Chapter 20

Manifolds Arising from Group Actions

20.1 Proper Maps

We saw in Chapter 5 that many topological spaces arise from a group action.

The scenario is that we have a smooth action $\varphi\colon G \times M \to M$ of a Lie group G acting on a manifold M.

If G acts transitively on M, then for any point $x \in M$, if G_x is the stabilizer of x, Theorem 5.14 ensures that M is homeomorphic to G/G_x.

For simplicity of notation, write $H = G_x$.

What we would really like is that G/H actually be a manifold.

This is indeed the case, because the transitive action of G on G/H is equivalent to a *right action* of H on G which is no longer transitive, but which has some special properties (to be proper and free).

We are thus led to considering left (and right) actions $\varphi\colon G \times M \to M$ of a Lie group G on a manifold M that are not necessarily transitive.

If the action is not transitive, then we consider the *orbit space* M/G of orbits $G \cdot x$ $(x \in M)$.

However, in general, M/G is not even Hausdorff. It is thus desirable to look for sufficient conditions that ensure that M/G is Hausdorff.

A sufficient condition can be given using the notion of a *proper map*.

If our action is also *free*, then the orbit space M/G is indeed a smooth manifold.

Sharper results hold if we consider Riemannian manifolds.

Before we go any further, let us observe that the case where our action is transitive is subsumed by the more general situation of an orbit space.

Indeed, if our action is transitive, for any $x \in M$, we know that the stabilizer $H = G_x$ of x is a closed subgroup of G.

Then, we can consider the *right* action $G \times H \to G$ of H on G given by

$$g \cdot h = gh, \quad g \in G, h \in H.$$

The orbits of this (right) action are precisely the *left cosets* gH of H.

Therefore, the set of left cosets G/H (the homogeneous space induced by the action $\cdot \colon G \times M \to M$) is the set of orbits of the right action $G \times H \to G$.

Observe that we have a transitive left action of G on the space G/H of left cosets, given by

$$g_1 \cdot g_2 H = g_1 g_2 H.$$

The stabilizer of $1H$ is obviously H itself.

Thus, we recover the original transitive left action of G on $M = G/H$.

Now, it turns out that an action of the form $G \times H \to G$, where H is a closed subgroup of a Lie group G, is a special case of a free and proper action $M \times G \to G$, in which case the orbit space M/G is a manifold, and the projection $\pi\colon G \to M/G$ is a submersion.

Let us now define proper maps.

Definition 20.1. If X and Y are two Hausdorff topological spaces,[1] a continuous map $\varphi\colon X \to Y$ is *proper* iff for every topological space Z, the map
$\varphi \times \mathrm{id}\colon X \times Z \to Y \times Z$ is a *closed map* (recall that f is a closed map iff the image of any closed set by f is a closed set).

If we let Z be a one-point space, we see that a proper map is closed.

[1]It is not necessary to assume that X and Y are Hausdorff but, if X and/or Y are not Hausdorff, we have to replace "compact" by "quasi-compact." We have no need for this extra generality.

At first glance, it is not obvious how to check that a map is proper just from Definition 20.1. Proposition 20.2 gives a more palatable criterion.

The following proposition is easy to prove .

Proposition 20.1. *If $\varphi\colon X \to Y$ is any proper map, then for any closed subset F of X, the restriction of φ to F is proper.*

The following result providing a "good" criterion for checking that a map is proper can be shown (see Bourbaki, General Topology [9], Chapter 1, Section 10).

Proposition 20.2. *A continuous map $\varphi\colon X \to Y$ is proper iff φ is closed and if $\varphi^{-1}(y)$ is compact for every $y \in Y$.*

Proposition 20.2 shows that a homeomorphism (or a diffeomorphism) is proper.

If φ is proper, it is easy to show that $\varphi^{-1}(K)$ is compact in X whenever K is compact in Y.

Moreover, if Y is also locally compact, then we have the following result (see Bourbaki, General Topology [9], Chapter 1, Section 10).

Proposition 20.3. *If Y is locally compact, a map $\varphi\colon X \to Y$ is a proper map iff $\varphi^{-1}(K)$ is compact in X whenever K is compact in Y*

In particular, this is true if Y is a manifold since manifolds are locally compact.

This explains why Lee [31] (Chapter 9) takes the property stated in Proposition 20.3 as the definition of a proper map (because he only deals with manifolds).

Finally, we can define proper actions.

