all $a_i = O$ $(1 \leq i \leq p)$. If we pick O as an origin, then every point a_j can be identified with a vector in $\mathbb{E}^n$ and O corresponds to the zero vector, 0. Observe that any set of p points, $a_j \in \mathbb{E}^n$, corresponds to the $n \times p$ matrix, A, whose j^{th} column is a_j. Then, the equation of the the polar hyperplane, $a_j^\dagger$, of any a_j $(\neq 0)$ is $a_j \cdot x = 1$, that is

$$a_j^\top x = 1.$$

Consequently, the system of inequalities defining $\mathrm{conv}(\{a_1, \ldots, a_p\})^*$ can be written in matrix form as

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\},$$

where $\mathbf{1}$ denotes the vector of $\mathbb{R}^p$ with all coordinates equal to 1. We write $P(A^\top, \mathbf{1}) = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\}$. There is a useful converse of this property as proved in the next proposition.

Proposition 4.4 *Given any set of p points, $\{a_1, \ldots, a_p\}$, in $\mathbb{R}^n$ with $\{a_1, \ldots, a_p\} \neq \{0\}$, if A is the $n \times p$ matrix whose j^{th} column is a_j, then*

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = P(A^\top, \mathbf{1}),$$

with $P(A^\top, \mathbf{1}) = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\}$.

Conversely, given any $p \times n$ matrix, A, not equal to the zero matrix, we have

$$P(A, \mathbf{1})^* = \mathrm{conv}(\{a_1, \ldots, a_p\} \cup \{0\}),$$

where $a_i \in \mathbb{R}^n$ is the i^{th} row of A or, equivalently,

$$P(A, \mathbf{1})^* = \{x \in \mathbb{R}^n \mid x = A^\top t,\ t \in \mathbb{R}^p,\ t \geq 0,\ \mathbb{I}t = 1\},$$

where $\mathbb{I}$ is the row vector of length p whose coordinates are all equal to 1.

Proof. Only the second part needs a proof. Let $B = \mathrm{conv}(\{a_1, \ldots, a_p\} \cup \{0\})$, where $a_i \in \mathbb{R}^n$ is the i^{th} *row* of A. Then, by the first part,

$$B^* = P(A, \mathbf{1}).$$

As $0 \in B$, by Proposition 3.21, we have $B = B^{**} = P(A, \mathbf{1})^*$, as claimed. $\square$

Remark: Proposition 4.4 still holds if A is the zero matrix because then, the inequalities $A^\top x \leq \mathbf{1}$ (or $Ax \leq \mathbf{1}$) are trivially satisfied. In the first case, $P(A^\top, \mathbf{1}) = \mathbb{E}^d$ and in the second case, $P(A, \mathbf{1}) = \mathbb{E}^d$.

Using the above, the reader should check that the dual of a simplex is a simplex and that the dual of an n-cube is an n-cross polytope.

Observe that not every $\mathcal{H}$-polyhedron is of the form $P(A, \mathbf{1})$. Firstly, 0 belongs to the interior of $P(A, \mathbf{1})$ and, secondly cones with apex 0 can't be described in this form. However, we will see in Section 4.3 that the full class of polyhedra can be captured is we allow inequalities of the form $a^\top x \leq 0$. In order to find the corresponding "$\mathcal{V}$-definition" we will need to add positive combinations of vectors to convex combinations of points. Intuitively, these vectors correspond to "points at infinity".

We will see shortly that if A is an $\mathcal{H}$-polytope and if $O \in \overset{\circ}{A}$, then A^* is also an $\mathcal{H}$-polytope. For this, we will prove first that an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. This requires taking a closer look at polyhedra.

Note that some of the hyperplanes cutting out a polyhedron may be redundant. If $A = \bigcap_{i=1}^t C_i$ is a polyhedron (where each closed half-space, C_i, is associated with a hyperplane, H_i, so that $\partial C_i = H_i$), we say that $\bigcap_{i=1}^t C_i$ *is an irredundant decomposition of A* if

A cannot be expressed as $A = \bigcap_{i=1}^{m} C_i'$ with $m < t$ (for some closed half-spaces, C_i'). The following proposition shows that the C_i in an irredundant decomposition of A are uniquely determined by A.

Proposition 4.5 *Let A be a polyhedron with nonempty interior and assume that $A = \bigcap_{i=1}^{t} C_i$ is an irredundant decomposition of A. Then,*

(i) Up to order, the C_i's are uniquely determined by A.

(ii) If $H_i = \partial C_i$ is the boundary of C_i, then $H_i \cap A$ is a polyhedron with nonempty interior in H_i, denoted $\mathrm{Facet}_i\, A$, *and called a* **facet** *of A.*

(iii) We have $\partial A = \bigcup_{i=1}^{t} \mathrm{Facet}_i\, A$, where the union is irredundant, i.e., $\mathrm{Facet}_i\, A$ *is not a subset of* $\mathrm{Facet}_j\, A$, *for all $i \neq j$.*

Proof. (ii) Fix any i and consider $A_i = \bigcap_{j \neq i} C_j$. As $A = \bigcap_{i=1}^{t} C_i$ is an irredundant decomposition, there is some $x \in A_i - C_i$. Pick any $a \in \overset{\circ}{A}$. By Lemma 3.1, we get $b = [a, x] \cap H_i \in \overset{\circ}{A}_i$, so b belongs to the interior of $H_i \cap A_i$ in H_i.

(iii) As $\partial A = A - \overset{\circ}{A} = A \cap (\overset{\circ}{A})^c$ (where B^c denotes the complement of a subset B of $\mathbb{E}^n$) and $\partial C_i = H_i$, we get

$$
\begin{aligned}
\partial A &= \left(\bigcap_{i=1}^{t} C_i\right) - \left(\overset{\circ}{\overline{\bigcap_{j=1}^{t} C_j}}\right) \\
&= \left(\bigcap_{i=1}^{t} C_i\right) - \left(\bigcap_{j=1}^{t} \overset{\circ}{C}_j\right) \\
&= \left(\bigcap_{i=1}^{t} C_i\right) \cap \left(\bigcap_{j=1}^{t} \overset{\circ}{C}_j\right)^c \\
&= \left(\bigcap_{i=1}^{t} C_i\right) \cap \left(\bigcup_{j=1}^{t} (\overset{\circ}{C}_j)^c\right) \\
&= \bigcup_{j=1}^{t} \left(\left(\bigcap_{i=1}^{t} C_i\right) \cap (\overset{\circ}{C}_j)^c\right) \\
&= \bigcup_{j=1}^{t} \left(\partial C_j \cap \left(\bigcap_{i \neq j} C_i\right)\right) \\
&= \bigcup_{j=1}^{t} (H_j \cap A) = \bigcup_{j=1}^{t} \mathrm{Facet}_j\, A.
\end{aligned}
$$

If we had $\mathrm{Facet}_i\, A \subseteq \mathrm{Facet}_j\, A$, for some $i \neq j$, then, by (ii), as the affine hull of $\mathrm{Facet}_i\, A$ is H_i and the affine hull of $\mathrm{Facet}_j\, A$ is H_j, we would have $H_i \subseteq H_j$, a contradiction.

(i) As the decomposition is irredundant, the H_i are pairwise distinct. Also, by (ii), each facet, $\mathrm{Facet}_i\, A$, has dimension $d-1$ (where $d = \dim A$). Then, in (iii), we can show that the decomposition of ∂A as a union of polytopes of dimension $d-1$ whose pairwise nonempty intersections have dimension at most $d-2$ (since they are contained in pairwise distinct hyperplanes) is unique up to permutation. Indeed, assume that

$$\partial A = F_1 \cup \cdots \cup F_m = G_1 \cup \cdots \cup G_n,$$

where the F_i's and G_j' are polyhedra of dimension $d-1$ and each of the unions is irredundant. Then, we claim that for each F_i, there is some $G_{\varphi(i)}$ such that $F_i \subseteq G_{\varphi(i)}$. If not, F_i would be expressed as a union

$$F_i = (F_i \cap G_{i_1}) \cup \cdots \cup (F_i \cap G_{i_k})$$

where $\dim(F_i \cap G_{i_j}) \leq d-2$, since the hyperplanes containing F_i and the G_j's are pairwise distinct, which is absurd, since $\dim(F_i) = d-1$. By symmetry, for each G_j, there is some $F_{\psi(j)}$ such that $G_j \subseteq F_{\psi(j)}$. But then, $F_i \subseteq F_{\psi(\varphi(i))}$ for all i and $G_j \subseteq G_{\varphi(\psi(j))}$ for all j which implies $\psi(\varphi(i)) = i$ for all i and $\varphi(\psi(j)) = j$ for all j since the unions are irredundant. Thus, φ and ψ are mutual inverses and the B_j's are just a permutation of the A_i's, as claimed. Therefore, the facets, $\mathrm{Facet}_i\, A$, are uniquely determined by A and so are the hyperplanes, $H_i = \mathrm{aff}(\mathrm{Facet}_i\, A)$, and the half-spaces, C_i, that they determine. $\square$

As a consequence, if A is a polyhedron, then so are its facets and the same holds for $\mathcal{H}$-polytopes. If A is an $\mathcal{H}$-polytope and H is a hyperplane with $H \cap \overset{\circ}{A} \neq \emptyset$, then $H \cap A$ is an $\mathcal{H}$-polytope whose facets are of the form $H \cap F$, where F is a facet of A.

We can use induction and define k-faces, for $0 \leq k \leq n-1$.

Definition 4.2 Let $A \subseteq \mathbb{E}^n$ be a polyhedron with nonempty interior. We define a *k-face of A* to be a facet of a $(k+1)$-face of A, for $k = 0, \ldots, n-2$, where an *$(n-1)$-face* is just a facet of A. The 1-faces are called *edges*. Two k-faces are *adjacent* if their intersection is a $(k-1)$-face.

The polyhedron A itself is also called a *face* (of itself) or *n-face* and the k-faces of A with $k \leq n-1$ are called *proper faces* of A. If $A = \bigcap_{i=1}^t C_i$ is an irredundant decomposition of A and H_i is the boundary of C_i, then the hyperplane, H_i, is called the *supporting hyperplane* of the facet $H_i \cap A$. We suspect that the 0-faces of a polyhedron are vertices in the sense of Definition 2.5. This is true and, in fact, the vertices of a polyhedron coincide with its extreme points (see Definition 2.6).

Proposition 4.6 *Let $A \subseteq \mathbb{E}^n$ be a polyhedron with nonempty interior.*

(1) For any point, $a \in \partial A$, on the boundary of A, the intersection of all the supporting hyperplanes to A at a coincides with the intersection of all the faces that contain a. In particular, points of order k of A are those points in the relative interior of the k-faces of A[2]; thus, 0-faces coincide with the vertices of A.

[2]Given a convex set, S, in $\mathbb{A}^n$, its *relative interior* is its interior in the affine hull of S (which might be of dimension strictly less than n).

(2) The vertices of A coincide with the extreme points of A.

Proof. (1) If H is a supporting hyperplane to A at a, then, one of the half-spaces, C, determined by H, satisfies $A = A \cap C$. It follows from Proposition 4.5 that if $H \neq H_i$ (where the hyperplanes H_i are the supporting hyperplanes of the facets of A), then C is redundant, from which (1) follows.

(2) If $a \in \partial A$ is not extreme, then $a \in [y, z]$, where $y, z \in \partial A$. However, this implies that a has order $k \geq 1$, i.e, a is not a vertex. $\square$

4.2 The Equivalence of $\mathcal{H}$-Polytopes and $\mathcal{V}$-Polytopes

We are now ready for the theorem showing the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes. This is a nontrivial theorem usually attributed to Weyl and Minkowski (for example, see Barvinok [3]).

Theorem 4.7 *(Weyl-Minkowski) If A is an $\mathcal{H}$-polytope, then A has a finite number of extreme points (equal to its vertices) and A is the convex hull of its set of vertices; thus, an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. Moreover, A has a finite number of k-faces (for $k = 0, \ldots, d-2$, where $d = \mathrm{dim}(A)$). Conversely, the convex hull of a finite set of points is an $\mathcal{H}$-polytope. As a consequence, a $\mathcal{V}$-polytope is an $\mathcal{H}$-polytope.*

Proof. By restricting ourselves to the affine hull of A (some $\mathbb{E}^d$, with $d \leq n$) we may assume that A has nonempty interior. Since an $\mathcal{H}$-polytope has finitely many facets, we deduce by induction that an $\mathcal{H}$-polytope has a finite number of k-faces, for $k = 0, \ldots, d-2$. In particular, an $\mathcal{H}$-polytope has finitely many vertices. By proposition 4.6, these vertices are the extreme points of A and since an $\mathcal{H}$-polytope is compact and convex, by the theorem of Krein and Milman (Theorem 2.8), A is the convex hull of its set of vertices.

Conversely, again, we may assume that A has nonempty interior by restricting ourselves to the affine hull of A. Then, pick an origin, O, in the interior of A and consider the dual, A^*, of A. By Proposition 4.3, the convex set A^* is an $\mathcal{H}$-polytope. By the first part of the proof of Theorem 4.7, the $\mathcal{H}$-polytope, A^*, is the convex hull of its vertices. Finally, as the hypotheses of Proposition 3.21 and Proposition 4.3 (again) hold, we deduce that $A = A^{**}$ is an $\mathcal{H}$-polytope. $\square$

In view of Theorem 4.7, we are justified in dropping the $\mathcal{V}$ or $\mathcal{H}$ in front of polytope, and will do so from now on. Theorem 4.7 has some interesting corollaries regarding the dual of a polytope.

Corollary 4.8 *If A is any polytope in $\mathbb{E}^n$ such that the interior of A contains the origin, O, then the dual, A^*, of A is also a polytope whose interior contains O and $A^{**} = A$.*

Corollary 4.9 *If A is any polytope in $\mathbb{E}^n$ whose interior contains the origin, O, then the k-faces of A are in bijection with the $(n-k-1)$-faces of the dual polytope, A^*. This correspondence is as follows: If $Y = \mathrm{aff}(F)$ is the k-dimensional subspace determining the k-face, F, of A then the subspace, $Y^* = \mathrm{aff}(F^*)$, determining the corresponding face, F^*, of A^*, is the intersection of the polar hyperplanes of points in Y.*

Proof. Immediate from Proposition 4.6 and Proposition 3.22. □

We also have the following proposition whose proof would not be that simple if we only had the notion of an $\mathcal{H}$-polytope (as a matter of fact, there is a way of proving Theorem 4.7 using Proposition 4.10)

Proposition 4.10 *If $A \subseteq \mathbb{E}^n$ is a polytope and $f\colon \mathbb{E}^n \to \mathbb{E}^m$ is an affine map, then $f(A)$ is a polytope in $\mathbb{E}^m$.*

Proof. Immediate, since an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope and since affine maps send convex sets to convex sets. □

The reader should check that the Minkowski sum of polytopes is a polytope.

We were able to give a short proof of Theorem 4.7 because we relied on a powerful theorem, namely, Krein and Milman. A drawback of this approach is that it bypasses the interesting and important problem of designing algorithms for finding the vertices of an $\mathcal{H}$-polyhedron from the sets of inequalities defining it. A method for doing this is Fourier-Motzkin elimination, see Ziegler [45] (Chapter 1) and Section 4.3. This is also a special case of *linear programming*.

It is also possible to generalize the notion of $\mathcal{V}$-polytope to polyhedra using the notion of cone and to generalize the equivalence theorem to $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra.

4.3 The Equivalence of $\mathcal{H}$-Polyhedra and $\mathcal{V}$-Polyhedra

The equivalence of $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes can be generalized to polyhedral sets, *i.e.*, finite intersections of closed half-spaces that are not necessarily bounded. This equivalence was first proved by Motzkin in the early 1930's. It can be proved in several ways, some involving cones.

Definition 4.3 Let $\mathcal{E}$ be any affine Euclidean space of finite dimension, n (with associated vector space, $\overrightarrow{\mathcal{E}}$). A subset, $C \subseteq \overrightarrow{\mathcal{E}}$, is a *cone* if C is closed under linear combinations involving only nonnegative scalars called *positive combinations*. Given a subset, $V \subseteq \overrightarrow{\mathcal{E}}$, the *conical hull* or *positive hull* of V is the set

$$\mathrm{cone}(V) = \Big\{\sum_I \lambda_i v_i \mid \{v_i\}_{i\in I} \subseteq V,\ \lambda_i \geq 0 \quad \text{for all } i \in I\Big\}.$$

A $\mathcal{V}$*-polyhedron* or *polyhedral set* is a subset, $A \subseteq \mathcal{E}$, such that

$$A = \mathrm{conv}(Y) + \mathrm{cone}(V) = \{a + v \mid a \in \mathrm{conv}(Y),\ v \in \mathrm{cone}(V)\},$$

where $V \subseteq \overrightarrow{\mathcal{E}}$ is a finite set of vectors and $Y \subseteq \mathcal{E}$ is a finite set of points.

A set, $C \subseteq \overrightarrow{\mathcal{E}}$, is a $\mathcal{V}$*-cone* or *polyhedral cone* if C is the positive hull of a finite set of vectors, that is,

$$C = \mathrm{cone}(\{u_1, \ldots, u_p\}),$$

for some vectors, $u_1, \ldots, u_p \in \overrightarrow{\mathcal{E}}$. An $\mathcal{H}$*-cone* is any subset of $\overrightarrow{\mathcal{E}}$ given by a finite intersection of closed half-spaces cut out by hyperplanes through 0.

The positive hull, $\mathrm{cone}(V)$, of V is also denoted $\mathrm{pos}(V)$. Observe that a $\mathcal{V}$-cone can be viewed as a polyhedral set for which $Y = \{O\}$, a single point. However, if we take the point O as the origin, we may view a $\mathcal{V}$-polyhedron, A, for which $Y = \{O\}$, as a $\mathcal{V}$-cone. We will switch back and forth between these two views of cones as we find it convenient, this should not cause any confusion. In this section, we favor the view that $\mathcal{V}$-cones are special kinds of $\mathcal{V}$-polyhedra. As a consequence, a ($\mathcal{V}$ or $\mathcal{H}$)-cone always contains 0, sometimes called an *apex* of the cone.