20.2 Proper and Free Actions

Definition 20.2. Given a Hausdorff topological group G and a topological space M, a left action $\cdot\colon G \times M \to M$ is *proper* if it is continuous and if the map

$$\theta\colon G \times M \longrightarrow M \times M, \quad (g, x) \mapsto (g \cdot x, x)$$

is proper.

The right actions associated with the transitive actions presented in Section 5.2 are examples of proper actions.

Proposition 20.4. *The action $\cdot\colon H \times G \to G$ of a closed subgroup H of a group G on G (given by $(h, g) \mapsto hg$) is proper. The same is true for the right action of H on G.*

As desired, proper actions yield Hausdorff orbit spaces.

Proposition 20.5. *If the action* $\cdot \colon G \times M \to M$ *is proper (where* G *is Hausdorff), then the orbit space* M/G *is Hausdorff. Furthermore,* M *is also Hausdorff.*

We also have the following properties (see Bourbaki, General Topology [9], Chapter 3, Section 4).

Proposition 20.6. *Let* $\cdot \colon G \times M \to M$ *be a proper action, with* G *Hausdorff. For any* $x \in M$*, let* $G \cdot x$ *be the orbit of* x *and let* G_x *be the stabilizer of* x*. Then:*

(a) The map $g \mapsto g \cdot x$ *is a proper map from* G *to* M*.*

(b) G_x *is compact.*

(c) The canonical map from G/G_x *to* $G \cdot x$ *is a homeomorphism.*

(d) The orbit $G \cdot x$ *is closed in* M*.*

If G is locally compact, we have the following characterization of being proper (see Bourbaki, General Topology [9], Chapter 3, Section 4).

Proposition 20.7. *If G and M are Hausdorff and G is locally compact, then the action $\cdot\colon G \times M \to M$ is proper iff for all $x, y \in M$, there exist some open sets, V_x and V_y in M, with $x \in V_x$ and $y \in V_y$, so that the closure $\overline{K}$ of the set $K = \{g \in G \mid (g \cdot V_x) \cap V_y \neq \emptyset\}$, is compact in G.*

In particular, if G has the discrete topology, the above condition holds iff the sets $\{g \in G \mid (g \cdot V_x) \cap V_y \neq \emptyset\}$ are finite.

Also, if G is compact, then $\overline{K}$ is automatically compact, so every compact group acts properly.

If M is locally compact, we have the following characterization of being proper (see Bourbaki, General Topology [9], Chapter 3, Section 4).

Proposition 20.8. *Let $\cdot\colon G \times M \to M$ be a continuous action, with G and M Hausdorff. For any compact subset K of M we have:*

(a) The set $G_K = \{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$ is closed.

(b) If M is locally compact, then the action is proper iff G_K is compact for every compact subset K of M.

In the special case where G is discrete (and M is locally compact), condition (b) says that the action is proper iff G_K is finite.

Remark: If G is a Hausdorff topological group and if H is a subgroup of G, then it can be shown that the action of G on G/H ($(g_1, g_2H) \mapsto g_1g_2H$) is proper iff H is compact in G.

Definition 20.3. An action $\cdot\colon G \times M \to M$ is *free* if for all $g \in G$ and all $x \in M$, if $g \neq 1$ then $g \cdot x \neq x$.

An equivalent way to state that an action $\cdot\colon G \times M \to M$ is free is as follows. For every $g \in G$, let $\tau_g\colon M \to M$ be the diffeomorphism of M given by

$$\tau_g(x) = g \cdot x, \quad x \in M.$$

Then, the action $\cdot\colon G \times M \to M$ is free iff for all $g \in G$, if $g \neq 1$ then τ_g has no fixed point.

Another equivalent statement is that for every $x \in M$, the stabilizer G_x of x is reduced to the trivial group $\{1\}$.

For example, the action of $\mathbf{SO}(3)$ on S^2 given by Example 5.1 of Section 5.2 is not free since any rotation of S^2 fixes the two points of the rotation axis.

If H is a subgroup of G, obviously H acts freely on G (by multiplication on the left or on the right). This fact together with Proposition 20.4 yields the following corollary which provides a large supply of free and proper actions.

Corollary 20.9. *The action* $\cdot\colon H \times G \to G$ *of a closed subgroup* H *of a group* G *on* G *(given by* $(h, g) \mapsto hg$*) is free and proper. The same is true for the right action of* H *on* G*.*

There is a stronger version of the results that we are going to state next that involves the notion of principal bundle.

Since this notion is not discussed until Section **??**, we state weaker versions not dealing with principal bundles.

The weaker version that does not mention principal bundles is usually stated for left actions; for instance, in Lee [31] (Chapter 9, Theorem 9.16). We formulate both a left and a right version.

Theorem 20.10. *Let M be a smooth manifold, G be a Lie group, and let $\cdot\colon G \times M \to M$ be a left smooth action (resp. right smooth action $\cdot\colon M \times G \to M$) which is proper and free. Then the canonical projection $\pi\colon G \to M/G$ is a submersion (which means that $d\pi_g$ is surjective for all $g \in G$), and there is a unique manifold structure on M/G with this property.*

Theorem 20.10 has some interesting corollaries.

Because a closed subgroup H of a Lie group G is a Lie group, and because the action of a closed subgroup is free and proper, if we apply Theorem 20.10 to the right action $\cdot\colon G \times H \to G$ (here $M = G$ and $G = H$), we get the following result.

This is the result we use to verify reductive homogeneous spaces are indeed manifolds.

Theorem 20.11. *If G is a Lie group and H is a closed subgroup of G, then the canonical projection $\pi\colon G \to G/H$ is a submersion (which means that $d\pi_g$ is surjective for all $g \in G$), and there is a unique manifold structure on G/H with this property.*

In the special case where G acts transitively on M, for any $x \in M$, if G_x is the stabilizer of x, then with $H = G_x$,

Theorem 20.11 shows that there is a manifold structure on G/H such that $\pi\colon G \to G/H$ is a submersion.

Actually, G/H is diffeomorphic to M, as shown by the following theorem whose proof can be found in Lee [31] (Chapter 9, Theorem 9.24).

Theorem 20.12. *Let* $\cdot\colon G \times M \to M$ *be a smooth transitive action of a Lie group* G *on a smooth manifold* M *(so that* M *is a homogeneous space). For any* $x \in M$*, if* G_x *is the stabilizer of* x *and if we write* $H = G_x$*, then the map* $\overline{\pi}_x\colon G/H \to M$ *given by*

$$\overline{\pi}_x(gH) = g \cdot x$$

is a diffeomorphism and an equivariant map (with respect to the action of G *on* G/H *and the action of* G *on* M*).*

By Theorem 20.11 and Theorem 20.12, every homogeneous space M (with a smooth G-action) is equivalent to a manifold G/H as above.

This is an important and very useful result that reduces the study of homogeneous spaces to the study of coset manifolds of the form G/H where G is a Lie group and H is a closed subgroup of G.

Here is a simple example of Theorem 20.11. Let $G = \mathbf{SO}(3)$ and

$$H = \left\{ M \in \mathbf{SO}(3) \mid M = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix},\ S \in \mathbf{SO}(2) \right\}.$$

The right action $\cdot \colon \mathbf{SO}(3) \times H \to \mathbf{SO}(3)$ given by the matrix multiplication

$$g \cdot h = gh, \qquad g \in \mathbf{SO}(3),\ \ h \in H,$$

yields the left cosets gH, and the orbit space $\mathbf{SO}(3)/\mathbf{SO}(2)$, which by Theorem 20.11 and Theorem 20.12 is diffeomorphic to S^2.

20.3 Riemannian Submersions and Coverings Induced by Group Actions ⊛

The purpose of this section is to equip the orbit space M/G of Theorem 20.10 with the inner product structure of a Riemannian manifold.

Because we provide a different proof for the reason why reductive homogeneous manifolds are Riemannian manifolds, namely Proposition 20.22, this section is not necessary for understanding the material in Section 20.4 and may be skipped on the first reading.

Definition 20.4. Given a Riemannian manifold (N, h), we say that a Lie group G *acts by isometries on N* if for every $g \in G$, the diffeomorphism $\tau_g\colon N \to N$ given by

$$\tau_g(p) = g \cdot p, \quad p \in N,$$

is an isometry ($(d\tau_g)_p\colon T_pN \to T_{\tau_g(p)}N$ is an isometry for all $p \in N$).

If (N, h) is a Riemannian manifold and if G is a Lie group, then $\pi\colon N \to N/G$ can be made into a Riemannian submersion.

Theorem 20.13. *Let (N, h) be a Riemannian manifold and let $\cdot\colon G \times N \to N$ be a smooth, free and proper proper action, with G a Lie group acting by isometries of N. Then, there is a unique Riemannian metric g on $M = N/G$ such that $\pi\colon N \to M$ is a Riemannian submersion.*

As an example, if $N = S^{2n+1}$, then the group $G = S^1 = \mathbf{SU}(1)$ acts by isometries on S^{2n+1}, and we obtain a submersion $\pi\colon S^{2n+1} \to \mathbb{CP}^n$.