A set of the form $\{a + tu \mid t \geq 0\}$, where $a \in \mathcal{E}$ is a point and $u \in \overrightarrow{\mathcal{E}}$ is a nonzero vector, is called a *half-line* or *ray*. Then, we see that a $\mathcal{V}$-polyhedron, $A = \mathrm{conv}(Y) + \mathrm{cone}(V)$, is the convex hull of the union of a finite set of points with a finite set of rays. In the case of a $\mathcal{V}$-cone, all these rays meet in a common point, an apex of the cone.

Propositions 4.1 and 4.2 generalize easily to $\mathcal{V}$-polyhedra and cones.

Proposition 4.11 *Let $A \subseteq \mathcal{E}$ be a $\mathcal{V}$-polyhedron or an $\mathcal{H}$-polyhedron, let $E = \mathrm{aff}(A)$ be the affine hull of A in $\mathcal{E}$ (with the Euclidean structure on E induced by the Euclidean structure on $\mathcal{E}$) and write $d = \dim(E)$. Then, the following assertions hold:*

(1) The set, A, is a $\mathcal{V}$-polyhedron in E (i.e., viewed as a subset of E) iff A is a $\mathcal{V}$-polyhedron in $\mathcal{E}$.

(2) The set, A, is an $\mathcal{H}$-polyhedron in E (i.e., viewed as a subset of E) iff A is an $\mathcal{H}$-polyhedron in $\mathcal{E}$.

Proof. We already proved (2) in Proposition 4.1. For (1), observe that the direction, $\overrightarrow{E}$, of E is a linear subspace of $\overrightarrow{\mathcal{E}}$. Consequently, E is closed under affine combinations and $\overrightarrow{E}$ is closed under linear combinations and the result follows immediately. $\square$

Proposition 4.12 *Given any two affine Euclidean spaces, E and F, if $h\colon E \to F$ is any affine map then:*

(1) If A is any $\mathcal{V}$-polyhedron in E, then $h(E)$ is a $\mathcal{V}$-polyhedron in F.

(2) If $g\colon \overrightarrow{E} \to \overrightarrow{F}$ is any linear map and if C is any $\mathcal{V}$-cone in $\overrightarrow{E}$, then $g(C)$ is a $\mathcal{V}$-cone in $\overrightarrow{F}$.

(3) If h is bijective and A is any $\mathcal{H}$-polyhedron in E, then $h(E)$ is an $\mathcal{H}$-polyhedron in F.

Proof. We already proved (3) in Proposition 4.2. For (1), using the fact that $h(a+u) = h(a) + \overrightarrow{h}(u)$ for any affine map, h, where $\overrightarrow{h}$ is the linear map associated with h, we get

$$h(\mathrm{conv}(Y) + \mathrm{cone}(V)) = \mathrm{conv}(h(Y)) + \mathrm{cone}(\overrightarrow{h}(V)).$$

For (2), as g is linear, we get

$$g(\mathrm{cone}(V)) = \mathrm{cone}(g(V)),$$

establishing the proposition. $\square$

Propositions 4.11 and 4.12 allow us to assume that $\mathcal{E} = \mathbb{E}^d$ and that our ($\mathcal{V}$ or $\mathcal{H}$)-polyhedra, $A \subseteq \mathbb{E}^d$, have nonempty interior (*i.e.* $\dim(A) = d$).

The generalization of Theorem 4.7 is that every $\mathcal{V}$-polyhedron, A, is an $\mathcal{H}$-polyhedron and conversely. At first glance, it may seem that there is a small problem when $A = \mathbb{E}^d$. Indeed, Definition 4.3 allows the possibility that $\mathrm{cone}(V) = \mathbb{E}^d$ for some finite subset, $V \subseteq \mathbb{R}^d$. This is because it is possible to generate a basis of $\mathbb{R}^d$ using finitely many positive combinations. On the other hand the definition of an $\mathcal{H}$-polyhedron, A, (Definition 4.1) assumes that $A \subseteq \mathbb{E}^n$ is cut out by *at least one* hyperplane. So, A is always contained in some half-space of $\mathbb{E}^n$ and strictly speaking, $\mathbb{E}^n$ is not an $\mathcal{H}$-polyhedron! The simplest way to circumvent this difficulty is to decree that $\mathbb{E}^n$ itself is a polyhedron, which we do.

Yet another solution is to assume that, unless stated otherwise, every finite set of vectors, V, that we consider when defining a polyhedron has the property that there is some hyperplane, H, through the origin so that all the vectors in V lie in only one of the two closed half-spaces determined by H. But then, the polar dual of a polyhedron can't be a single point! Therefore, we stick to our decision that $\mathbb{E}^n$ itself is a polyhedron.

To prove the equivalence of $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra, Ziegler proceeds as follows: First, he shows that the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra reduces to the equivalence of $\mathcal{V}$-cones and $\mathcal{H}$-cones using an "old trick" of projective geometry, namely, "homogenizing" [45] (Chapter 1). Then, he uses two dual versions of Fourier-Motzkin elimination to pass from $\mathcal{V}$-cones to $\mathcal{H}$-cones and conversely.

Since the homogenization method is an important technique we will describe it in some detail. However, it turns out that the double dualization technique used in the proof of Theorem 4.7 can be easily adapted to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron. Moreover, it can also be used to prove that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron! So, we will not describe the version of Fourier-Motzkin elimination used to go from $\mathcal{V}$-cones to $\mathcal{H}$-cones. However, we will present the Fourier-Motzkin elimination method used to go from $\mathcal{H}$-cones to $\mathcal{V}$-cones.

Here is the generalization of Proposition 4.3 to polyhedral sets. In order to avoid confusion between the origin of $\mathbb{E}^d$ and the center of polar duality we will denote the origin by O and the center of our polar duality by Ω. Given any nonzero vector, $u \in \mathbb{R}^d$, let $u_-^\dagger$ be the closed half-space

$$u_-^\dagger = \{x \in \mathbb{R}^d \mid x \cdot u \leq 0\}.$$

In other words, $u_-^\dagger$ is the closed half-space bounded by the hyperplane through Ω normal to u and on the "opposite side" of u.

Proposition 4.13 *Let $A = \mathrm{conv}(Y) + \mathrm{cone}(V) \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron with $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Then, for any point, Ω, if $A \neq \{\Omega\}$, then the polar dual, A^*, of A w.r.t. Ω is the $\mathcal{H}$-polyhedron given by*

$$A^* = \bigcap_{i=1}^{p} (y_i^\dagger)_- \cap \bigcap_{j=1}^{q} (v_j^\dagger)_-.$$

Furthermore, if A has nonempty interior and Ω belongs to the interior of A, then A^* is bounded, that is, A^* is an $\mathcal{H}$-polytope. If $A = \{\Omega\}$, then A^* is the special polyhedron, $A^* = \mathbb{E}^d$.

Proof. By definition of A^* w.r.t. Ω, we have

$$\begin{aligned}
A^* &= \left\{ x \in \mathbb{E}^d \;\middle|\; \mathbf{\Omega x} \cdot \mathbf{\Omega} \left(\sum_{\mathbf{i=1}}^{\mathbf{p}} \lambda_{\mathbf{i}} \mathbf{y_i} + \sum_{\mathbf{j=1}}^{\mathbf{q}} \mu_{\mathbf{j}} \mathbf{v_j} \right) \leq 1,\ \lambda_i \geq 0,\ \sum_{i=1}^{p} \lambda_i = 1,\ \mu_j \geq 0 \right\} \\
&= \left\{ x \in \mathbb{E}^d \;\middle|\; \sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} + \sum_{j=1}^{q} \mu_j \mathbf{\Omega x} \cdot v_j \leq 1,\ \lambda_i \geq 0,\ \sum_{i=1}^{p} \lambda_i = 1,\ \mu_j \geq 0 \right\}.
\end{aligned}$$

When $\mu_j = 0$ for $j = 1, \ldots, q$, we get

$$\sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} \leq 1, \quad \lambda_i \geq 0,\ \sum_{i=1}^{p} \lambda_i = 1$$

and we check that

$$\begin{aligned}
\left\{ x \in \mathbb{E}^d \;\middle|\; \sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} \leq 1,\ \lambda_i \geq 0,\ \sum_{i=1}^{p} \lambda_i = 1 \right\} &= \bigcap_{i=1}^{p} \{x \in \mathbb{E}^d \mid \mathbf{\Omega x} \cdot \mathbf{\Omega y_i} \leq 1\} \\
&= \bigcap_{i=1}^{p} (y_i^\dagger)_-.
\end{aligned}$$

The points in A^* must also satisfy the conditions

$$\sum_{j=1}^{q} \mu_j \mathbf{\Omega x} \cdot v_j \leq 1 - \alpha, \quad \mu_j \geq 0,\ \mu_j > 0 \text{ for some } j,\ 1 \leq j \leq q,$$

with $\alpha \leq 1$ (here $\alpha = \sum_{i=1}^{p} \lambda_i \mathbf{\Omega x} \cdot \mathbf{\Omega y_i}$). In particular, for every $j \in \{1, \ldots, q\}$, if we set $\mu_k = 0$ for $k \in \{1, \ldots, q\} - \{j\}$, we should have

$$\mu_j \mathbf{\Omega x} \cdot v_j \leq 1 - \alpha \quad \text{for all} \quad \mu_j > 0,$$

that is,

$$\mathbf{\Omega x} \cdot v_j \leq \frac{1-\alpha}{\mu_j} \quad \text{for all} \quad \mu_j > 0,$$

which is equivalent to

$$\mathbf{\Omega x} \cdot v_j \leq 0.$$

Consequently, if $x \in A^*$, we must also have

$$x \in \bigcap_{j=1}^{q} \{x \in \mathbb{E}^d \mid \mathbf{\Omega x} \cdot v_j \leq 0\} = \bigcap_{j=1}^{q} (v_j^\dagger)_-.$$

Therefore,

$$A^* \subseteq \bigcap_{i=1}^{p} (y_i^\dagger)_- \cap \bigcap_{j=1}^{q} (v_j^\dagger)_-.$$

However, the reverse inclusion is obvious and thus, we have equality. If Ω belongs to the interior of A, we know from Proposition 3.21 that A^* is bounded. Therefore, A^* is indeed an $\mathcal{H}$-polytope of the above form. $\square$

It is fruitful to restate Proposition 4.13 in terms of matrices (as we did for Proposition 4.3). First, observe that

$$(\mathrm{conv}(Y) + \mathrm{cone}(V))^* = (\mathrm{conv}(Y \cup \{\Omega\}) + \mathrm{cone}(V))^*.$$

If we pick Ω as an origin then we can represent the points in Y as vectors. The old origin is still denoted O and Ω is now denoted 0. The zero vector is denoted $\mathbf{0}$.

If $A = \mathrm{conv}(Y) + \mathrm{cone}(V) \neq \{0\}$, let Y be the $d \times p$ matrix whose i^{th} column is y_i and let V is the $d \times q$ matrix whose j^{th} column is v_j. Then Proposition 4.13 says that

$$(\mathrm{conv}(Y) + \mathrm{cone}(V))^* = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}.$$

We write $P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}) = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}$.

If $A = \mathrm{conv}(Y) + \mathrm{cone}(V) = \{0\}$, then both Y and V must be zero matrices but then, the inequalities $Y^\top x \leq \mathbf{1}$ and $V^\top x \leq \mathbf{0}$ are trivially satisfied by all $x \in \mathbb{E}^d$, so even in this case we have

$$\mathbb{E}^d = (\mathrm{conv}(Y) + \mathrm{cone}(V))^* = P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}).$$

The converse of Proposition 4.13 also holds as shown below.

Proposition 4.14 *Let $\{y_1, \ldots, y_p\}$ be any set of points in $\mathbb{E}^d$ and let $\{v_1, \ldots, v_q\}$ be any set of nonzero vectors in $\mathbb{R}^d$. If Y is the $d \times p$ matrix whose i^{th} column is y_i and V is the $d \times q$ matrix whose j^{th} column is v_j, then*

$$(\mathrm{conv}(\{y_1, \ldots, y_p\}) + \mathrm{cone}(\{v_1, \ldots, v_q\}))^* = P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}),$$

with $P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}) = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}$.

Conversely, given any $p \times d$ matrix, Y, and any $q \times d$ matrix, V, we have

$$P(Y, \mathbf{1}; V, \mathbf{0})^* = \mathrm{conv}(\{y_1, \ldots, y_p\} \cup \{0\}) + \mathrm{cone}(\{v_1, \ldots, v_q\}),$$

where $y_i \in \mathbb{R}^n$ is the i^{th} row of Y and $v_j \in \mathbb{R}^n$ is the j^{th} row of V or, equivalently,

$$P(Y, \mathbf{1}; V, \mathbf{0})^* = \{x \in \mathbb{R}^d \mid x = Y^\top u + V^\top t,\ u \in \mathbb{R}^p,\ t \in \mathbb{R}^q,\ u, t \geq 0,\ \mathbb{I}u = 1\},$$

where $\mathbb{I}$ is the row vector of length p whose coordinates are all equal to 1.

Proof. Only the second part needs a proof. Let

$$B = \mathrm{conv}(\{y_1, \ldots, y_p\} \cup \{0\}) + \mathrm{cone}(\{v_1, \ldots, v_q\}),$$

where $y_i \in \mathbb{R}^p$ is the i^{th} *row* of Y and $v_j \in \mathbb{R}^q$ is the j^{th} *row* of V. Then, by the first part,

$$B^* = P(Y, \mathbf{1}; V, \mathbf{0}).$$

As $0 \in B$, by Proposition 3.21, we have $B = B^{**} = P(Y, \mathbf{1}; V, \mathbf{0})$, as claimed. $\square$

Proposition 4.14 has the following important Corollary:

Proposition 4.15 *The following assertions hold:*

(1) The polar dual, A^*, of every $\mathcal{H}$-polyhedron, is a $\mathcal{V}$-polyhedron.

(2) The polar dual, A^*, of every $\mathcal{V}$-polyhedron, is an $\mathcal{H}$-polyhedron.

Proof. (1) We may assume that $0 \in A$, in which case, A is of the form $A = P(Y, \mathbf{1}; V, \mathbf{0})$. By the second part of Proposition 4.14, A^* is a $\mathcal{V}$-polyhedron.

(2) This is the first part of Proposition 4.14. $\square$

We can now use Proposition 4.13, Proposition 3.21 and Krein and Milman's Theorem to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron.

Proposition 4.16 *Every $\mathcal{V}$-polyhedron, A, is an $\mathcal{H}$-polyhedron. Furthermore, if $A \neq \mathbb{E}^d$, then A is of the form $A = P(Y, \mathbf{1})$.*

Proof. Let A be a $\mathcal{V}$-polyhedron of dimension d. Thus, $A \subseteq \mathbb{E}^d$ has nonempty interior so we can pick some point, Ω, in the interior of A. If $d = 0$, then $A = \{0\} = \mathbb{E}^0$ and we are done. Otherwise, by Proposition 4.13, the polar dual, A^*, of A w.r.t. Ω is an $\mathcal{H}$-polytope. Then, as in the proof of Theorem 4.7, using Krein and Milman's Theorem we deduce that A^* is a $\mathcal{V}$-polytope. Now, as Ω belongs to A, by Proposition 3.21 (even if A is not bounded) we have $A = A^{**}$ and by Proposition 4.3 (or Proposition 4.13) we conclude that $A = A^{**}$ is an $\mathcal{H}$-polyhedron of the form $A = P(Y, \mathbf{1})$. $\square$

Interestingly, we can now prove easily that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.

Proposition 4.17 *Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.*

Proof. Let A be an $\mathcal{H}$-polyhedron of dimension d. By Proposition 4.15, the polar dual, A^*, of A is a $\mathcal{V}$-polyhedron. By Proposition 4.16, A^* is an $\mathcal{H}$-polyhedron and again, by Proposition 4.15, we deduce that $A^{**} = A$ is a $\mathcal{V}$-polyhedron ($A = A^{**}$ because $0 \in A$). $\square$

Putting together Propositions 4.16 and 4.17 we obtain our main theorem:

Theorem 4.18 *(Equivalence of $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra) Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron and conversely.*

Even though we proved the main result of this section, it is instructive to consider a more computational proof making use of cones and an elimination method known as *Fourier-Motzkin elimination*.

4.4 Fourier-Motzkin Elimination and the Polyhedron-Cone Correspondence

The problem with the converse of Proposition 4.16 when A is unbounded (*i.e.*, not compact) is that Krein and Milman's Theorem does not apply. We need to take into account "points at infinity" corresponding to certain vectors. The trick we used in Proposition 4.16 is that the polar dual of a $\mathcal{V}$-polyhedron with nonempty interior is an *$\mathcal{H}$-polytope*. This reduction to polytopes allowed us to use Krein and Milman to convert an $\mathcal{H}$-polytope to a $\mathcal{V}$-polytope and then again we took the polar dual.