If we pick the canonical metric on S^{2n+1}, by Theorem 20.13, we obtain a Riemannian metric on $\mathbb{CP}^n$ known as the *Fubini–Study metric*.

Using Proposition 15.8, it is possible to describe the geodesics of $\mathbb{CP}^n$; see Gallot, Hulin, Lafontaine [19] (Chapter 2).

Another situation where a group action yields a Riemannian submersion is the case where a transitive action is reductive, considered in the next section.

We now consider the case of a smooth action $\cdot\colon G \times M \to M$, where G is a discrete group (and M is a manifold). In this case, we will see that $\pi\colon M \to M/G$ is a Riemannian covering map.

Assume G is a discrete group. By Proposition 20.7, the action $\cdot\colon G\times M \to M$ is proper iff for all $x, y \in M$, there exist some open sets, V_x and V_y in M, with $x \in V_x$ and $y \in V_y$, so that the set $K = \{g \in G \mid (g \cdot V_x) \cap V_y \neq \emptyset\}$ is finite.

By Proposition 20.8, the action $\cdot\colon G\times M \to M$ is proper iff $G_K = \{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$ is finite for every compact subset K of M.

It is shown in Lee [31] (Chapter 9) that the above conditions are equivalent to the conditions below.

Proposition 20.14. *If* $\cdot\colon G \times M \to M$ *is a smooth action of a discrete group* G *on a manifold* M*, then this action is proper iff*

(i) For every $x \in M$*, there is some open subset* V *with* $x \in V$ *such that* $gV \cap V \neq \emptyset$ *for only finitely many* $g \in G$*.*

(ii) For all $x, y \in M$*, if* $y \notin G \cdot x$ *(*y *is not in the orbit of* x*), then there exist some open sets* V, W *with* $x \in V$ *and* $y \in W$ *such that* $gV \cap W = 0$ *for all* $g \in G$*.*

The following proposition gives necessary and sufficient conditions for a discrete group to act freely and properly often found in the literature (for instance, O'Neill [38], Berger and Gostiaux [6], and do Carmo [13], but beware that in this last reference Hausdorff separation is not required!).

Proposition 20.15. *If X is a locally compact space and G is a discrete group, then a smooth action of G on X is free and proper iff the following conditions hold:*

(i) For every $x \in X$, there is some open subset V with $x \in V$ such that $gV \cap V = \emptyset$ for all $g \in G$ such that $g \neq 1$.

(ii) For all $x, y \in X$, if $y \notin G \cdot x$ (y is not in the orbit of x), then there exist some open sets V, W with $x \in V$ and $y \in W$ such that $gV \cap W = 0$ for all $g \in G$.

Remark: The action of a discrete group satisfying the properties of Proposition 20.15 is often called "properly discontinuous."

However, as pointed out by Lee ([31], just before Proposition 9.18), this term is self-contradictory since such actions are smooth, and thus continuous!

Then, we have the following useful result.

Theorem 20.16. *Let N be a smooth manifold and let G be discrete group acting smoothly, freely and properly on N. Then, there is a unique structure of smooth manifold on N/G such that the projection map $\pi\colon N \to N/G$ is a covering map.*

Real projective spaces are illustrations of Theorem 20.16.

Indeed, if N is the unit n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ and $G = \{I, -I\}$, where $-I$ is the antipodal map, then the conditions of Proposition 20.15 are easily checked (since S^n is compact), and consequently the quotient

$$\mathbb{RP}^n = S^n/G$$

is a smooth manifold and the projection map $\pi\colon S^n \to \mathbb{RP}^n$ is a covering map.

The fiber $\pi^{-1}([x])$ of every point $[x] \in \mathbb{RP}^n$ consists of two antipodal points: $x, -x \in S^n$.

The next step is to see how a Riemannian metric on N induces a Riemannian metric on the quotient manifold N/G. The following theorem is the Riemannian version of Theorem 20.16.

Theorem 20.17. *Let (N, h) be a Riemannian manifold and let G be discrete group acting smoothly, freely and properly on N, and such that the map $x \mapsto \sigma \cdot x$ is an isometry for all $\sigma \in G$. Then there is a unique structure of Riemannian manifold on $M = N/G$ such that the projection map $\pi\colon N \to M$ is a Riemannian covering map.*

Theorem 20.17 implies that every Riemannian metric g on the sphere S^n induces a Riemannian metric $\widehat{g}$ on the projective space $\mathbb{RP}^n$, in such a way that the projection $\pi\colon S^n \to \mathbb{RP}^n$ is a Riemannian covering.