Another trick is to switch to cones by "homogenizing". Given any subset, $S \subseteq \mathbb{E}^d$, we can form the cone, $C(S) \subseteq \mathbb{E}^{d+1}$, by "placing" a copy of S in the hyperplane, $H_{d+1} \subseteq \mathbb{E}^{d+1}$, of equation $x_{d+1} = 1$, and drawing all the half-lines from the origin through any point of S. If S is given by m polynomial inequalities of the form

$$P_i(x_1, \ldots, x_d) \leq b_i,$$

where $P_i(x_1, \ldots, x_d)$ is a polynomial of total degree n_i, this amounts to forming the new homogeneous inequalities

$$x_{d+1}^{n_i} P_i\left(\frac{x_1}{x_{d+1}}, \ldots, \frac{x_d}{x_{d+1}}\right) - b_i x_{d+1}^{n_i} \leq 0$$

together with $x_{d+1} \geq 0$. In particular, if the P_i's are linear forms (which means that $n_i = 1$), then our inequalities are of the form

$$a_i \cdot x \leq b_i,$$

where $a_i \in \mathbb{R}^d$ is some vector and the homogenized inequalities are

$$a_i \cdot x - b_i x_{d+1} \leq 0.$$

It will be convenient to formalize the process of putting a copy of a subset, $S \subseteq \mathbb{E}^d$, in the hyperplane, $H_{d+1} \subseteq \mathbb{E}^{d+1}$, of equation $x_{d+1} = 1$, as follows: For every point, $a \in \mathbb{E}^d$, let

$$\widehat{a} = \begin{pmatrix} a \\ 1 \end{pmatrix} \in \mathbb{E}^{d+1}$$

and let $\widehat{S} = \{\widehat{a} \mid a \in S\}$. Obviously, the map $S \mapsto \widehat{S}$ is a bijection between the subsets of $\mathbb{E}^d$ and the subsets of H_{d+1}. We will use this bijection to identify S and $\widehat{S}$ and use the simpler notation, S, unless confusion arises. Let's apply this to polyhedra.

Let $P \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-polyhedron. Then, P is cut out by m hyperplanes, H_i, and for each H_i, there is a nonzero vector, a_i, and some $b_i \in \mathbb{R}$ so that

$$H_i = \{x \in \mathbb{E}^d \mid a_i \cdot x = b_i\}$$

and P is given by

$$P = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid a_i \cdot x \leq b_i\}.$$

If A denotes the $m \times d$ matrix whose i-th row is a_i and b is the vector $b = (b_1, \ldots, b_m)$, then we can write

$$P = P(A, b) = \{x \in \mathbb{E}^d \mid Ax \leq b\}.$$

We "homogenize" $P(A, b)$ as follows: Let $C(P)$ be the subset of $\mathbb{E}^{d+1}$ defined by

$$\begin{aligned} C(P) &= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax \leq x_{d+1} b,\ x_{d+1} \geq 0 \right\} \\ &= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax - x_{d+1} b \leq 0,\ -x_{d+1} \leq 0 \right\}. \end{aligned}$$

Thus, we see that $C(P)$ is the $\mathcal{H}$-cone given by the system of inequalities

$$\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and that

$$\widehat{P} = C(P) \cap H_{d+1}.$$

Conversely, if Q is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), it is clear that $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $H_{d+1} \approx \mathbb{E}^d$.

Let us now assume that $P \subseteq \mathbb{E}^d$ is a $\mathcal{V}$-polyhedron, $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$, where $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Define $\widehat{Y} = \{\widehat{y}_1, \ldots, \widehat{y}_p\} \subseteq \mathbb{E}^{d+1}$, and $\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_q\} \subseteq \mathbb{E}^{d+1}$, by

$$\widehat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \qquad \widehat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

We check immediately that

$$C(P) = \mathrm{cone}(\{\widehat{Y} \cup \widehat{V}\})$$

is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$ such that

$$\widehat{P} = C(P) \cap H_{d+1},$$

where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$.

Conversely, if $C = \mathrm{cone}(W)$ is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$, with $w_{i\,d+1} \geq 0$ for every $w_i \in W$, we prove next that $P = C \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron.

Proposition 4.19 *(Polyhedron–Cone Correspondence) We have the following correspondence between polyhedra in $\mathbb{E}^d$ and cones in $\mathbb{E}^{d+1}$:*

(1) For any $\mathcal{H}$-polyhedron, $P \subseteq \mathbb{E}^d$, if $P = P(A, b) = \{x \in \mathbb{E}^d \mid Ax \leq b\}$, where A is an $m \times d$-matrix and $b \in \mathbb{R}^m$, then $C(P)$ given by

$$\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ and $\widehat{P} = C(P) \cap H_{d+1}$, where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$. Conversely, if Q is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), then $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $H_{d+1} \approx \mathbb{E}^d$.

(2) Let $P \subseteq \mathbb{E}^d$ be any $\mathcal{V}$-polyhedron, where $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$ with $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Define $\widehat{Y} = \{\widehat{y}_1, \ldots, \widehat{y}_p\} \subseteq \mathbb{E}^{d+1}$, and $\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_q\} \subseteq \mathbb{E}^{d+1}$, by

$$\widehat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \qquad \widehat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

Then,

$$C(P) = \mathrm{cone}(\{\widehat{Y} \cup \widehat{V}\})$$

is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$ such that

$$\widehat{P} = C(P) \cap H_{d+1},$$

Conversely, if $C = \mathrm{cone}(W)$ is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$, with $w_{i\,d+1} \geq 0$ for every $w_i \in W$, then $P = C \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron in $H_{d+1} \approx \mathbb{E}^d$.

In both (1) and (2), $\widehat{P} = \{\widehat{p} \mid p \in P\}$, with

$$\widehat{p} = \begin{pmatrix} p \\ 1 \end{pmatrix} \in \mathbb{E}^{d+1}.$$

Proof. We already proved everything except the last part of the proposition. Let

$$\widehat{Y} = \left\{ \frac{w_i}{w_{i\,d+1}} \;\middle|\; w_i \in W,\; w_{i\,d+1} > 0 \right\}$$

and

$$\widehat{V} = \{w_j \in W \mid w_{j\,d+1} = 0\}.$$

We claim that

$$P = C \cap H_{d+1} = \mathrm{conv}(\widehat{Y}) + \mathrm{cone}(\widehat{V}),$$

and thus, P is a $\mathcal{V}$-polyhedron.

Recall that any element, $z \in C$, can be written as

$$z = \sum_{k=1}^{s} \mu_k w_k, \quad \mu_k \geq 0.$$

Thus, we have

$$\begin{aligned}
z &= \sum_{k=1}^{s} \mu_k w_k \quad \mu_k \geq 0 \\
&= \sum_{w_{i\,d+1} > 0} \mu_i w_i + \sum_{w_{j\,d+1} = 0} \mu_j w_j \quad \mu_i, \mu_j \geq 0 \\
&= \sum_{w_{i\,d+1} > 0} w_{i\,d+1} \mu_i \frac{w_i}{w_{i\,d+1}} + \sum_{w_{j\,d+1} = 0} \mu_j w_j, \quad \mu_i, \mu_j \geq 0 \\
&= \sum_{w_{i\,d+1} > 0} \lambda_i \frac{w_i}{w_{i\,d+1}} + \sum_{w_{j\,d+1} = 0} \mu_j w_j, \quad \lambda_i, \mu_j \geq 0.
\end{aligned}$$

Now, $z \in C \cap H_{d+1}$ iff $z_{d+1} = 1$ iff $\sum_{i=1}^{p} \lambda_i = 1$ (where p is the number of elements in $\widehat{Y}$), since the $(d+1)^{\text{th}}$ coordinate of $\frac{w_i}{w_{i\,d+1}}$ is equal to 1, and the above shows that

$$P = C \cap H_{d+1} = \mathrm{conv}(\widehat{Y}) + \mathrm{cone}(\widehat{V}),$$

as claimed. $\square$

By Proposition 4.19, if P is an $\mathcal{H}$-polyhedron, then $C(P)$ is an $\mathcal{H}$-cone. If we can prove that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone, then again, Proposition 4.19 shows that $\widehat{P} = C(P) \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron and so, P is a $\mathcal{V}$-polyhedron. Therefore, in order to prove that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron it suffices to show that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone.

By a similar argument, Proposition 4.19 shows that in order to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron it suffices to show that every $\mathcal{V}$-cone is an $\mathcal{H}$-cone. We will not prove this direction again since we already have it by Proposition 4.16.

It remains to prove that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone. Let $C \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-cone. Then, C is cut out by m hyperplanes, H_i, through 0. For each H_i, there is a nonzero vector, u_i, so that

$$H_i = \{x \in \mathbb{E}^d \mid u_i \cdot x = 0\}$$

and C is given by

$$C = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid u_i \cdot x \leq 0\}.$$

If A denotes the $m \times d$ matrix whose i-th row is u_i, then we can write

$$C = P(A, 0) = \{x \in \mathbb{E}^d \mid Ax \leq 0\}.$$

Observe that $C = C_0(A) \cap H_w$, where

$$C_0(A) = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid Ax \leq w \right\}$$

is an $\mathcal{H}$-cone in $\mathbb{E}^{d+m}$ and

$$H_w = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid w = 0 \right\}$$

is an affine subspace in $\mathbb{E}^{d+m}$.

We claim that $C_0(A)$ is a $\mathcal{V}$-cone. This follows by observing that for every $\binom{x}{w}$ satisfying $Ax \leq w$, we can write

$$\begin{pmatrix} x \\ w \end{pmatrix} = \sum_{i=1}^{d} |x_i| (\mathrm{sign}(x_i)) \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} + \sum_{j=1}^{m} (w_j - (Ax)_j) \begin{pmatrix} 0 \\ e_j \end{pmatrix},$$

and then

$$C_0(A) = \mathrm{cone}\left(\left\{ \pm \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} \mid 1 \leq i \leq d \right\} \cup \left\{ \begin{pmatrix} 0 \\ e_j \end{pmatrix} \mid 1 \leq j \leq m \right\} \right).$$

Since $C = C_0(A) \cap H_w$ is now the intersection of a $\mathcal{V}$-cone with an affine subspace, to prove that C is a $\mathcal{V}$-cone it is enough to prove that the intersection of a $\mathcal{V}$-cone with a hyperplane is also a $\mathcal{V}$-cone. For this, we use *Fourier-Motzkin elimination*. It suffices to prove the result for a hyperplane, H_k, in $\mathbb{E}^{d+m}$ of equation $y_k = 0$ $(1 \leq k \leq d + m)$.

Proposition 4.20 *(Fourier-Motzkin Elimination) Say $C = \mathrm{cone}(Y) \subseteq \mathbb{E}^d$ is a $\mathcal{V}$-cone. Then, the intersection $C \cap H_k$ (where H_k is the hyperplane of equation $y_k = 0$) is a $\mathcal{V}$-cone, $C \cap H_k = \mathrm{cone}(Y^{/k})$, with*

$$Y^{/k} = \{y_i \mid y_{ik} = 0\} \cup \{y_{ik}y_j - y_{jk}y_i \mid y_{ik} > 0,\ y_{jk} < 0\},$$

the set of vectors obtained from Y by "eliminating the k-th coordinate". Here, each y_i is a vector in $\mathbb{R}^d$.

Proof. The only nontrivial direction is to prove that $C \cap H_k \subseteq \mathrm{cone}(Y^{/k})$. For this, consider any $v = \sum_{i=1}^d t_i y_i \in C \cap H_k$, with $t_i \geq 0$ and $v_k = 0$. Such a v can be written

$$v = \sum_{i \mid y_{ik}=0} t_i y_i + \sum_{i \mid y_{ik}>0} t_i y_i + \sum_{j \mid y_{jk}<0} t_j y_j$$

and as $v_k = 0$, we have

$$\sum_{i \mid y_{ik}>0} t_i y_{ik} + \sum_{j \mid y_{jk}<0} t_j y_{jk} = 0.$$

If $t_i y_{ik} = 0$ for $i = 1, \ldots, d$, we are done. Otherwise, we can write

$$\Lambda = \sum_{i \mid y_{ik}>0} t_i y_{ik} = \sum_{j \mid y_{jk}<0} -t_j y_{jk} > 0.$$

Then,

$$\begin{aligned} v &= \sum_{i \mid y_{ik}=0} t_i y_i + \frac{1}{\Lambda} \sum_{i \mid y_{ik}>0} \left(\sum_{j \mid y_{jk}<0} -t_j y_{jk} \right) t_i y_i + \frac{1}{\Lambda} \sum_{j \mid y_{jk}<0} \left(\sum_{i \mid y_{ik}>0} t_i y_{ik} \right) t_j y_j \\ &= \sum_{i \mid y_{ik}=0} t_i y_i + \sum_{\substack{i \mid y_{ik}>0 \\ j \mid y_{jk}<0}} \frac{t_i t_j}{\Lambda} \left(y_{ik} y_j - y_{jk} y_i \right). \end{aligned}$$

Since the k^{th} coordinate of $y_{ik}y_j - y_{jk}y_i$ is 0, the above shows that any $v \in C \cap H_k$ can be written as a positive combination of vectors in $Y^{/k}$. $\square$

As discussed above, Proposition 4.20 implies (again!)

Corollary 4.21 *Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.*

Another way of proving that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron is to use cones. This method is interesting in its own right so we discuss it briefly.

Let $P = \mathrm{conv}(Y) + \mathrm{cone}(V) \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron. We can view Y as a $d \times p$ matrix whose ith column is the ith vector in Y and V as $d \times q$ matrix whose jth column is the jth vector in V. Then, we can write

$$P = \{x \in \mathbb{R}^d \mid (\exists u \in \mathbb{R}^p)(\exists t \in \mathbb{R}^d)(x = Yu + Vt,\ u \geq 0,\ \mathbb{I}u = 1,\ t \geq 0)\},$$

where $\mathbb{I}$ is the row vector

$$\mathbb{I} = \underbrace{(1, \ldots, 1)}_{p}.$$

Now, observe that P can be interpreted as the projection of the $\mathcal{H}$-polyhedron, $\widetilde{P} \subseteq \mathbb{E}^{d+p+q}$, given by

$$\widetilde{P} = \{(x, u, t) \in \mathbb{R}^{d+p+q} \mid x = Yu + Vt,\ u \geq 0,\ \mathbb{I}u = 1,\ t \geq 0\}$$

onto $\mathbb{R}^d$. Consequently, if we can prove that the projection of an $\mathcal{H}$-polyhedron is also an $\mathcal{H}$-polyhedron, then we will have proved that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron. Here again, it is possible that $P = \mathbb{E}^d$, but that's fine since $\mathbb{E}^d$ has been declared to be an $\mathcal{H}$-polyhedron.

In view of Proposition 4.19 and the discussion that followed, it is enough to prove that the projection of any $\mathcal{H}$-cone is an $\mathcal{H}$-cone. This can be done by using a type of Fourier-Motzkin elimination dual to the method used in Proposition 4.20. We state the result without proof and refer the interested reader to Ziegler [45], Section 1.2–1.3, for full details.

Proposition 4.22 *If $C = P(A, 0) \subseteq \mathbb{E}^d$ is an $\mathcal{H}$-cone, then the projection,* $\mathrm{proj}_k(C)$, *onto the hyperplane, H_k, of equation $y_k = 0$ is given by* $\mathrm{proj}_k(C) = \mathrm{elim}_k(C) \cap H_k$, *with* $\mathrm{elim}_k(C) = \{x \in \mathbb{R}^d \mid (\exists t \in \mathbb{R})(x + te_k \in P)\} = \{z - te_k \mid z \in P,\ t \in \mathbb{R}\} = P(A^{/k}, 0)$ *and where the rows of $A^{/k}$ are given by*

$$A^{/k} = \{a_i \mid a_{i\,k} = 0\} \cup \{a_{i\,k}a_j - a_{j\,k}a_i \mid a_{i\,k} > 0,\ a_{j\,k} < 0\}.$$

It should be noted that both Fourier-Motzkin elimination methods generate a quadratic number of new vectors or inequalities at each step and thus they lead to a combinatorial explosion. Therefore, these methods become intractable rather quickly. The problem is that many of the new vectors or inequalities are redundant. Therefore, it is important to find ways of detecting redundancies and there are various methods for doing so. Again, the interested reader should consult Ziegler [45], Chapter 1.

There is yet another way of proving that an $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron without using Fourier-Motzkin elimination. As we already observed, Krein and Milman's theorem does not apply if our polyhedron is unbounded. Actually, the full power of Krein and Milman's theorem is not needed to show that an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. The crucial point is that if P is an $\mathcal{H}$-polytope with nonempty interior, then *every* line, ℓ, through any point, a, in the interior of P intersects P in a line segment. This is because P is compact and ℓ is closed, so $P \cap \ell$ is a compact subset of a line thus, a closed interval $[b, c]$ with $b < a < c$, as a is in the interior of P. Therefore, we can use induction on the dimension of P to show that every point in P is a convex combination of vertices of the facets of P. Now, if P is unbounded and cut out by at least two half-spaces (so, P is not a half-space), then we claim that for every point, a, in the interior of P, there is *some* line through a that intersects two facets of P. This is because if we pick the origin in the interior of P, we may assume that P is given by an irredundant intersection, $P = \bigcap_{i=1}^{t} (H_i)_-$, and for any point, a, in the interior of P, there is a line, ℓ, through P *in general position w.r.t.* P, which means that ℓ is not parallel to any of the hyperplanes H_i and intersects all of them in distinct points (see Definition 7.2). Fortunately, lines in general position always exist, as shown in Proposition 7.3. Using this fact, we can prove the following result:

Proposition 4.23 *Let $P \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-polyhedron, $P = \bigcap_{i=1}^t (H_i)_-$ (an irredundant decomposition), with nonempty interior. If $t = 1$, that is, $P = (H_1)_-$ is a half-space, then*

$$P = a + \mathrm{cone}(u_1, \ldots, u_{d-1}, -u_1, \ldots, -u_{d-1}, u_d),$$

where a is any point in H_1, the vectors $u_1, \ldots, u_{d-1}$ form a basis of the direction of H_1 and u_d is normal to (the direction of) H_1. (When $d = 1$, P is the half-line, $P = \{a + tu_1 \mid t \geq 0\}$.) If $t \geq 2$, then every point, $a \in P$, can be written as a convex combination, $a = (1-\alpha)b + \alpha c$ ($0 \leq \alpha \leq 1$), where b and c belong to two distinct facets, F and G, of P and where

$$F = \mathrm{conv}(Y_F) + \mathrm{cone}(V_F) \quad \text{and} \quad G = \mathrm{conv}(Y_G) + \mathrm{cone}(V_G),$$

for some finite sets of points, Y_F and Y_G and some finite sets of vectors, V_F and V_G. (Note: $\alpha = 0$ or $\alpha = 1$ is allowed.) Consequently, every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.

Proof. We proceed by induction on the dimension, d, of P. If $d = 1$, then P is either a closed interval, $[b, c]$, or a half-line, $\{a + tu \mid t \geq 0\}$, where $u \neq 0$. In either case, the proposition is clear.

For the induction step, assume $d > 1$. Since every facet, F, of P has dimension $d-1$, the induction hypothesis holds for F, that is, there exist a finite set of points, Y_F, and a finite set of vectors, V_F, so that

$$F = \mathrm{conv}(Y_F) + \mathrm{cone}(V_F).$$

Thus, every point on the boundary of P is of the desired form. Next, pick any point, a, in the interior of P. Then, from our previous discussion, there is a line, ℓ, through a in general position w.r.t. P. Consequently, the intersection points, $z_i = \ell \cap H_i$, of the line ℓ with the hyperplanes supporting the facets of P exist and are all distinct. If we give ℓ an orientation, the z_i's can be sorted and there is a unique k such that a lies between $b = z_k$ and $c = z_{k+1}$. But then, $b \in F_k = F$ and $c \in F_{k+1} = G$, where F and G the facets of P supported by H_k and H_{k+1}, and $a = (1-\alpha)b + \alpha c$, a convex combination. Consequently, every point in P is a mixed convex + positive combination of finitely many points associated with the facets of P and finitely many vectors associated with the directions of the supporting hyperplanes of the facets P. Conversely, it is easy to see that any such mixed combination must belong to P and therefore, P is a $\mathcal{V}$-polyhedron. $\square$

We conclude this section with a version of Farkas Lemma for polyhedral sets.

Lemma 4.24 *(Farkas Lemma, Version IV) Let Y be any $d \times p$ matrix and V be any $d \times q$ matrix. For every $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:*

(a) *There exist* $u \in \mathbb{R}^p$ *and* $t \in \mathbb{R}^q$, *with* $u \geq 0$, $t \geq 0$, $\mathbb{I}u = 1$ *and* $z = Yu + Vt$.

(b) *There is some vector,* $(\alpha, c) \in \mathbb{R}^{d+1}$, *such that* $c^\top y_i \geq \alpha$ *for all* i *with* $1 \leq i \leq p$, $c^\top v_j \geq 0$ *for all* j *with* $1 \leq j \leq q$, *and* $c^\top z < \alpha$.

Proof. We use Farkas Lemma, Version II (Lemma 3.13). Observe that (a) is equivalent to the problem: Find $(u, t) \in \mathbb{R}^{p+q}$, so that

$$\begin{pmatrix} u \\ t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ Y & V \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ z \end{pmatrix},$$

which is exactly Farkas II(a). Now, the second alternative of Farkas II says that there is no solution as above if there is some $(-\alpha, c) \in \mathbb{R}^{d+1}$ so that

$$(-\alpha, c^\top) \begin{pmatrix} 1 \\ z \end{pmatrix} < 0 \quad \text{and} \quad (-\alpha, c^\top) \begin{pmatrix} \mathbb{I} & 0 \\ Y & V \end{pmatrix} \geq (\mathbb{O}, \mathbb{O}).$$

These are equivalent to

$$-\alpha + c^\top z < 0, \quad -\alpha\mathbb{I} + c^\top Y \geq \mathbb{O}, \quad c^\top V \geq \mathbb{O},$$

namely, $c^\top z < \alpha$, $c^\top Y \geq \alpha\mathbb{I}$ and $c^\top V \geq \mathbb{O}$, which are indeed the conditions of Farkas IV(b), in matrix form. $\square$

Observe that Farkas IV can be viewed as a separation criterion for polyhedral sets. This version subsumes Farkas I and Farkas II.

Chapter 5

Projective Spaces, Projective Polyhedra, Polar Duality w.r.t. a Nondegenerate Quadric

5.1 Projective Spaces

The fact that not just points but also vectors are needed to deal with unbounded polyhedra is a hint that perhaps the notions of polytope and polyhedra can be unified by "going projective". However, we have to be careful because projective geometry does not accomodate well the notion of convexity. This is because convexity has to do with convex combinations, but the essense of projective geometry is that everything is defined up to *non-zero* scalars, without any requirement that these scalars be positive.

It is possible to develop a theory of *oriented projective geometry* (due to J. Stolfi [38]) in wich convexity is nicely accomodated. However, in this approach, every point comes as a pair, (positive point, negative point), and although it is a very elegant theory, we find it a bit unwieldy. However, since all we really want is to "embed" $\mathbb{E}^d$ into its *projective completion*, $\mathbb{P}^d$, so that we can deal with "points at infinity" and "normal points" in a uniform manner in particular, with respect to projective transformations, we will content ourselves with a definition of the notion of a projective polyhedron using the notion of polyhedral cone. Thus, we will not attempt to define a general notion of convexity.

We begin with a "crash course" on (real) projective spaces. There are many texts on projective geometry. We suggest starting with Gallier [20] and then move on to far more comprehensive treatments such as Berger (Geometry II) [6] or Samuel [35].

Definition 5.1 The *(real) projective space*, $\mathbb{RP}^n$, is the set of all lines through the origin in $\mathbb{R}^{n+1}$, i.e., the set of one-dimensional subspaces of $\mathbb{R}^{n+1}$ (where $n \geq 0$). Since a one-dimensional subspace, $L \subseteq \mathbb{R}^{n+1}$, is spanned by any nonzero vector, $u \in L$, we can view $\mathbb{RP}^n$ as the set of equivalence classes of nonzero vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \in \mathbb{R},\ \lambda \neq 0.$$

We have the projection, $p\colon (\mathbb{R}^{n+1} - \{0\}) \to \mathbb{RP}^n$, given by $p(u) = [u]_\sim$, the equivalence class of u modulo $\sim$. Write $[u]$ (or $\langle u \rangle$) for the line,

$$[u] = \{\lambda u \mid \lambda \in \mathbb{R}\},$$

defined by the nonzero vector, u. Note that $[u]_\sim = [u] - \{0\}$, for every $u \neq 0$, so the map $[u]_\sim \mapsto [u]$ is a bijection which allows us to identify $[u]_\sim$ and $[u]$. Thus, we will use both notations interchangeably as convenient.

The projective space, $\mathbb{RP}^n$, is sometimes denoted $\mathbb{P}(\mathbb{R}^{n+1})$. Since every line, L, in $\mathbb{R}^{n+1}$ intersects the sphere S^n in two antipodal points, we can view $\mathbb{RP}^n$ as the quotient of the sphere S^n by identification of antipodal points. We call this the *spherical model* of $\mathbb{RP}^n$.

A more subtle construction consists in considering the (upper) half-sphere instead of the sphere, where the upper half-sphere S^n_+ is set of points on the sphere S^n such that $x_{n+1} \geq 0$. This time, every line through the center intersects the (upper) half-sphere in a single point, except on the boundary of the half-sphere, where it intersects in two antipodal points a_+ and a_-. Thus, the projective space $\mathbb{RP}^n$ is the quotient space obtained from the (upper) half-sphere S^n_+ by identifying antipodal points a_+ and a_- on the boundary of the half-sphere. We call this model of $\mathbb{RP}^n$ the *half-spherical model*.

When $n = 2$, we get a circle. When $n = 3$, the upper half-sphere is homeomorphic to a closed disk (say, by orthogonal projection onto the xy-plane), and $\mathbb{RP}^2$ is in bijection with a closed disk in which antipodal points on its boundary (a unit circle) have been identified. This is hard to visualize! In this model of the real projective space, projective lines are great semicircles on the upper half-sphere, with antipodal points on the boundary identified. Boundary points correspond to points at infinity. By orthogonal projection, these great semicircles correspond to semiellipses, with antipodal points on the boundary identified. Traveling along such a projective "line," when we reach a boundary point, we "wrap around"! In general, the upper half-sphere S^n_+ is homeomorphic to the closed unit ball in $\mathbb{R}^n$, whose boundary is the $(n-1)$-sphere S^{n-1}. For example, the projective space $\mathbb{RP}^3$ is in bijection with the closed unit ball in $\mathbb{R}^3$, with antipodal points on its boundary (the sphere S^2) identified!

Another useful way of "visualizing" $\mathbb{RP}^n$ is to use the hyperplane, $H_{n+1} \subseteq \mathbb{R}^{n+1}$, of equation $x_{n+1} = 1$. Observe that for every line, $[u]$, through the origin in $\mathbb{R}^{n+1}$, if u does not belong to the hyperplane, $H_{n+1}(0) \cong \mathbb{R}^n$, of equation $x_{n+1} = 0$, then $[u]$ intersects H_{n+1} is a unique point, namely,

$$\left(\frac{u_1}{u_{n+1}}, \ldots, \frac{u_n}{u_{n+1}}, 1\right),$$

where $u = (u_1, \ldots, u_{n+1})$. The lines, $[u]$, for which $u_{n+1} = 0$ are "points at infinity". Observe that the set of lines in $H_{n+1}(0) \cong \mathbb{R}^n$ is the set of points of the projective space, $\mathbb{RP}^{n-1}$, and so, $\mathbb{RP}^n$ can be written as the disjoint union

$$\mathbb{RP}^n = \mathbb{R}^n \amalg \mathbb{RP}^{n-1}.$$

We can repeat the above analysis on $\mathbb{RP}^{n-1}$ and so we can think of $\mathbb{RP}^n$ as the disjoint union

$$\mathbb{RP}^n = \mathbb{R}^n \amalg \mathbb{R}^{n-1} \amalg \cdots \amalg \mathbb{R}^1 \amalg \mathbb{R}^0,$$

where $\mathbb{R}^0 = \{0\}$ consist of a single point. The above shows that there is an embedding, $\mathbb{R}^n \hookrightarrow \mathbb{RP}^n$, given by $(u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n, 1)$.

It will also be very useful to use homogeneous coordinates. Given any point, $p = [u]_\sim \in \mathbb{RP}^n$, the set

$$\{(\lambda u_1, \ldots, \lambda u_{n+1}) \mid \lambda \neq 0\}$$

is called the set of *homogeneous coordinates* of p. Since $u \neq 0$, observe that for all homogeneous coordinates, $(u_1, \ldots, u_{n+1})$, for p, some u_i must be non-zero. The traditional notation for the homogeneous coordinates of a point, $p = [u]_\sim$, is

$$(u_1 \colon \cdots \colon u_n \colon u_{n+1}).$$

There is a useful bijection between certain kinds of subsets of $\mathbb{R}^{d+1}$ and subsets of $\mathbb{RP}^d$. For any subset, S, of $\mathbb{R}^{d+1}$, let

$$-S = \{-u \mid u \in S\}.$$

Geometrically, $-S$ is the reflexion of S about 0. Note that for *any* nonempty subset, $S \subseteq \mathbb{R}^{d+1}$, with $S \neq \{0\}$, the sets S, $-S$ and $S \cup -S$ all induce the *same* set of points in projective space, $\mathbb{RP}^d$, since

$$\begin{aligned}
p(S - \{0\}) &= \{[u]_\sim \mid u \in S - \{0\}\} \\
&= \{[-u]_\sim \mid u \in S - \{0\}\} \\
&= \{[u]_\sim \mid u \in -S - \{0\}\} = p((-S) - \{0\}) \\
&= \{[u]_\sim \mid u \in S - \{0\}\} \cup \{[u]_\sim \mid u \in (-S) - \{0\}\} = p((S \cup -S) - \{0\}),
\end{aligned}$$

because $[u]_\sim = [-u]_\sim$. Using these facts we obtain a bijection between subsets of $\mathbb{RP}^d$ and certain subsets of $\mathbb{R}^{d+1}$.

We say that a set, $S \subseteq \mathbb{R}^{d+1}$, is *symmetric* iff $S = -S$. Obviously, $S \cup -S$ is symmetric for any set, S. Say that a subset, $C \subseteq \mathbb{R}^{d+1}$, is a *double cone* iff for every $u \in C - \{0\}$, the entire line, $[u]$, spanned by u is contained in C. We exclude the trivial double cone, $C = \{0\}$, since the trivial vector space does not yield a projective space. Thus, every double cone can be viewed as a set of lines through 0. Note that a double cone is symmetric. Given any nonempty subset, $S \subseteq \mathbb{RP}^d$, let $v(S) \subseteq \mathbb{R}^{d+1}$ be the set of vectors,

$$v(S) = \bigcup_{[u]_\sim \in S} [u]_\sim \cup \{0\}.$$

Note that $v(S)$ is a double cone.

Proposition 5.1 *The map, $v\colon S \mapsto v(S)$, from the set of nonempty subsets of $\mathbb{RP}^d$ to the set of nonempty, nontrivial double cones in $\mathbb{R}^{d+1}$ is a bijection.*

Proof. We already noted that $v(S)$ is nontrivial double cone. Consider the map,

$$\mathrm{ps}\colon S \mapsto p(S) = \{[u]_\sim \in \mathbb{RP}^d \mid u \in S - \{0\}\}.$$

We leave it as an easy exercise to check that $\mathrm{ps} \circ v = \mathrm{id}$ and $v \circ \mathrm{ps} = \mathrm{id}$, which shows that v and ps are mutual inverses. $\square$

Given any subspace, $X \subseteq \mathbb{R}^{n+1}$, with $\dim X = k+1 \geq 1$ and $0 \leq k \leq n$, a *k-dimensional projective subspace* of $\mathbb{RP}^n$ is the image, $Y = p(X - \{0\})$, of $X - \{0\}$ under the projection p. We often write $Y = \mathbb{P}(X)$. When $k = n-1$, we say that Y is a *projective hyperplane* or simply a *hyperplane*. When $k = 1$, we say that Y is a *projective line* or simply a *line*. It is easy to see that every projective hyperplane, H, is the kernel (zero set) of some linear equation of the form

$$a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0,$$

where one of the a_i is nonzero, in the sense that

$$H = \{(x_1\colon \cdots : x_{n+1}) \in \mathbb{RP}^n \mid a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0\}.$$

Conversely, the kernel of any such linear equation defines a projective hyperplane. Furthermore, given a projective hyperplane, $H \subseteq \mathbb{RP}^n$, the linear equation defining H is unique up to a nonzero scalar.

For any i, with $1 \leq i \leq n+1$, the set

$$U_i = \{(x_1\colon \cdots : x_{n+1}) \in \mathbb{RP}^n \mid x_i \neq 0\}$$

is a subset of $\mathbb{RP}^n$ called an *affine patch* of $\mathbb{RP}^n$. We have a bijection, $\varphi_i\colon U_i \to \mathbb{R}^n$, between U_i and $\mathbb{R}^n$ given by

$$\varphi_i\colon (x_1\colon \cdots : x_{n+1}) \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

This map is well defined because if $(y_1, \ldots, y_{n+1}) \sim (x_1, \ldots, x_{n+1})$, that is, $(y_1, \ldots, y_{n+1}) = \lambda(x_1, \ldots, x_{n+1})$, with $\lambda \neq 0$, then

$$\frac{y_j}{y_i} = \frac{\lambda x_j}{\lambda x_i} = \frac{x_j}{x_i} \qquad (1 \leq j \leq n+1),$$

since $\lambda \neq 0$ and $x_i, y_i \neq 0$. The inverse, $\psi_i\colon \mathbb{R}^n \to U_i \subseteq \mathbb{RP}^n$, of φ_i is given by

$$\psi_i\colon (x_1, \cdots, x_n) \mapsto (x_1\colon \cdots x_{i-1}\colon 1\colon x_i\colon \cdots : x_n).$$

Observe that the bijection, φ_i, between U_i and $\mathbb{R}^n$ can also be viewed as the bijection

$$(x_1 \colon \cdots : x_{n+1}) \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right),$$

between U_i and the hyperplane, $H_i \subseteq \mathbb{R}^{n+1}$, of equation $x_i = 1$. We will make heavy use of these bijections. For example, for any subset, $S \subseteq \mathbb{RP}^n$, the "view of S from the patch U_i", $S \upharpoonright U_i$, is in bijection with $v(S) \cap H_i$, where $v(S)$ is the double cone associated with S (see Proposition 5.1).

The affine patches, $U_1, \ldots, U_{n+1}$, cover the projective space $\mathbb{RP}^n$, in the sense that every $(x_1 \colon \cdots : x_{n+1}) \in \mathbb{RP}^n$ belongs to one of the U_i's, as not all $x_i = 0$. The U_i's turn out to be open subsets of $\mathbb{RP}^n$ and they have nonempty overlaps. When we restrict ourselves to one of the U_i, we have an "affine view of $\mathbb{RP}^n$ from U_i". In particular, on the affine patch U_{n+1}, we have the "standard view" of $\mathbb{R}^n$ embedded into $\mathbb{RP}^n$ as H_{n+1}, the hyperplane of equation $x_{n+1} = 1$. The complement, $H_i(0)$, of U_i in $\mathbb{RP}^n$ is the (projective) hyperplane of equation $x_i = 0$ (a copy of $\mathbb{RP}^{n-1}$). With respect to the affine patch, U_i, the hyperplane, $H_i(0)$, plays the role of *hyperplane (of points) at infinity*.

From now on, for simplicity of notation, we will write $\mathbb{P}^n$ for $\mathbb{RP}^n$. We need to define projective maps. Such maps are induced by linear maps.

Definition 5.2 Any injective linear map, $h \colon \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$, induces a map, $\mathbb{P}(h) \colon \mathbb{P}^m \to \mathbb{P}^n$, defined by

$$\mathbb{P}(h)([u]_\sim) = [h(u)]_\sim$$

and called a *projective map*. When $m = n$ and h is bijective, the map $\mathbb{P}(h)$ is also bijective and it is called a *projectivity*.