In particular, if U is an open hemisphere obtained by removing its boundary S^{n-1} from a closed hemisphere, then π is an isometry between U and its image $\mathbb{RP}^n - \pi(S^{n-1}) \approx \mathbb{RP}^n - \mathbb{RP}^{n-1}$.

In summary, given a Riemannian manifold N and a group G acting on N, Theorem 20.13 gives us a method for obtaining a Riemannian manifold N/G such that $\pi\colon N \to N/G$ is a Riemannian submersion ($\cdot\colon G \times N \to N$ is a free and proper action and G acts by isometries).

Theorem 20.17 gives us a method for obtaining a Riemannian manifold N/G such that $\pi\colon N \to N/G$ is a Riemannian covering ($\cdot\colon G \times N \to N$ is a free and proper action of a discrete group G acting by isometries).

In the next section, we show that Riemannian submersions arise from a reductive homogeneous space.

20.4 Reductive Homogeneous Spaces

If $\cdot\colon G \times M \to M$ is a smooth action of a Lie group G on a manifold M, then a certain class of Riemannian metrics on M is particularly interesting.

Recall that for every $g \in G$, $\tau_g\colon M \to M$ is the diffeomorphism of M given by

$$\tau_g(p) = g \cdot p, \quad \text{for all } p \in M.$$

Definition 20.5. Given a smooth action $\cdot\colon G \times M \to M$, a metric $\langle -, - \rangle$ on M is *G-invariant* if τ_g is an isometry for all $g \in G$; that is, for all $p \in M$, we have

$$\langle d(\tau_g)_p(u), d(\tau_g)_p(v) \rangle_p = \langle u, v \rangle_p \quad \text{for all } u, v \in T_pM.$$

If the action is transitive, then for any fixed $p_0 \in M$ and for every $p \in M$, there is some $g \in G$ such that $p = g \cdot p_0$, so it is sufficient to require that $d(\tau_g)_{p_0}$ be an isometry for every $g \in G$.

From now on we are dealing with a *smooth transitive action* $\cdot\colon G \times M \to M$, and for any given $p_0 \in M$, if $H = G_{p_0}$ is the stabilizer of p_0, then by Theorem 20.12, M is diffeomorphic to G/H.

The existence of G-invariant metrics on G/H depends on properties of a certain representation of H called the isotropy representation (see Proposition 20.20).

The isotropy representation is equivalent to another representation $\mathrm{Ad}^{G/H}\colon H \to \mathbf{GL}(\mathfrak{g}/\mathfrak{h})$ of H involving the quotient algebra $\mathfrak{g}/\mathfrak{h}$.

This representation is too complicated to deal with, so we consider the more tractable situation where the Lie algebra $\mathfrak{g}$ of G factors as a direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

for some subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$ for all $h \in H$, where $\mathfrak{h}$ is the Lie algebra of H.

Then $\mathfrak{g}/\mathfrak{h}$ is isomorphic to $\mathfrak{m}$, and the representation $\mathrm{Ad}^{G/H}\colon H \to \mathbf{GL}(\mathfrak{g}/\mathfrak{h})$ becomes the representation $\mathrm{Ad}\colon H \to \mathbf{GL}(\mathfrak{m})$, where Ad_h is the restriction of Ad_h to $\mathfrak{m}$ for every $h \in H$.

In this situation there is an isomorphism between $T_{p_0}M \cong T_o(G/H)$ and $\mathfrak{m}$ (where o denotes the point in G/H corresponding to the coset H).

It is also the case that if H is "nice" (for example, compact), then $M = G/H$ will carry G-invariant metrics, and that under such metrics, the projection $\pi\colon G \to G/H$ is a Riemannian submersion.

In order to proceed it is necessary to express the derivative $d\pi_1\colon \mathfrak{g} \to T_o(G/H)$ of the projection map $\pi\colon G \to G/H$ in terms of certain vector fields.

This is a special case of a process in which an action $\cdot\colon G \times M \to M$ associates a vector field X^* on M to every vector $X \in \mathfrak{g}$ in the Lie algebra of G.