We have to check that this definition makes sense, that is, it is compatible with the equivalence relation, $\sim$. For this, assume that $u \sim v$, that is

$$v = \lambda u,$$

with $\lambda \neq 0$ (of course, $u, v \neq 0$). As h is linear, we get

$$h(v) = h(\lambda u) = \lambda h(u),$$

that is, $h(u) \sim h(v)$, which shows that $[h(u)]_\sim$ does not depend on the representative chosen in the equivalence class of $[u]_\sim$. It is also easy to check that whenever two linear maps, h_1 and h_2, induce the same projective map, *i.e.*, if $\mathbb{P}(h_1) = \mathbb{P}(h_2)$, then there is a nonzero scalar, λ, so that $h_2 = \lambda h_1$.

Why did we require h to be injective? Because if h has a nontrivial kernel, then, any nonzero vector, $u \in \mathrm{Ker}\,(h)$, is mapped to 0, but as 0 does **not** correspond to any point of $\mathbb{P}^n$, the map $\mathbb{P}(h)$ is undefined on $\mathbb{P}(\mathrm{Ker}\,(h))$.

In some case, we allow projective maps induced by non-injective linear maps h. In this case, $\mathbb{P}(h)$ is a map whose domain is $\mathbb{P}^n - \mathbb{P}(\mathrm{Ker}\,(h))$. An example is the map, $\sigma_N \colon \mathbb{P}^3 \to \mathbb{P}^2$, given by

$$(x_1 \colon x_2 \colon x_3 \colon x_4) \mapsto (x_1 \colon x_2 \colon x_4 - x_3),$$

which is undefined at the point $(0\colon 0\colon 1\colon 1)$. This map is the "homogenization" of the central projection (from the north pole, $N = (0, 0, 1)$) from $\mathbb{E}^3$ onto $\mathbb{E}^2$.

Although a projective map, $f \colon \mathbb{P}^m \to \mathbb{P}^n$, is induced by some linear map, h, the map f is **not** linear! This is because linear combinations of points in $\mathbb{P}^m$ *do not make any sense!*

Another way of defining functions (possibly partial) between projective spaces involves using homogeneous polynomials. If $p_1(x_1, \ldots, x_{m+1}), \ldots, p_{n+1}(x_1, \ldots, x_{m+1})$ are $n+1$ homogeneous polynomials all of the same degree, d, and if these $n+1$ polynomials do not vanish simultaneously, then we claim that the function, f, given by

$$f(x_1 \colon \cdots \colon x_{m+1}) = (p_1(x_1, \ldots, x_{m+1}) \colon \cdots \colon p_{n+1}(x_1, \ldots, x_{m+1}))$$

is indeed a well-defined function from $\mathbb{P}^m$ to $\mathbb{P}^n$. Indeed, if $(y_1, \ldots, y_{m+1}) \sim (x_1, \ldots, x_{m+1})$, that is, $(y_1, \ldots, y_{m+1}) = \lambda(x_1, \ldots, x_{m+1})$, with $\lambda \neq 0$, as the p_i are homogeneous of degree d,

$$p_i(y_1, \ldots, y_{m+1}) = p_i(\lambda x_1, \ldots, \lambda x_{m+1}) = \lambda^d p_i(x_1, \ldots, x_{m+1}),$$

and so,

$$\begin{aligned}
f(y_1 \colon \cdots \colon y_{m+1}) &= (p_1(y_1, \ldots, y_{m+1}) \colon \cdots \colon p_{n+1}(y_1, \ldots, y_{m+1})) \\
&= (\lambda^d p_1(x_1, \ldots, x_{m+1}) \colon \cdots \colon \lambda^d p_{n+1}(x_1, \ldots, x_{m+1})) \\
&= \lambda^d (p_1(x_1, \ldots, x_{m+1}) \colon \cdots \colon p_{n+1}(x_1, \ldots, x_{m+1})) \\
&= \lambda^d f(x_1 \colon \cdots \colon x_{m+1}),
\end{aligned}$$

which shows that $f(y_1 \colon \cdots \colon y_{m+1}) \sim f(x_1 \colon \cdots \colon x_{m+1})$, as required.

For example, the map, $\tau_N \colon \mathbb{P}^2 \to \mathbb{P}^3$, given by

$$(x_1 \colon x_2, \colon x_3) \mapsto (2x_1x_3 \colon 2x_2x_3 \colon x_1^2 + x_2^2 - x_3^2 \colon x_1^2 + x_2^2 + x_3^2),$$

is well-defined. It turns out to be the "homogenization" of the inverse stereographic map from $\mathbb{E}^2$ to S^2 (see Section 8.5). Observe that

$$\tau_N(x_1 \colon x_2 \colon 0) = (0 \colon 0 \colon x_1^2 + x_2^2 \colon x_1^2 + x_2^2) = (0 \colon 0 \colon 1 \colon 1),$$

that is, τ_N maps all the points at infinity (in $H_3(0)$) to the "north pole", $(0\colon 0\colon 1\colon 1)$. However, when $x_3 \neq 0$, we can prove that τ_N is injective (in fact, its inverse is σ_N, defined earlier).

Most interesting subsets of projective space arise as the collection of zeros of a (finite) set of homogeneous polynomials. Let us begin with a single homogeneous polynomial, $p(x_1, \ldots, x_{n+1})$, of degree d and set

$$V(p) = \{(x_1 \colon \cdots \colon x_{n+1}) \in \mathbb{P}^n \mid p(x_1, \ldots, x_{n+1}) = 0\}.$$

As usual, we need to check that this definition does not depend on the specific representative chosen in the equivalence class of $[(x_1, \ldots, x_{n+1})]_\sim$. If $(y_1, \ldots, y_{n+1}) \sim (x_1, \ldots, x_{n+1})$, that is, $(y_1, \ldots, y_{n+1}) = \lambda(x_1, \ldots, x_{n+1})$, with $\lambda \neq 0$, as p is homogeneous of degree d,

$$p(y_1, \ldots, y_{n+1}) = p(\lambda x_1, \ldots, \lambda x_{n+1}) = \lambda^d p(x_1, \ldots, x_{n+1}),$$

and as $\lambda \neq 0$,

$$p(y_1, \ldots, y_{n+1}) = 0 \quad \text{iff} \quad p(x_1, \ldots, x_{n+1}) = 0,$$

which shows that $V(p)$ is well defined. For a set of homogeneous polynomials (not necessarily of the same degree), $\mathcal{E} = \{p_1(x_1, \ldots, x_{n+1}), \ldots, p_s(x_1, \ldots, x_{n+1})\}$, we set

$$V(\mathcal{E}) = \bigcap_{i=1}^{s} V(p_i) = \{(x_1 \colon \cdots \colon x_{n+1}) \in \mathbb{P}^n \mid p_i(x_1, \ldots, x_{n+1}) = 0,\ i = 1 \ldots, s\}.$$

The set, $V(\mathcal{E})$, is usually called the *projective variety* defined by $\mathcal{E}$ (or *cut out by* $\mathcal{E}$). When $\mathcal{E}$ consists of a single polynomial, p, the set $V(p)$ is called a (projective) *hypersurface*. For example, if

$$p(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2,$$

then $V(p)$ is the *projective sphere* in $\mathbb{P}^3$, also denoted S^2. Indeed, if we "look" at $V(p)$ on the affine patch U_4, where $x_4 \neq 0$, we know that this amounts to setting $x_4 = 1$, and we do get the set of points $(x_1, x_2, x_3, 1) \in U_4$ satisfying $x_1^2 + x_2^2 + x_3^2 - 1 = 0$, our usual 2-sphere! However, if we look at $V(p)$ on the patch U_1, where $x_1 \neq 0$, we see the quadric of equation $1 + x_2^2 + x_3^2 = x_4^2$, which is not a sphere but a hyperboloid of two sheets! Nevertheless, if we pick $x_4 = 0$ as the plane at infinity, note that the projective sphere does not have points at infinity since the only *real* solution of $x_1^2 + x_2^2 + x_3^2 = 0$ is $(0, 0, 0)$, but $(0, 0, 0, 0)$ does not correspond to any point of $\mathbb{P}^3$.

Another example is given by

$$q = (x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3 x_4,$$

for which $V(q)$ corresponds to a paraboloid in the patch U_4. Indeed, if we set $x_4 = 1$, we get the set of points in U_4 satisfying $x_3 = x_1^2 + x_2^2$. For this reason, we denote $V(q)$ by $\mathcal{P}$ and called it a *(projective) paraboloid*.

Given any homogeneous polynomial, $F(x_1, \ldots, x_{d+1})$, we will also make use of the *hypersurface cone*, $C(F) \subseteq \mathbb{R}^{d+1}$, defined by

$$C(F) = \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid F(x_1, \ldots, x_{d+1}) = 0\}.$$

Observe that $V(F) = \mathbb{P}(C(F))$.

Remark: Every variety, $V(\mathcal{E})$, defined by a *set* of polynomials, $\mathcal{E} = \{p_1(x_1, \ldots, x_{n+1}), \ldots, p_s(x_1, \ldots, x_{n+1})\}$, is also the hypersurface defined by the *single* polynomial equation,

$$p_1^2 + \cdots + p_s^2 = 0.$$

This fact, peculiar to the real field, $\mathbb{R}$, is a mixed blessing. On the one-hand, the study of varieties is reduced to the study of hypersurfaces. On the other-hand, this is a hint that we should expect that such a study will be hard.

Perhaps to the surprise of the novice, there is a bijective projective map (a projectivity) sending S^2 to $\mathcal{P}$. This map, θ, is given by

$$\theta(x_1 \colon x_2 \colon x_3 \colon x_4) = (x_1 \colon x_2 \colon x_3 + x_4 \colon x_4 - x_3),$$

whose inverse is given by

$$\theta^{-1}(x_1 \colon x_2 \colon x_3 \colon x_4) = \left(x_1 \colon x_2 \colon \frac{x_3 - x_4}{2} \colon \frac{x_3 + x_4}{2}\right).$$

Indeed, if $(x_1 \colon x_2 \colon x_3 \colon x_4)$ satisfies

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0,$$

and if $(z_1 \colon z_2 \colon z_3 \colon z_4) = \theta(x_1 \colon x_2 \colon x_3 \colon x_4)$, then from above,

$$(x_1 \colon x_2 \colon x_3 \colon x_4) = \left(z_1 \colon z_2 \colon \frac{z_3 - z_4}{2} \colon \frac{z_3 + z_4}{2}\right),$$

and by plugging the right-hand sides in the equation of the sphere, we get

$$\begin{aligned} z_1^2 + z_2^2 + \left(\frac{z_3 - z_4}{2}\right)^2 - \left(\frac{z_3 + z_4}{2}\right)^2 &= z_1^2 + z_2^2 + \frac{1}{4}(z_3^2 + z_4^2 - 2z_3z_4 - (z_3^2 + z_4^2 + 2z_3z_4)) \\ &= z_1^2 + z_2^2 - z_3z_4 = 0, \end{aligned}$$

which is the equation of the paraboloid, $\mathcal{P}$.

5.2 Projective Polyhedra

Following the "projective doctrine" which consists in replacing points by lines through the origin, that is, to "conify" everything, we will define a projective polyhedron as any set of points in $\mathbb{P}^d$ induced by a polyhedral cone in $\mathbb{R}^{d+1}$. To do so, it is preferable to consider cones as sets of positive combinations of vectors (see Definition 4.3). Just to refresh our memory, a set, $C \subseteq \mathbb{R}^d$, is a $\mathcal{V}$-*cone* or *polyhedral cone* if C is the positive hull of a finite set of vectors, that is,

$$C = \mathrm{cone}(\{u_1, \ldots, u_p\}),$$

for some vectors, $u_1, \ldots, u_p \in \mathbb{R}^d$. An $\mathcal{H}$-*cone* is any subset of $\mathbb{R}^d$ given by a finite intersection of closed half-spaces cut out by hyperplanes through 0.

A good place to learn about cones (and much more) is Fulton [19]. See also Ewald [18].

By Theorem 4.18, $\mathcal{V}$-cones and $\mathcal{H}$-cones form the same collection of convex sets (for every $d \geq 0$). Naturally, we can think of these cones as sets of rays (half-lines) of the form

$$\langle u \rangle_+ = \{\lambda u \mid \lambda \in \mathbb{R},\ \lambda \geq 0\},$$

where $u \in \mathbb{R}^d$ is any *nonzero* vector. We exclude the trivial cone, $\{0\}$, since 0 does not define any point in projective space. When we "go projective", each ray corresponds to the full line, $\langle u \rangle$, spanned by u which can be expressed as

$$\langle u \rangle = \langle u \rangle_+ \cup -\langle u \rangle_+,$$

where $-\langle u \rangle_+ = \langle u \rangle_- = \{\lambda u \mid \lambda \in \mathbb{R},\ \lambda \leq 0\}$. Now, if $C \subseteq \mathbb{R}^d$ is a polyhedral cone, obviously $-C$ is also a polyhedral cone and the set $C \cup -C$ consists of the union of the two polyhedral cones C and $-C$. Note that $C \cup -C$ can be viewed as the set of all lines determined by the nonzero vectors in C (and $-C$). It is a double cone. Unless C is a closed half-space, $C \cup -C$ is not convex. It seems perfectly natural to define a projective polyhedron as any set of lines induced by a set of the form $C \cup -C$, where C is a polyhedral cone.

Definition 5.3 A *projective polyhedron* is any subset, $P \subseteq \mathbb{P}^d$, of the form

$$P = p((C \cup -C) - \{0\}) = p(C - \{0\}),$$

where C is any polyhedral cone ($\mathcal{V}$ or $\mathcal{H}$ cone) in $\mathbb{R}^{d+1}$ (with $C \neq \{0\}$). We write $P = \mathbb{P}(C \cup -C)$ or $P = \mathbb{P}(C)$.

It is important to observe that because $C \cup -C$ is a double cone there is a bijection between nontrivial double polyhedral cones and projective polyhedra. So, projective polyhedra are equivalent to double polyhedral cones. However, the projective interpretation of the lines induced by $C \cup -C$ as points in $\mathbb{P}^d$ makes the study of projective polyhedra geometrically more interesting.

Projective polyhedra inherit many of the properties of cones but we have to be careful because we are really dealing with double cones, $C \cup -C$, and not cones. As a consequence, there are a few unpleasant surprises, for example, the fact that the collection of projective polyhedra is **not** closed under intersection!

Before dealing with these issues, let us show that every "standard" polyhedron, $P \subseteq \mathbb{E}^d$, has a natural projective completion, $\widetilde{P} \subseteq \mathbb{P}^d$, such that on the affine patch U_{d+1} (where $x_{d+1} \neq$

0), $\widetilde{P} \restriction U_{d+1} = P$. For this, we use our theorem on the Polyhedron–Cone Correspondence (Theorem 4.19, part (2)).

Let $A = X + U$, where X is a set of points in $\mathbb{E}^d$ and U is a cone in $\mathbb{R}^d$. For every point, $x \in X$, and every vector, $u \in U$, let

$$\widehat{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \qquad \widehat{u} = \begin{pmatrix} u \\ 0 \end{pmatrix},$$

and let $\widehat{X} = \{\widehat{x} \mid x \in X\}$, $\widehat{U} = \{\widehat{u} \mid u \in U\}$ and $\widehat{A} = \{\widehat{a} \mid a \in A\}$, with

$$\widehat{a} = \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

Then,

$$C(A) = \mathrm{cone}(\{\widehat{X} \cup \widehat{U}\})$$

is a cone in $\mathbb{R}^{d+1}$ such that

$$\widehat{A} = C(A) \cap H_{d+1},$$

where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$. If we set $\widetilde{A} = \mathbb{P}(C(A))$, then we get a subset of $\mathbb{P}^d$ and in the patch U_{d+1}, the set $\widetilde{A} \restriction U_{d+1}$ is in bijection with the intersection $(C(A) \cup -C(A)) \cap H_{d+1} = \widehat{A}$, and thus, in bijection with A. We call $\widetilde{A}$ the *projective completion of* A. We have an injection, $A \longrightarrow \widetilde{A}$, given by

$$(a_1, \ldots, a_d) \mapsto (a_1 \colon \cdots \colon a_d \colon 1),$$

which is just the map, $\psi_{d+1} \colon \mathbb{R}^d \to U_{d+1}$. What the projective completion does is to add to A the "points at infinity" corresponding to the vectors in U, that is, the points of $\mathbb{P}^d$ corresponding to the lines in the cone, U. In particular, if $X = \mathrm{conv}(Y)$ and $U = \mathrm{cone}(V)$, for some finite sets $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$, then $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$ is a $\mathcal{V}$-polyhedron and $\widetilde{P} = \mathbb{P}(C(P))$ is a projective polyhedron. The projective polyhedron, $\widetilde{P} = \mathbb{P}(C(P))$, is called the *projective completion* of P.

Observe that if C is a closed half-space in $\mathbb{R}^{d+1}$, then $P = \mathbb{P}(C \cup -C) = \mathbb{P}^d$. Now, if $C \subseteq \mathbb{R}^{d+1}$ is a polyhedral cone and C is contained in a closed half-space, it is still possible that C contains some nontrivial linear subspace and we would like to understand this situation.

The first thing to observe is that $U = C \cap (-C)$ is the largest linear subspace contained in C. If $C \cap (-C) = \{0\}$, we say that C is a *pointed* or *strongly convex* cone. In this case, one immediately realizes that 0 is an extreme point of C and so, there is a hyperplane, H, through 0 so that $C \cap H = \{0\}$, that is, except for its apex, C lies in one of the open half-spaces determined by H. As a consequence, by a linear change of coordinates, we may assume that this hyperplane is H_{d+1} and so, for every projective polyhedron, $P = \mathbb{P}(C)$, if C is pointed then there is an affine patch (say, U_{d+1}) where P has no points at infinity, that is, P is a polytope! On the other hand, from another patch, U_i, as $P \restriction U_i$ is in bijection with $(C \cup -C) \cap H_i$, the projective polyhedron P viewed on U_i may consist of *two* disjoint polyhedra.

The situation is very similar to the classical theory of projective conics or quadrics (for example, see Brannan, Esplen and Gray, [10]). The case where C is a pointed cone corresponds to the nondegenerate conics or quadrics. In the case of the conics, depending how we slice a cone, we see an ellipse, a parabola or a hyperbola. For projective polyhedra, when we slice a polyhedral double cone, $C \cup -C$, we may see a polytope (*elliptic type*) a single unbounded polyhedron (*parabolic type*) or two unbounded polyhedra (*hyperbolic type*).

Now, when $U = C \cap (-C) \neq \{0\}$, the polyhedral cone, C, contains the linear subspace, U, and if $C \neq \mathbb{R}^{d+1}$, then for every hyperplane, H, such that C is contained in one of the two closed half-spaces determined by H, the subspace $U \cap H$ is nontrivial. An example is the cone, $C \subseteq \mathbb{R}^3$, determined by the intersection of two planes through 0 (a wedge). In this case, U is equal to the line of intersection of these two planes. Also observe that $C \cap (-C) = C$ iff $C = -C$, that is, iff C is a linear subspace.

The situation where $C \cap (-C) \neq \{0\}$ is reminiscent of the case of cylinders in the theory of quadric surfaces (see [10] or Berger [6]). Now, every cylinder can be viewed as the ruled surface defined as the family of lines orthogonal to a plane and touching some nondegenerate conic. A similar decomposition holds for polyhedral cones as shown below in a proposition borrowed from Ewald [18] (Chapter V, Lemma 1.6). We should warn the reader that we have some doubts about the proof given there and so, we offer a different proof adapted from the proof of Lemma 16.2 in Barvinok [3]. Given any two subsets, $V, W \subseteq \mathbb{R}^d$, as usual, we write $V + W = \{v + w \mid v \in V,\ w \in W\}$ and $v + W = \{v + w \mid w \in W\}$, for any $v \in \mathbb{R}^d$.

Proposition 5.2 *For every polyhedral cone, $C \subseteq \mathbb{R}^d$, if $U = C \cap (-C)$, then there is some pointed cone, C_0, so that U and C_0 are orthogonal and*

$$C = U + C_0,$$

with $\dim(U) + \dim(C_0) = \dim(C)$.

Proof. We already know that $U = C \cap (-C)$ is the largest linear subspace of C. Let $U^\perp$ be the orthogonal complement of U in $\mathbb{R}^d$ and let $\pi\colon \mathbb{R}^d \to U^\perp$ be the orthogonal projection onto $U^\perp$. By Proposition 4.12, the projection, $C_0 = \pi(C)$, of C onto $U^\perp$ is a polyhedral cone. We claim that C_0 is pointed and that

$$C = U + C_0.$$

Since $\pi^{-1}(v) = v + U$ for every $v \in C_0$, we have $U + C_0 \subseteq C$. On the other hand, by definition of C_0, we also have $C \subseteq U + C_0$, so $C = U + C_0$. If C_0 was not pointed, then it would contain a linear subspace, V, of dimension at least 1 but then, $U + V$ would be a linear subspace of C of dimension strictly greater than U, which is impossible. Finally, $\dim(U) + \dim(C_0) = \dim(C)$ is obvious by orthogonality. $\square$

The linear subspace, $U = C \cap (-C)$, is called the *cospan* of C. Both U and C_0 are uniquely determined by C. To a great extent, Proposition reduces the study of non-pointed cones to the study of pointed cones. We propose to call the projective polyhedra of the form $P = \mathbb{P}(C)$, where C is a cone with a non-trivial cospan (a non-pointed cone) a *projective polyhedral cylinder*, by analogy with the quadric surfaces. We also propose to call the projective polyhedra of the form $P = \mathbb{P}(C)$, where C is a pointed cone, a *projective polytope* (or *nondegenerate projective polyhedron*).

The following propositions show that projective polyhedra behave well under projective maps and intersection with a hyperplane:

Proposition 5.3 *Given any projective map, $h\colon \mathbb{P}^m \to \mathbb{P}^n$, for any projective polyhedron, $P \subseteq \mathbb{P}^m$, the image, $h(P)$, of P is a projective polyhedron in $\mathbb{P}^n$. Even if $h\colon \mathbb{P}^m \to \mathbb{P}^n$ is a partial map but h is defined on P, then $h(P)$ is a projective polyhedron.*

Proof. The projective map, $h\colon \mathbb{P}^m \to \mathbb{P}^n$, is of the form $h = \mathbb{P}(\widehat{h})$, for some injective linear map, $\widehat{h}\colon \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$. Moreover, the projective polyhedron, P, is of the form $P = \mathbb{P}(C)$, for some polyhedral cone, $C \subseteq \mathbb{R}^{m+1}$, with $C = \mathrm{cone}(\{u_1, \ldots, u_p\})$, for some nonzero vector $u_i \in \mathbb{R}^{m+1}$. By definition,

$$\mathbb{P}(h)(P) = \mathbb{P}(h)(\mathbb{P}(C)) = \mathbb{P}(\widehat{h}(C)).$$

As $\widehat{h}$ is linear,

$$\widehat{h}(C) = \widehat{h}(\mathrm{cone}(\{u_1, \ldots, u_p\})) = \mathrm{cone}(\{\widehat{h}(u_1), \ldots, \widehat{h}(u_p)\}).$$

If we let $\widehat{C} = \mathrm{cone}(\{\widehat{h}(u_1), \ldots, \widehat{h}(u_p)\})$, then $\widehat{h}(C) = \widehat{C}$ is a polyhedral cone and so,

$$\mathbb{P}(h)(P) = \mathbb{P}(\widehat{h}(C)) = \mathbb{P}(\widehat{C})$$

is a projective cone. This argument does not depend on the injectivity of $\widehat{h}$, as long as $C \cap \mathrm{Ker}\,(\widehat{h}) = \{0\}$. $\square$

Proposition 5.3 together with earlier arguments shows that every projective polytope, $P \subseteq \mathbb{P}^d$, is equivalent under some suitable projectivity to another projective polytope, P', which is a polytope when viewed in the affine patch, U_{d+1}. This property is similar to the fact that every (non-degenerate) projective conic is projectively equivalent to an ellipse.

Since the notion of a face is defined for arbitrary polyhedra it is also defined for cones. Consequently, we can define the notion of a face for projective polyhedra. Given a projective polyhedron, $P \subseteq \mathbb{P}^d$, where $P = \mathbb{P}(C)$ for some polyhedral cone (uniquely determined by P), $C \subseteq \mathbb{R}^{d+1}$, a *face of* P is any subset of P of the form $\mathbb{P}(F) = p(F - \{0\})$, for any nontrivial face, $F \subseteq C$, of C ($F \neq \{0\}$). Consequently, we say that $\mathbb{P}(F)$ is a *vertex* iff $\dim(F) = 1$, an *edge* iff $\dim(F) = 2$ and a *facet* iff $\dim(F) = \dim(C) - 1$. The projective polyhedron, P, and the empty set are the *improper* faces of P. If C is strongly convex, then it is easy to prove that C is generated by its edges (its one-dimensional faces, these are rays) in the sense that any set of nonzero vectors spanning these edges generates C (using positive linear combinations). As a consequence, if C is strongly convex, we may say that P is "spanned" by its vertices, since P is equal to $\mathbb{P}$(all positive combinations of vectors representing its edges).

Remark: Even though we did not define the notion of convex combination of points in $\mathbb{P}^d$, the notion of projective polyhedron gives us a way to mimic certain properties of convex sets in the framework of projective geometry. That's because every projective polyhedron corresponds to a unique polyhedral cone.

If our projective polyhedron is the completion, $\widetilde{P} = \mathbb{P}(C(P)) \subseteq \mathbb{P}^d$, of some polyhedron, $P \subseteq \mathbb{R}^d$, then each face of the cone, $C(P)$, is of the form $C(F)$, where F is a face of P and so, each face of $\widetilde{P}$ is of the form $\mathbb{P}(C(F))$, for some face, F, of P. In particular, in the affine patch, U_{d+1}, the face, $\mathbb{P}(C(F))$, is in bijection with the face, F, of P. We will usually identify $\mathbb{P}(C(F))$ and F.

We now consider the intersection of projective polyhedra but first, let us make some general remarks about the intersection of subsets of $\mathbb{P}^d$. Given any two nonempty subsets, $\mathbb{P}(S)$ and $\mathbb{P}(S')$, of $\mathbb{P}^d$ what is $\mathbb{P}(S) \cap \mathbb{P}(S')$? It is tempting to say that

$$\mathbb{P}(S) \cap \mathbb{P}(S') = \mathbb{P}(S \cap S'),$$

but unfortunately this is generally false! The problem is that $\mathbb{P}(S) \cap \mathbb{P}(S')$ is the set of *all lines* determined by vectors both in S and S' but there may be some line spanned by some vector $u \in (-S) \cap S'$ or $u \in S \cap (-S')$ such that u does not belong to $S \cap S'$ or $-(S \cap S')$.

Observe that

$$\begin{aligned} -(-S) &= S \\ -(S \cap S') &= (-S) \cap (-S'). \end{aligned}$$

Then, the correct intersection is given by

$$\begin{aligned} (S \cup -S) \cap (S' \cup -S') &= (S \cap S') \cup ((-S) \cap (-S')) \cup (S \cap (-S')) \cup ((-S) \cap S') \\ &= (S \cap S') \cup -(S \cap S') \cup (S \cap (-S')) \cup -(S \cap (-S')), \end{aligned}$$

which is the union of two double cones (except for 0, which belongs to both). Therefore,

$$\mathbb{P}(S) \cap \mathbb{P}(S') = \mathbb{P}(S \cap S') \cup \mathbb{P}(S \cap (-S')) = \mathbb{P}(S \cap S') \cup \mathbb{P}((-S) \cap S'),$$

since $\mathbb{P}(S \cap (-S')) = \mathbb{P}((-S) \cap S')$.

Furthermore, if S' is symmetric (*i.e.*, $S' = -S'$), then

$$\begin{aligned} (S \cup -S) \cap (S' \cup -S') &= (S \cup -S) \cap S' \\ &= (S \cap S') \cup ((-S) \cap S') \\ &= (S \cap S') \cup -(S \cap (-S')) \\ &= (S \cap S') \cup -(S \cap S'). \end{aligned}$$

Thus, if either S or S' is symmetric, it is true that

$$\mathbb{P}(S) \cap \mathbb{P}(S') = \mathbb{P}(S \cap S').$$

Now, if C is a pointed polyhedral cone then $C \cap (-C) = \{0\}$. Consequently, for any other polyhedral cone, C', we have $(C \cap C') \cap ((-C) \cap C') = \{0\}$. Using these facts we obtain the following result:

Proposition 5.4 *Let $P = \mathbb{P}(C)$ and $P' = \mathbb{P}(C')$ be any two projective polyhedra in $\mathbb{P}^d$. If $\mathbb{P}(C) \cap \mathbb{P}(C') \neq \emptyset$, then the following properties hold:*

(1)
$$\mathbb{P}(C) \cap \mathbb{P}(C') = \mathbb{P}(C \cap C') \cup \mathbb{P}(C \cap (-C')),$$
the union of two projective polyhedra. If C or C' is a pointed cone i.e., P or P' is a projective polytope, then $\mathbb{P}(C \cap C')$ and $\mathbb{P}(C \cap (-C'))$ are disjoint.

(2) If $P' = H$, for some hyperplane, $H \subseteq \mathbb{P}^d$, then $P \cap H$ is a projective polyhedron.

Proof. We already proved (1) so only (2) remains to be proved. Of course, we may assume that $P \neq \mathbb{P}^d$. This time, using the equivalence theorem of $\mathcal{V}$-cones and $\mathcal{H}$-cones (Theorem 4.18), we know that P is of the form $P = \mathbb{P}(C)$, with $C = \bigcap_{i=1}^p C_i$, where the C_i are closed half-spaces in $\mathbb{R}^{d+1}$. Moreover, $H = \mathbb{P}(\widehat{H})$, for some hyperplane, $\widehat{H} \subseteq \mathbb{R}^{d+1}$, through 0. Now, as $\widehat{H}$ is symmetric,

$$P \cap H = \mathbb{P}(C) \cap \mathbb{P}(\widehat{H}) = \mathbb{P}(C \cap \widehat{H}).$$

Consequently,

$$\begin{aligned} P \cap H &= \mathbb{P}(C \cap \widehat{H}) \\ &= \mathbb{P}\left(\left(\bigcap_{i=1}^p C_i\right) \cap \widehat{H}\right). \end{aligned}$$

However, $\widehat{H} = \widehat{H}_+ \cap \widehat{H}_-$, where $\widehat{H}_+$ and $\widehat{H}_-$ are the two closed half-spaces determined by $\widehat{H}$ and so,

$$\widehat{C} = \left(\bigcap_{i=1}^p C_i\right) \cap \widehat{H} = \left(\bigcap_{i=1}^p C_i\right) \cap \widehat{H}_+ \cap \widehat{H}_-$$

is a polyhedral cone. Therefore, $P \cap H = \mathbb{P}(\widehat{C})$ is a projective polyhedron. $\square$

We leave it as an instructive exercise to find explicit examples where $P \cap P'$ consists of two disjoint projective polyhedra in $\mathbb{P}^1$ (or $\mathbb{P}^2$).

Proposition 5.4 can be sharpened a little.

Proposition 5.5 *Let $P = \mathbb{P}(C)$ and $P' = \mathbb{P}(C')$ be any two projective polyhedra in $\mathbb{P}^d$. If $\mathbb{P}(C) \cap \mathbb{P}(C') \neq \emptyset$, then*

$$\mathbb{P}(C) \cap \mathbb{P}(C') = \mathbb{P}(C \cap C') \cup \mathbb{P}(C \cap (-C')),$$

the union of two projective polyhedra. If $C = -C$, i.e., C is a linear subspace (or if C' is a linear subspace), then

$$\mathbb{P}(C) \cap \mathbb{P}(C') = \mathbb{P}(C \cap C').$$

Furthermore, if either C or C' is pointed, the above projective polyhedra are disjoint, else if C and C' both have nontrivial cospan and $\mathbb{P}(C \cap C')$ and $\mathbb{P}(C \cap (-C'))$ intersect then

$$\mathbb{P}(C \cap C') \cap \mathbb{P}(C \cap (-C')) = \mathbb{P}(C \cap (C' \cap (-C'))) \cup \mathbb{P}(C' \cap (C \cap (-C))).$$

Finally, if the two projective polyhedra on the right-hand side intersect, then

$$\mathbb{P}(C \cap (C' \cap (-C'))) \cap \mathbb{P}(C' \cap (C \cap (-C))) = \mathbb{P}((C \cap (-C)) \cap (C' \cap (-C'))).$$

Proof. Left as a simple exercise in boolean algebra. $\square$

In preparation for Section 8.6, we also need the notion of tangent space at a point of a variety.

5.3 Tangent Spaces of Hypersurfaces and Projective Hypersurfaces

Since we only need to consider the case of hypersurfaces we restrict attention to this case (but the general case is a straightforward generalization). Let us begin with a hypersurface of equation $p(x_1, \ldots, x_d) = 0$ in $\mathbb{R}^d$, that is, the set

$$S = V(p) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid p(x_1, \ldots, x_d) = 0\},$$

where $p(x_1, \ldots, x_d)$ is a polynomial of total degree, m.

Pick any point $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Recall that there is a version of the Taylor expansion formula for polynomials such that, for any polynomial, $p(x_1, \ldots, x_d)$, of total degree m, for every $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$, we have

$$\begin{aligned} p(a+h) &= p(a) + \sum_{1 \leq |\alpha| \leq m} \frac{D^\alpha p(a)}{\alpha!} h^\alpha \\ &= p(a) + \sum_{i=1}^{d} p_{x_i}(a) h_i + \sum_{2 \leq |\alpha| \leq m} \frac{D^\alpha p(a)}{\alpha!} h^\alpha, \end{aligned}$$

where we use the *multi-index notation*, with $\alpha = (i_1, \ldots, i_d) \in \mathbb{N}^d$, $|\alpha| = i_1 + \cdots + i_d$, $\alpha! = i_1! \cdots i_d!$, $h^\alpha = h_1^{i_1} \cdots h_d^{i_d}$,

$$D^\alpha p(a) = \frac{\partial^{i_1 + \cdots + i_d} p}{\partial x_1^{i_1} \cdots \partial x_d^{i_d}}\,(a),$$

and

$$p_{x_i}(a) = \frac{\partial p}{\partial x_i}\,(a).$$

Consider any line, ℓ, through a, given parametrically by

$$\ell = \{a + th \mid t \in \mathbb{R}\},$$

with $h \neq 0$ and say $a \in S$ is a point on the hypersurface, $S = V(p)$, which means that $p(a) = 0$. The intuitive idea behind the notion of the tangent space to S at a is that it is the set of lines that intersect S at a in a point of *multiplicity at least two*, which means that the equation giving the intersection, $S \cap \ell$, namely

$$p(a + th) = p(a_1 + th_1, \ldots, a_d + th_d) = 0,$$

is of the form

$$t^2 q(a, h)(t) = 0,$$

where $q(a, h)(t)$ is some polynomial in t. Using Taylor's formula, as $p(a) = 0$, we have

$$p(a + th) = t \sum_{i=1}^{d} p_{x_i}(a) h_i + t^2 q(a, h)(t),$$

for some polynomial, $q(a, h)(t)$. From this, we see that a is an intersection point of multiplicity at least 2 iff

$$\sum_{i=1}^{d} p_{x_i}(a) h_i = 0. \qquad (\dagger)$$

Consequently, if $\nabla p(a) = (p_{x_1}(a), \ldots, p_{x_d}(a)) \neq 0$ (that is, if the gradient of p at a is nonzero), we see that ℓ intersects S at a in a point of multiplicity at least 2 iff h belongs to the hyperplane of equation $(\dagger)$.