Definition 20.6. Given a smooth action $\varphi\colon G \times M \to M$ of a Lie group on a manifold M, for every $X \in \mathfrak{g}$, we define the vector field X^* (or X_M) on M called an *action field* or *infinitesimal generator* of the action corresponding to X, by

$$X^*(p) = \frac{d}{dt}(\exp(tX) \cdot p)\bigg|_{t=0}, \quad p \in M.$$

For a fixed $X \in \mathfrak{g}$, the map $t \mapsto \exp(tX)$ is a curve through 1 in G, so the map $t \mapsto \exp(tX) \cdot p$ is a curve through p in M, and $X^*(p)$ is the tangent vector to this curve at p.

For example, in the case of the adjoint action $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$, for every $X \in \mathfrak{g}$, we have

$$X^*(Y) = [X, Y],$$

so $X^* = \mathrm{ad}(X)$.

For any $p_0 \in M$, there is a diffeomorphism $G/G_{p_0} \to G \cdot p_0$ onto the orbit $G \cdot p_0$ of p_0 viewed as a manifold, and it is not hard to show that for any $p \in G \cdot p_0$, we have an isomorphism

$$T_p(G \cdot p_0) = \{X^*(p) \mid X \in \mathfrak{g}\};$$

see Marsden and Ratiu [32] (Chapter 9, Section 9.3).

It can also be shown that the Lie algebra $\mathfrak{g}_p$ of the stabilizer G_p of p is given by

$$\mathfrak{g}_p = \{X \in \mathfrak{g} \mid X^*(p) = 0\}.$$

The following technical proposition is shown in Marsden and Ratiu [32] (Chapter 9, Proposition 9.3.6 and lemma 9.3.7).

Proposition 20.18. *Given a smooth action $\varphi\colon G \times M \to M$ of a Lie group on a manifold M, the following properties hold:*

(1) For every $X \in \mathfrak{g}$, we have

$$(\mathrm{Ad}_g X)^* = \tau_{g^{-1}}^* X^* = (\tau_g)_* X^*, \quad \textit{for every } g \in G;$$

Here, $\tau_{g^{-1}}^*$ is the pullback associated with $\tau_{g^{-1}}$, and $(\tau_g)_*$ is the push-forward associated with τ_g.

(2) The map $X \mapsto X^*$ from $\mathfrak{g}$ to $\mathfrak{X}(M)$ is a Lie algebra anti-homomorphism, which means that

$$[X^*, Y^*] = -[X, Y]^* \quad \textit{for all } X, Y \in \mathfrak{g}.$$

Remark: If the metric on M is G-invariant (that is, every τ_g is an isometry of M), then the vector field X^* is a Killing vector field on M for every $X \in \mathfrak{g}$.

Given a pair (G, H), where G is a Lie group and H is a closed subgroup of G, it turns out that there is a criterion for the existence of some G-invariant metric on the homogeneous space G/H in terms of a certain representation of H called the isotropy representation.

Let us explain what this representation is.

Recall that G acts on the left on G/H *via*

$$g_1 \cdot (g_2 H) = g_1 g_2 H, \quad g_1, g_2 \in G.$$

For any $g_1 \in G$, the diffeomorphism $\tau_{g_1} \colon G/H \to G/H$ is left coset multiplication, given by

$$\tau_{g_1}(g_2 H) = g_1 \cdot (g_2 H) = g_1 g_2 H.$$

In this situation, Part (1) of Proposition 20.18 is easily proved as follows.

Proposition 20.19. *For any $X \in \mathfrak{g}$ and any $g \in G$, we have*

$$(\tau_g)_* X^* = (\mathrm{Ad}_g(X))^*.$$

Denote the point in G/H corresponding to the coset $1H = H$ by o. Then, we have a homomorphism

$$\chi^{G/H} \colon H \to \mathbf{GL}(T_o(G/H)),$$

given by

$$\chi^{G/H}(h) = (d\tau_h)_o, \quad \text{for all } \ h \in H.$$

Definition 20.7. The homomorphism $\chi^{G/H}$ is called the *isotropy representation* of the homogeneous space G/H.

Actually, we have an isomorphism

$$T_o(G/H) \cong \mathfrak{g}/\mathfrak{h}$$

induced by $d\pi_1 \colon \mathfrak{g} \to T_o(G/H)$, where $\pi \colon G \to G/H$ is the canonical projection.

The homomorphism $\chi^{G/H}$ is a representation of the group H, and since we can view H as the isotropy group (the stabilizer) of the element $o \in G/H$ corresponding to the coset H, it makes sense to call it the isotropy representation.

It is not easy to deal with the isotropy representation directly.

Fortunately, the isotropy representation is *equivalent* to another representation $\mathrm{Ad}^{G/H} \colon H \to \mathbf{GL}(\mathfrak{g}/\mathfrak{h})$ obtained from the representation $\mathrm{Ad} \colon G \to \mathbf{GL}(\mathfrak{g})$ by a quotient process.