Definition 5.4 Let $S = V(p)$ be a hypersurface in $\mathbb{R}^d$. For any point, $a \in S$, if $\nabla p(a) \neq 0$, then we say that a is a *non-singular* point of S. When a is nonsingular, the (*affine*) *tangent space*, $T_a(S)$ (or simply, $T_a S$), to S at a is the hyperplane through a of equation

$$\sum_{i=1}^{d} p_{x_i}(a)(x_i - a_i) = 0.$$

Observe that the hyperplane of the direction of T_aS is the hyperplane through 0 and parallel to T_aS given by

$$\sum_{i=1}^{d} p_{x_i}(a)x_i = 0.$$

When $\nabla p(a) = 0$, we either say that T_aS is undefined or we set $T_aS = \mathbb{R}^d$.

We now extend the notion of tangent space to projective varieties. As we will see, this amounts to homogenizing and the result turns out to be simpler than the affine case!

So, let $S = V(F) \subseteq \mathbb{P}^d$ be a projective hypersurface, which means that

$$S = V(F) = \{(x_1\colon \cdots : x_{d+1}) \in \mathbb{P}^d \mid F(x_1, \ldots, x_{d+1}) = 0\},$$

where $F(x_1, \ldots, x_{d+1})$ is a homogeneous polynomial of total degree, m. Again, we say that a point, $a \in S$, is *non-singular* iff $\nabla F(a) = (F_{x_1}(a), \ldots, F_{x_{d+1}}(a)) \neq 0$. For every $i = 1, \ldots, d+1$, let

$$z_j^{\restriction i} = \frac{x_j}{x_i},$$

where $j = 1, \ldots, d+1$ and $j \neq i$, and let $f^{\restriction i}$ be the result of "dehomogenizing" F at i, that is,

$$f^{\restriction i}(z_1^{\restriction i}, \ldots, z_{i-1}^{\restriction i}, z_{i+1}^{\restriction i}, \ldots, z_{d+1}^{\restriction i}) = F(z_1^{\restriction i}, \ldots, z_{i-1}^{\restriction i}, 1, z_{i+1}^{\restriction i}, \ldots, z_{d+1}^{\restriction i}).$$

We define the *(projective) tangent space*, T_aS, to a at S as the hyperplane, H, such that for each affine patch, U_i where $a_i \neq 0$, if we let

$$a_j^{\restriction i} = \frac{a_j}{a_i},$$

where $j = 1, \ldots, d+1$ and $j \neq i$, then the restriction, $H \restriction U_i$, of H to U_i is the affine hyperplane tangent to $S \restriction U_i$ given by

$$\sum_{\substack{j=1\\ j\neq i}}^{d+1} f_{z_j^{\restriction i}}^{\restriction i}(a^{\restriction i})(z_j^{\restriction i} - a_j^{\restriction i}) = 0.$$

Thus, on the affine patch, U_i, the tangent space, T_aS, is given by the homogeneous equation

$$\sum_{\substack{j=1\\ j\neq i}}^{d+1} f_{z_j^{\restriction i}}^{\restriction i}(a^{\restriction i})(x_j - a_j^{\restriction i}x_i) = 0.$$

This looks awful but we can make it pretty if we remember that F is a homogeneous polynomial of degree m and that we have the *Euler relation*:

$$\sum_{j=1}^{d+1} F_{x_j}(x)x_j = mF,$$

for every $x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1}$. Using this, we can come up with a clean equation for our projective tangent hyperplane. It is enough to carry out the computations for $i = d+1$. Our tangent hyperplane has the equation

$$\sum_{j=1}^{d} F_{x_j}(a_1^{\restriction d+1}, \ldots, a_d^{\restriction d+1}, 1)(x_j - a_j^{\restriction d+1} x_{d+1}) = 0,$$

that is,

$$\sum_{j=1}^{d} F_{x_j}(a_1^{\restriction d+1}, \ldots, a_d^{\restriction d+1}, 1)x_j + \sum_{j=1}^{d} F_{x_j}(a_1^{\restriction d+1}, \ldots, a_d^{\restriction d+1}, 1)(-a_j^{\restriction d+1} x_{d+1}) = 0.$$

As $F(x_1, \ldots, x_{d+1})$ is homogeneous of degree m, and as $a_{d+1} \neq 0$ on U_{d+1}, we have

$$a_{d+1}^m F(a_1^{\restriction d+1}, \ldots, a_d^{\restriction d+1}, 1) = F(a_1, \ldots, a_d, a_{d+1}),$$

so from the above equation we get

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})x_j + \sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})(-a_j^{\restriction d+1} x_{d+1}) = 0. \qquad (*)$$

Since $a \in S$, we have $F(a) = 0$, so the Euler relation yields

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})a_j + F_{x_{d+1}}(a_1, \ldots, a_{d+1})a_{d+1} = 0,$$

which, by dividing by a_{d+1} and multiplying by x_{d+1}, yields

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})(-a_j^{\restriction d+1} x_{d+1}) = F_{x_{d+1}}(a_1, \ldots, a_{d+1})x_{d+1},$$

and by plugging this in $(*)$, we get

$$\sum_{j=1}^{d} F_{x_j}(a_1, \ldots, a_{d+1})x_j + F_{x_{d+1}}(a_1, \ldots, a_{d+1})x_{d+1} = 0.$$

Consequently, the tangent hyperplane to S at a is given by the equation

$$\sum_{j=1}^{d+1} F_{x_j}(a)x_j = 0.$$

Definition 5.5 Let $S = V(F)$ be a hypersurface in $\mathbb{P}^d$, where $F(x_1, \ldots, x_{d+1})$ is a homogeneous polynomial. For any point, $a \in S$, if $\nabla F(a) \neq 0$, then we say that a is a *non-singular* point of S. When a is nonsingular, the (*projective*) *tangent space*, $T_a(S)$ (or simply, T_aS), to S at a is the hyperplane through a of equation

$$\sum_{i=1}^{d+1} F_{x_i}(a)x_i = 0.$$

For example, if we consider the sphere, $S^2 \subseteq \mathbb{P}^3$, of equation

$$x^2 + y^2 + z^2 - w^2 = 0,$$

the tangent plane to S^2 at $a = (a_1, a_2, a_3, a_4)$ is given by

$$a_1x + a_2y + a_3z - a_4w = 0.$$

Remark: If $a \in S = V(F)$, as $F(a) = \sum_{i=1}^{d+1} F_{x_i}(a)a_i = 0$ (by Euler), the equation of the tangent plane, T_aS, to S at a can also be written as

$$\sum_{i=1}^{d+1} F_{x_i}(a)(x_i - a_i) = 0.$$

Now, if $a = (a_1 \colon \cdots \colon a_d \colon 1)$ is a point in the affine patch U_{d+1}, then the equation of the intersection of T_aS with U_{d+1} is obtained by setting $a_{d+1} = x_{d+1} = 1$, that is

$$\sum_{i=1}^{d} F_{x_i}(a_1, \ldots, a_d, 1)(x_i - a_i) = 0,$$

which is just the equation of the affine hyperplane to $S \cap U_{d+1}$ at $a \in U_{d+1}$.

It will be convenient to adopt the following notational convention: Given any point, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, written as a row vector, we let $\mathbf{x}$ denote the corresponding column vector such that $\mathbf{x}^\top = x$.

Projectivities behave well with respect to hypersurfaces and their tangent spaces. Let $S = V(F) \subseteq \mathbb{P}^d$ be a projective hypersurface, where F is a homogeneous polynomial of degree m and let $h \colon \mathbb{P}^d \to \mathbb{P}^d$ be a projectivity (a bijective projective map). Assume that h is induced by the invertible $(d+1) \times (d+1)$ matrix, $A = (a_{i\,j})$, and write $A^{-1} = (a^{-1}_{i\,j})$. For any hyperplane, $H \subseteq \mathbb{R}^{d+1}$, if φ is any linear from defining φ, *i.e.*, $H = \mathrm{Ker}\,(\varphi)$, then

$$\begin{aligned} h(H) &= \{h(x) \in \mathbb{R}^{d+1} \mid \varphi(x) = 0\} \\ &= \{y \in \mathbb{R}^{d+1} \mid (\exists x \in \mathbb{R}^{d+1})(y = h(x),\ \varphi(x) = 0)\} \\ &= \{y \in \mathbb{R}^{d+1} \mid (\varphi \circ h^{-1})(y) = 0\}. \end{aligned}$$

Consequently, if H is given by

$$\alpha_1 x_1 + \cdots + \alpha_{d+1} x_{d+1} = 0$$

and if we write $\alpha = (\alpha_1, \ldots, \alpha_{d+1})$, then $h(H)$ is the hyperplane given by the equation

$$\alpha A^{-1}\mathbf{y} = 0.$$

Similarly,

$$\begin{aligned} h(S) &= \{h(x) \in \mathbb{R}^{d+1} \mid F(x) = 0\} \\ &= \{y \in \mathbb{R}^{d+1} \mid (\exists x \in \mathbb{R}^{d+1})(y = h(x),\ F(x) = 0)\} \\ &= \{y \in \mathbb{R}^{d+1} \mid F((A^{-1}\mathbf{y})^\top) = 0\} \end{aligned}$$

is the hypersurface defined by the polynomial

$$G(x_1, \ldots, x_{d+1}) = F\left(\sum_{j=1}^{d+1} a^{-1}_{1\,j} x_j, \ldots, \sum_{j=1}^{d+1} a^{-1}_{d+1\,j} x_j\right).$$

Furthermore, using the chain rule, we get

$$(G_{x_1}, \ldots, G_{x_{d+1}}) = (F_{x_1}, \ldots, F_{x_{d+1}})A^{-1},$$

which shows that a point, $a \in S$, is non-singular iff its image, $h(a) \in h(S)$, is non-singular on $h(S)$. This also shows that

$$h(T_a S) = T_{h(a)} h(S),$$

that is, the projectivity, h, preserves tangent spaces. In summary, we have

Proposition 5.6 *Let $S = V(F) \subseteq \mathbb{P}^d$ be a projective hypersurface, where F is a homogeneous polynomial of degree m and let $h\colon \mathbb{P}^d \to \mathbb{P}^d$ be a projectivity (a bijective projective map). Then, $h(S)$ is a hypersurface in $\mathbb{P}^d$ and a point, $a \in S$, is nonsingular for S iff $h(a)$ is nonsingular for $h(S)$. Furthermore,*

$$h(T_a S) = T_{h(a)} h(S),$$

that is, the projectivity, h, preserves tangent spaces.

Remark: If $h\colon \mathbb{P}^m \to \mathbb{P}^n$ is a projective map, say induced by an injective linear map given by the $(n+1) \times (m+1)$ matrix, $A = (a_{i\,j})$, given any hypersurface, $S = V(F) \subseteq \mathbb{P}^n$, we can define the *pull-back*, $h^*(S) \subseteq \mathbb{P}^m$, of S, by

$$h^*(S) = \{x \in \mathbb{P}^m \mid F(h(x)) = 0\}.$$

This is indeed a hypersurface because $F(x_1, \ldots, x_{n+1})$ is a homogenous polynomial and $h^*(S)$ is the zero locus of the homogeneous polynomial

$$G(x_1, \ldots, x_{m+1}) = F\left(\sum_{j=1}^{m+1} a_{1\,j} x_j, \ldots, \sum_{j=1}^{m+1} a_{n+1\,j} x_j\right).$$

If $m = n$ and h is a projectivity, then we have

$$h(S) = (h^{-1})^*(S).$$

5.4 Quadrics (Affine, Projective) and Polar Duality

The case where $S = V(\Phi) \subseteq \mathbb{P}^d$ is a hypersurface given by a homogeneous polynomial, $\Phi(x_1, \ldots, x_{d+1})$, of degree 2 will come up a lot and deserves a little more attention. In this case, if we write $x = (x_1, \ldots, x_{d+1})$, then $\Phi(x) = \Phi(x_1, \ldots, x_{d+1})$ is completely determined by a $(d+1) \times (d+1)$ symmetric matrix, say $F = (f_{i\,j})$, and we have

$$\Phi(x) = \mathbf{x}^\top F \mathbf{x} = \sum_{i,j=1}^{d+1} f_{i\,j} x_i x_j.$$

Since F is symmetric, we can write

$$\Phi(x) = \sum_{i,j=1}^{d+1} f_{i\,j} x_i x_j = \sum_{i=1}^{d+1} f_{i\,i} x_i^2 + 2 \sum_{\substack{i,j=1 \\ i<j}}^{d+1} f_{i\,j} x_i x_j.$$

The *polar form*, $\varphi(x, y)$, of $\Phi(x)$, is given by

$$\varphi(x, y) = \mathbf{x}^\top F \mathbf{y} = \sum_{i,j=1}^{d+1} f_{i\,j} x_i y_j,$$

where $x = (x_1, \ldots, x_{d+1})$ and $y = (y_1, \ldots, y_{d+1})$. Of course,

$$2\varphi(x, y) = \Phi(x + y) - \Phi(x) - \Phi(y).$$

We also check immediately that

$$2\varphi(x, y) = 2\mathbf{x}^\top F \mathbf{y} = \sum_{j=1}^{d+1} \frac{\partial \Phi(x)}{\partial x_j}\, y_j,$$

and so,

$$\left(\frac{\partial \Phi(x)}{\partial x_1}, \ldots, \frac{\partial \Phi(x)}{\partial x_{d+1}} \right) = 2\mathbf{x}^\top F.$$

The hypersurface, $S = V(\Phi) \subseteq \mathbb{P}^d$, is called a *(projective) (hyper-)quadric surface*. We say that a quadric surface, $S = V(\Phi)$, is *nondegenerate* iff the matrix, F, defining Φ is invertible.

For example, the sphere, $S^d \subseteq \mathbb{P}^{d+1}$, is the nondegenerate quadric given by

$$\mathbf{x}^\top \begin{pmatrix} I_{d+1} & \mathbf{0} \\ \mathbb{O} & -1 \end{pmatrix} \mathbf{x} = 0$$

and the paraboloid, $\mathcal{P} \subseteq \mathbb{P}^{d+1}$, is the nongenerate quadric given by

$$\mathbf{x}^\top \begin{pmatrix} I_d & \mathbf{0} & \mathbf{0} \\ \mathbb{O} & 0 & -\frac{1}{2} \\ \mathbb{O} & -\frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = 0.$$

If $h\colon \mathbb{P}^d \to \mathbb{P}^d$ is a projectivity induced by some invertible matrix, $A = (a_{i\,j})$, and if $S = V(\Phi)$ is a quadric defined by the matrix F, we immediately check that $h(S)$ is the quadric defined by the matrix $(A^{-1})^\top F A^{-1}$. Furthermore, as A is invertible, we see that S is nondegenerate iff $h(S)$ is nondegenerate.

Observe that polar duality w.r.t. the sphere, S^{d-1}, can be expressed by

$$X^* = \left\{ x \in \mathbb{R}^d \mid (\forall y \in X) \left((\mathbf{x}^\top, 1) \begin{pmatrix} I_d & \mathbf{0} \\ \mathbb{O} & -1 \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \leq 0 \right) \right\},$$

where X is any subset of $\mathbb{R}^d$. The above suggests generalizing polar duality with respect to any nondegenerate quadric.

Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form φ and matrix $F = (f_{i\,j})$. Then, we know that φ induces a natural duality between $\mathbb{R}^{d+1}$ and $(\mathbb{R}^{d+1})^*$, namely, for every $u \in \mathbb{R}^{d+1}$, if $\varphi_u \in (\mathbb{R}^{d+1})^*$ is the linear form given by

$$\varphi_u(v) = \varphi(u, v)$$

for every $v \in \mathbb{R}^{d+1}$, then the map $u \mapsto \varphi_u$, from $\mathbb{R}^{d+1}$ to $(\mathbb{R}^{d+1})^*$, is a linear isomorphism.

Definition 5.6 Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. For any $u \in \mathbb{R}^{d+1}$, with $u \neq 0$, the set

$$u^\dagger = \{v \in \mathbb{R}^{d+1} \mid \varphi(u, v) = 0\} = \{v \in \mathbb{R}^{d+1} \mid \varphi_u(v) = 0\} = \mathrm{Ker}\ \varphi_u$$

is a hyperplane called the *polar of u (w.r.t. Q)*.

In terms of the matrix representation of Q, the polar of u is given by the equation

$$\mathbf{u}^\top F \mathbf{x} = 0,$$

or

$$\sum_{j=1}^{d+1} \frac{\partial \Phi(u)}{\partial x_j}\, x_j = 0.$$

Going over to $\mathbb{P}^d$, we say that $\mathbb{P}(u^\dagger)$ is the *polar (hyperplane)* of the point $a = [u] \in \mathbb{P}^d$ and we write $a^\dagger$ for $\mathbb{P}(u^\dagger)$.

Note that the equation of the polar hyperplane, $a^\dagger$, of a point, $a \in \mathbb{P}^d$, is identical to the equation of the tangent plane to Q at a, except that a is not necessarily on Q. However, if $a \in Q$, then the polar of a is indeed the tangent hyperplane, $T_a Q$, to Q at a.

Proposition 5.7 *Let $Q = V(\Phi(x_1, \ldots, x_{d+1})) \subseteq \mathbb{P}^d$ be a nondegenerate quadric with corresponding polar form, φ, and matrix, F. Then, every point, $a \in Q$, is nonsingular.*

Proof. Since

$$\left(\frac{\partial\Phi(a)}{\partial x_1},\ldots,\frac{\partial\Phi(a)}{\partial x_{d+1}}\right) = 2\mathbf{a}^\top F,$$

if $a \in Q$ is singular, then $\mathbf{a}^\top F = 0$ with $a \neq 0$, contradicting the fact that F is invertible. $\square$

The reader should prove the following simple proposition:

Proposition 5.8 *Let $Q = V(\Phi(x_1,\ldots,x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. Then, the following properties hold: For any two points, $a, b \in \mathbb{P}^d$,*

(1) $a \in b^\dagger$ iff $b \in a^\dagger$;

(2) $a \in a^\dagger$ iff $a \in Q$;

(3) Q does not contain any hyperplane.