Recall that $\mathbf{Ad}_{g_1}(g_2) = g_1 g_2 g_1^{-1}$. Then, following O'Neill [38] (see Proposition 22, Chapter 11), observe that

$$\tau_h \circ \pi = \pi \circ \mathbf{Ad}_h \quad \text{for all } h \in H,$$

since $h \in H$ implies that $h^{-1}H = H$, so for all $g \in G$,

$$(\tau_h \circ \pi)(g) = hgH = hgh^{-1}H = (\pi \circ \mathbf{Ad}_h)(g).$$

By taking derivatives at 1, we get

$$(d\tau_h)_o \circ d\pi_1 = d\pi_1 \circ \mathrm{Ad}_h,$$

which is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{Ad}_h"] \arrow[d, "d\pi_1"'] & \mathfrak{g} \arrow[d, "d\pi_1"] \\
T_o(G/H) \arrow[r, "(d\tau_h)_o"'] & T_o(G/H).
\end{tikzcd}$$

Proposition 20.20. *Let (G, H) be a pair where G is a Lie group and H is a closed subgroup of G. The following properties hold:*

(1) The representations $\chi^{G/H} \colon H \to \mathbf{GL}(T_o(G/H))$ and $\mathrm{Ad}^{G/H} \colon H \to \mathbf{GL}(\mathfrak{g}/\mathfrak{h})$ are equivalent; this means that for every $h \in H$, we have the commutative diagram

$$\begin{tikzcd} \mathfrak{g}/\mathfrak{h} \arrow[r, "\mathrm{Ad}_h^{G/H}"] \arrow[d, "\varphi"'] & \mathfrak{g}/\mathfrak{h} \arrow[d, "\varphi"] \\ T_o(G/H) \arrow[r, "(d\tau_h)_o"'] & T_o(G/H), \end{tikzcd}$$

where the isomorphism $\varphi \colon \mathfrak{g}/\mathfrak{h} \to T_o(G/H)$ and the quotient map $\mathrm{Ad}_h^{G/H} \colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ are defined in the notes.

(2) The homogeneous space G/H has some G-invariant metric iff the closure of $\mathrm{Ad}^{G/H}(H)$ is compact in $\mathbf{GL}(\mathfrak{g}/\mathfrak{h})$. Furthermore, this metric is unique up to a scalar if the isotropy representation is irreducible.

The representation $\mathrm{Ad}^{G/H}\colon H \to \mathbf{GL}(\mathfrak{g}/\mathfrak{h})$ which involves the quotient algebra $\mathfrak{g}/\mathfrak{h}$ is hard to deal with.

To make things more tractable, it is natural to assume that $\mathfrak{g}$ splits as a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for some *well-behaved* subspace $\mathfrak{m}$ of $\mathfrak{g}$, so that $\mathfrak{g}/\mathfrak{h}$ is isomorphic to $\mathfrak{m}$.

Definition 20.8. Let (G, H) be a pair where G is a Lie group and H is a closed subgroup of G. We say that the homogeneous space G/H is *reductive* if there is some subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

and

$$\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m} \quad \text{for all } h \in H.$$

See Figure 20.1.