Remark: As in the case of the sphere, if Q is a nondegenerate quadric and $a \in \mathbb{P}^d$ is any point such that the polar hyperplane, $a^\dagger$, intersects Q, then there is a nice geometric interpretation for $a^\dagger$. Observe that for every $b \in Q \cap a^\dagger$, the polar hyperplane, $b^\dagger$, is the tangent hyperplane, T_bQ, to Q at b and that $a \in T_bQ$. Also, if $a \in T_bQ$ for any $b \in Q$, as $b^\dagger = T_bQ$, then $b \in a^\dagger$. Therefore, $Q \cap a^\dagger$ is the set of contact points of all the tangent hyperplanes to Q passing through a.

Every hyperplane, $H \subseteq \mathbb{P}^d$, is the polar of a single point, $a \in \mathbb{P}^d$. Indeed, if H is defined by a nonzero linear form, $f \in (\mathbb{R}^{d+1})^*$, as Φ is nondegenerate, there is a unique $u \in \mathbb{R}^{d+1}$, with $u \neq 0$, so that $f = \varphi_u$, and as φ_u vanishes on H, we see that H is the polar of the point $a = [u]$. If H is also the polar of another point, $b = [v]$, then φ_v vanishes on H, which means that

$$\varphi_v = \lambda\varphi_u = \varphi_{\lambda u},$$

with $\lambda \neq 0$ and this implies $v = \lambda u$, that is, $a = [u] = [v] = b$ and the pole of H is indeed unique.

Definition 5.7 Let $Q = V(\Phi(x_1,\ldots,x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. The *polar dual (w.r.t. Q)*, X^*, of a subset, $X \subseteq \mathbb{R}^{d+1}$, is given by

$$X^* = \{v \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\varphi(u,v) \leq 0)\}.$$

For every subset, $X \subseteq \mathbb{P}^d$, we let

$$X^* = \mathbb{P}((v(X))^*),$$

where $v(X)$ is the unique double cone associated with X as in Proposition 5.1.

Observe that X^* is always a double cone, even if $X \subseteq \mathbb{R}^{d+1}$ is not. By analogy with the Euclidean case, for any nonzero vector, $u \in \mathbb{R}^{d+1}$, let

$$(u^\dagger)_- = \{v \in \mathbb{R}^{d+1} \mid \varphi(u, v) \leq 0\}.$$

Now, we have the following version of Proposition 4.3:

Proposition 5.9 *Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ, and matrix, $F = (f_{i\,j})$. For any nontrivial polyhedral cone, $C = \mathrm{cone}(u_1, \ldots, u_p)$, where $u_i \in \mathbb{R}^{d+1}$, $u_i \neq 0$, we have*

$$C^* = \bigcap_{i=1}^{p} (u_i^\dagger)_-.$$

If U is the $(d+1) \times p$ matrix whose i^{th} column is u_i, then we can also write

$$C^* = P(U^\top F, \mathbf{0}),$$

where

$$P(U^\top F, \mathbf{0}) = \{v \in \mathbb{R}^{d+1} \mid U^\top F\mathbf{v} \leq \mathbf{0}\}.$$

Consequently, the polar dual of a polyhedral cone w.r.t. a nondegenerate quadric is a polyhedral cone.

Proof. The proof is essentially the same as the proof of Proposition 4.3. As

$$C = \mathrm{cone}(u_1, \ldots, u_p) = \{\lambda_1 u_1 + \cdots + \lambda_p u_p \mid \lambda_i \geq 0,\ 1 \leq i \leq p\},$$

we have

$$\begin{aligned}
C^* &= \{v \in \mathbb{R}^{d+1} \mid (\forall u \in C)(\varphi(u, v) \leq 0)\} \\
&= \{v \in \mathbb{R}^{d+1} \mid \varphi(\lambda_1 u_1 + \cdots + \lambda_p u_p, v) \leq 0,\ \lambda_i \geq 0,\ 1 \leq i \leq p\} \\
&= \{v \in \mathbb{R}^{d+1} \mid \lambda_1 \varphi(u_1, v) + \cdots + \lambda_p \varphi(u_p, v) \leq 0,\ \lambda_i \geq 0,\ 1 \leq i \leq p\} \\
&= \bigcap_{i=1}^{p} \{v \in \mathbb{R}^{d+1} \mid \varphi(u_i, v) \leq 0\} \\
&= \bigcap_{i=1}^{p} (u_i^\dagger)_-.
\end{aligned}$$

By the equivalence theorem for $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra, we conclude that C^* is a polyhedral cone. $\square$

Proposition 5.9 allows us to make the following definition:

Definition 5.8 Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ. Given any projective polyhedron, $P = \mathbb{P}(C)$, where C is a polyhedral cone, the *polar dual (w.r.t. Q)*, P^*, of P is the projective polyhedron

$$P^* = \mathbb{P}(C^*).$$

We also show that projectivities behave well with respect to polar duality.

Proposition 5.10 *Let $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ be a nondegenerate quadric with corresponding polar form, φ, and matrix, $F = (f_{i\,j})$. For every projectivity, $h\colon \mathbb{P}^d \to \mathbb{P}^d$, if h is induced by the linear map, $\widehat{h}$, given by the invertible matrix, $A = (a_{i\,j})$, for every subset, $X \subseteq \mathbb{R}^{d+1}$, we have*

$$\widehat{h}(X^*) = (\widehat{h}(X))^*,$$

where on the left-hand side, X^* is the polar dual of X w.r.t. Q and on the right-hand side, $(\widehat{h}(X))^*$ is the polar dual of $\widehat{h}(X)$ w.r.t. the nondegenerate quadric, $h(Q)$, given by the matrix $(A^{-1})^\top F A^{-1}$. Consequently, if $X \neq \{0\}$, then

$$h((\mathbb{P}(X))^*) = (h(\mathbb{P}(X)))^*$$

and for every projective polyhedron, P, we have

$$h(P^*) = (h(P))^*.$$

Proof. As

$$X^* = \{v \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\mathbf{u}^\top F \mathbf{v} \leq 0)\},$$

we have

$$\begin{aligned}
\widehat{h}(X^*) &= \{\widehat{h}(v) \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\mathbf{u}^\top F \mathbf{v} \leq 0)\} \\
&= \{y \in \mathbb{R}^{d+1} \mid (\forall u \in X)(\mathbf{u}^\top F A^{-1} \mathbf{y} \leq 0)\} \\
&= \{y \in \mathbb{R}^{d+1} \mid (\forall x \in \widehat{h}(X))(\mathbf{x}^\top (A^{-1})^\top F A^{-1} \mathbf{y} \leq 0)\} \\
&= (\widehat{h}(X))^*,
\end{aligned}$$

where $(\widehat{h}(X))^*$ is the polar dual of $\widehat{h}(X)$ w.r.t. the quadric whose matrix is $(A^{-1})^\top F A^{-1}$, that is, the polar dual w.r.t. $h(Q)$.

The second part of the proposition follows immediately by setting $X = C$, where C is the polyhedral cone defining the projective polyhedron, $P = \mathbb{P}(C)$. $\square$

We will also need the notion of an affine quadric and polar duality with respect to an affine quadric. Fortunately, the properties we need in the affine case are easily derived from the projective case using the "trick" that the affine space, $\mathbb{E}^d$, can be viewed as the hyperplane, $H_{d+1} \subseteq \mathbb{R}^{d+1}$, of equation, $x_{d+1} = 1$ and that its associated vector space, $\mathbb{R}^d$, can be viewed as the hyperplane, $H_{d+1}(0) \subseteq \mathbb{R}^{d+1}$, of equation $x_{d+1} = 0$. A point, $a \in \mathbb{A}^d$, corresponds to the vector, $\widehat{a} = \binom{a}{1} \in \mathbb{R}^{d+1}$, and a vector, $u \in \mathbb{R}^d$, corresponds to the vector, $\widehat{u} = \binom{u}{0} \in \mathbb{R}^{d+1}$. This way, the projective space, $\mathbb{P}^d = \mathbb{P}(\mathbb{R}^{d+1})$, is the natural *projective completion* of $\mathbb{E}^d$, which is isomorphic to the affine patch U_{d+1} where $x_{d+1} \neq 0$. The hyperplane, $x_{d+1} = 0$, is the "hyperplane at infinity" in $\mathbb{P}^d$.

If we write $x = (x_1, \ldots, x_d)$, a polynomial, $\Phi(x) = \Phi(x_1, \ldots, x_d)$, of degree 2 can be written as

$$\Phi(x) = \sum_{i,j=1}^{d} a_{i\,j} x_i x_j + 2 \sum_{i=1}^{d} b_i x_i + c,$$

where $A = (a_{i\,j})$ is a symmetric matrix. If we write $b^\top = (b_1, \ldots, b_d)$, then we have

$$\Phi(x) = (\mathbf{x}^\top, 1) \begin{pmatrix} A & b \\ b^\top & c \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \widehat{\mathbf{x}}^\top \begin{pmatrix} A & b \\ b^\top & c \end{pmatrix} \widehat{\mathbf{x}}.$$

Therefore, as in the projective case, Φ is completely determined by a $(d+1) \times (d+1)$ symmetric matrix, say $F = (f_{i\,j})$, and we have

$$\Phi(x) = (\mathbf{x}^\top, 1) F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \widehat{\mathbf{x}}^\top F \widehat{\mathbf{x}}.$$

We say that $Q \subseteq \mathbb{R}^d$ is a *nondegenerate affine quadric* iff

$$Q = V(\Phi) = \left\{ x \in \mathbb{R}^d \mid (\mathbf{x}^\top, 1) F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \right\},$$

where F is symmetric and invertible. Given any point $a \in \mathbb{R}^d$, the *polar hyperplane*, $a^\dagger$, of a w.r.t. Q is defined by

$$a^\dagger = \left\{ x \in \mathbb{R}^d \mid (\mathbf{a}^\top, 1) F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \right\}.$$

From a previous discussion, the equation of the polar hyperplane, $a^\dagger$, is

$$\sum_{i=1}^{d} \frac{\partial \Phi(a)}{\partial x_i} (x_i - a_i) = 0.$$

Given any subset, $X \subseteq \mathbb{R}^d$, the *polar dual*, X^*, of X is defined by

$$X^* = \left\{ y \in \mathbb{R}^d \mid (\forall x \in X) \left((\mathbf{x}^\top, 1) F \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \leq 0 \right) \right\}.$$

As noted before, polar duality with respect to the affine sphere, $S^d \subseteq \mathbb{R}^{d+1}$, corresponds to the case where

$$F = \begin{pmatrix} I_d & \mathbf{0} \\ \mathbb{O} & -1 \end{pmatrix}$$

and polar duality with respect to the affine paraboloid $\mathcal{P} \subseteq \mathbb{R}^{d+1}$, corresponds to the case where

$$F = \begin{pmatrix} I_{d-1} & \mathbf{0} & \mathbf{0} \\ \mathbb{O} & 0 & -\frac{1}{2} \\ \mathbb{O} & -\frac{1}{2} & 0 \end{pmatrix}.$$

We will need the following version of Proposition 4.14:

Proposition 5.11 *Let Q be a nondegenerate affine quadric given by the $(d+1)\times(d+1)$ symmetric matrix, F, let $\{y_1, \ldots, y_p\}$ be any set of points in $\mathbb{E}^d$ and let $\{v_1, \ldots, v_q\}$ be any set of nonzero vectors in $\mathbb{R}^d$. If $\widehat{Y}$ is the $(d+1)\times p$ matrix whose i^{th} column is $\widehat{y_i}$ and V is the $(d+1)\times q$ matrix whose j^{th} column is $\widehat{v_j}$, then*

$$(\mathrm{conv}(\{y_1, \ldots, y_p\}) \cup \mathrm{cone}(\{v_1, \ldots, v_q\}))^* = P(\widehat{Y}^\top F, \mathbf{0}; \widehat{V}^\top F, \mathbf{0}),$$

with

$$P(\widehat{Y}^\top F, \mathbf{0}; \widehat{V}^\top F, \mathbf{0}) = \left\{ x \in \mathbb{R}^d \mid \widehat{Y}^\top F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \leq \mathbf{0},\ \widehat{V}^\top F \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \leq \mathbf{0} \right\}.$$

Proof. The proof is immediately adpated from that of Proposition 4.14. $\square$

Using Proposition 5.11, we can prove the following Proposition showing that projective completion and polar duality commute:

Proposition 5.12 *Let Q be a nondegenerate affine quadric given by the $(d+1)\times(d+1)$ symmetric, invertible matrix, F. For every polyhedron, $P \subseteq \mathbb{R}^d$, we have*

$$\widetilde{P^*} = (\widetilde{P})^*,$$

where on the right-hand side, we use polar duality w.r.t. the nondegenerate projective quadric, $\widetilde{Q}$, defined by F.

Proof. By definition, we have $\widetilde{P} = \mathbb{P}(C(P))$, $(\widetilde{P})^* = \mathbb{P}((C(P))^*)$ and $\widetilde{P^*} = \mathbb{P}(C(P^*))$. Therefore, it suffices to prove that

$$(C(P))^* = C(P^*).$$

Now, $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$, for some finite set of points, Y, and some finite set of vectors, V, and we know that

$$C(P) = \mathrm{cone}(\widehat{Y} \cup \widehat{V}).$$

From Proposition 5.9,

$$(C(P))^* = \{v \in \mathbb{R}^{d+1} \mid \widehat{Y}^\top F\mathbf{v} \leq \mathbf{0},\ \widehat{V}^\top F\mathbf{v} \leq \mathbf{0}\}$$

and by Proposition 5.11,

$$P^* = \left\{ x \in \mathbb{R}^d \mid \widehat{Y}^\top F \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \leq \mathbf{0},\ \widehat{V}^\top F \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \leq \mathbf{0} \right\}.$$

But, by definition of $C(P^*)$ (see Section 4.4, especially Proposition 4.19), the hyperplanes cutting out $C(P^*)$ are obtained by homogenizing the equations of the hyperplanes cutting out P^* and so,

$$C(P^*) = \left\{ \begin{pmatrix} \mathbf{x} \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid \widehat{Y}^\top F \begin{pmatrix} \mathbf{x} \\ x_{d+1} \end{pmatrix} \leq \mathbf{0},\ \widehat{V}^\top F \begin{pmatrix} \mathbf{x} \\ x_{d+1} \end{pmatrix} \leq \mathbf{0} \right\} = (C(P))^*,$$

as claimed. $\square$

Remark: If $Q = V(\Phi(x_1, \ldots, x_{d+1}))$ is a projective or an affine quadric, it is obvious that

$$V(\Phi(x_1, \ldots, x_{d+1})) = V(\lambda\Phi(x_1, \ldots, x_{d+1}))$$

for every $\lambda \neq 0$. This raises the following question: If

$$Q = V(\Phi_1(x_1, \ldots, x_{d+1})) = V(\Phi_2(x_1, \ldots, x_{d+1})),$$

what is the relationship between Φ_1 and Φ_2?

The answer depends crucially on the field over which projective space or affine space is defined (*i.e.*, whether $Q \subseteq \mathbb{RP}^d$ or $Q \subseteq \mathbb{CP}^d$ in the projective case or whether $Q \subseteq \mathbb{R}^{d+1}$ or $Q \subseteq \mathbb{C}^{d+1}$ in the affine case). For example, over $\mathbb{R}$, the polynomials $\Phi_1(x_1, x_2, x_3) = x_1^2 + x_2^2$ and $\Phi_2(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2$ both define the point $(0\colon 0\colon 1) \in \mathbb{P}^2$, since the only real solution of Φ_1 and Φ_2 are of the form $(0, 0, z)$. However, if Q has some nonsingular point, the following can be proved (see Samuel [35], Theorem 46 (Chapter 3)):

Theorem 5.13 *Let $Q = V(\Phi(x_1, \ldots, x_{d+1})$ be a projective or an affine quadric, over $\mathbb{RP}^d$ or $\mathbb{R}^{d+1}$. If Q has a nonsingular point, then for every polynonial, Φ', such that $Q = V(\Phi'(x_1, \ldots, x_{d+1})$, there is some $\lambda \neq 0$ ($\lambda \in \mathbb{R}$) so that $\Phi' = \lambda\Phi$.*

In particular, Theorem 5.13 shows that the equation of a nondegenerate quadric is unique up to a scalar.

Actually, more is true. It turns out that if we allow complex solutions, that is, if $Q \subseteq \mathbb{CP}^d$ in the projective case or $Q \subseteq \mathbb{C}^{d+1}$ in the affine case, then $Q = V(\Phi_1) = V(\Phi_2)$ always implies $\Phi_2 = \lambda\Phi_1$ for some $\lambda \in \mathbb{C}$, with $\lambda \neq 0$. In the real case, the above holds (for some $\lambda \in \mathbb{R}$, with $\lambda \neq 0$) unless Q is an affine subspace (resp. a projective subspace) of dimension at most $d - 1$ (resp. of dimension at most $d - 2$). Even in this case, there is a bijective affine map, f, (resp. a bijective projective map, h), such that $\Phi_2 = \Phi_1 \circ f^{-1}$ (resp. $\Phi_2 = \Phi_1 \circ h^{-1}$). A proof of these facts (and more) can be found in Tisseron [42] (Chapter 3).

We now have everything we need for a rigorous presentation of the material of Section 8.6. For a comprehensive treatment of the affine and projective quadrics and related material, the reader should consult Berger (Geometry II) [6] or Samuel [35].