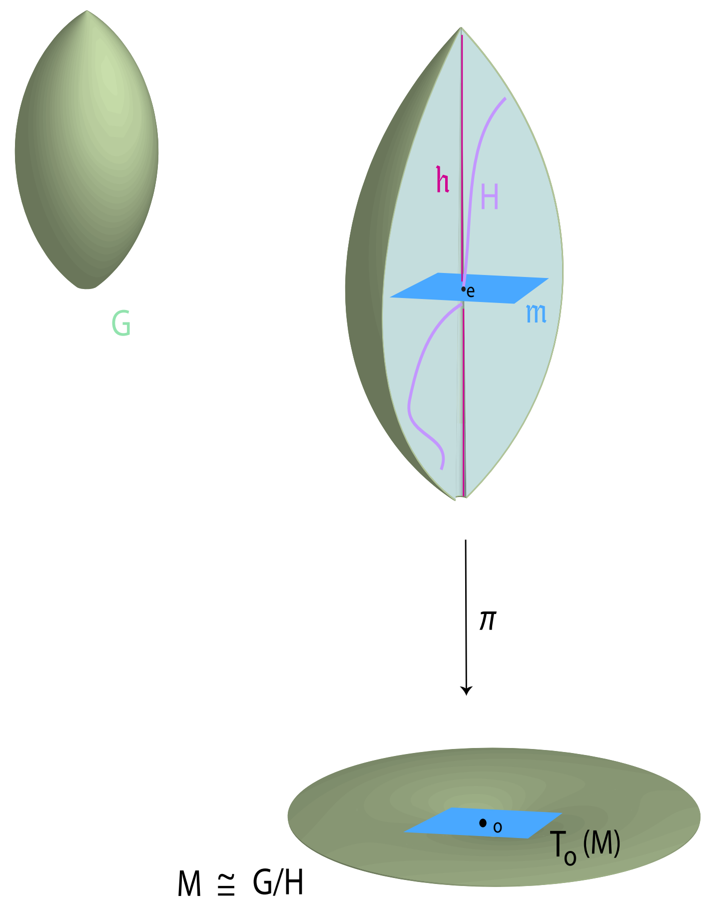

Figure 20.1: A schematic illustration of a reductive homogeneous manifold. Note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and that $T_o(M) \cong \mathfrak{m}$ via $d\pi_1$.

Observe that unlike $\mathfrak{h}$, which is a Lie subalgebra of $\mathfrak{g}$, the subspace $\mathfrak{m}$ is *not necessarily closed* under the Lie bracket, so in general it *is not* a Lie algebra.

Also, since $\mathfrak{m}$ is finite-dimensional and since Ad_h is an isomorphism, we actually have $\mathrm{Ad}_h(\mathfrak{m}) = \mathfrak{m}$.

Definition 20.8 allows us to deal with $\mathfrak{g}/\mathfrak{h}$ in a tractable manner, but does not provide any means of defining a metric on G/H.

We would like to define G-invariant metrics on G/H and a key property of a reductive spaces is that there is a criterion for the existence of G-invariant metrics on G/H in terms of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$.

Since $\mathfrak{g}/\mathfrak{h}$ is isomorphic to $\mathfrak{m}$, by the reasoning just before Proposition 20.20, the map $d\pi_1 \colon \mathfrak{g} \to T_o(G/H)$ restricts to an isomorphism between $\mathfrak{m}$ and $T_o(G/H)$ (where o denotes the point in G/H corresponding to the coset H).

The representation $\mathrm{Ad}^{G/H}\colon H \to \mathbf{GL}(\mathfrak{g}/\mathfrak{h})$ becomes the representation $\mathrm{Ad}\colon H \to \mathbf{GL}(\mathfrak{m})$, where Ad_h is the restriction of Ad_h to $\mathfrak{m}$ for every $h \in H$.

We also know that for any $X \in \mathfrak{g}$, we can express $d\pi_1(X)$ in terms of the vector field X^* introduced in Definition 20.6 by

$$d\pi_1(X) = X^*_o,$$

and that

$$\mathrm{Ker}\, d\pi_1 = \mathfrak{h}.$$

Thus, the *restriction* of $d\pi_1$ to $\mathfrak{m}$ is an isomorphism onto $T_o(G/H)$, given by $X \mapsto X^*_o$.

Also, for every $X \in \mathfrak{g}$, since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we can write $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$, for some unique $X_{\mathfrak{h}} \in \mathfrak{h}$ and some unique $X_{\mathfrak{m}} \in \mathfrak{m}$, and

$$d\pi_1(X) = d\pi_1(X_{\mathfrak{m}}) = X_o^*.$$

We use the isomorphism $d\pi_1$ to transfer any inner product $\langle -, - \rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ to an inner product $\langle -, - \rangle$ on $T_o(G/H)$, and vice-versa, by stating that

$$\langle X, Y \rangle_{\mathfrak{m}} = \langle X_o^*, Y_o^* \rangle, \quad \text{for all } X, Y \in \mathfrak{m};$$

that is, by declaring $d\pi_1$ to be an isometry between $\mathfrak{m}$ and $T_o(G/H)$. See Figure 20.1.

If the metric on G/H is G-invariant, then the map $p \mapsto \exp(tX) \cdot p = \exp(tX)aH$ (with $p = aH \in G/H$, $a \in G$) is an isometry of G/H for every $t \in \mathbb{R}$, so by Proposition 15.9, X^* is a Killing vector field. This fact is needed in Section 20.6.

Proposition 20.21. *Let (G, H) be a pair of Lie groups defining a reductive homogeneous space $M = G/H$, with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The following properties hold:*

(1) The isotropy representation $\chi^{G/H}\colon H \to \mathbf{GL}(T_o(G/H))$ is equivalent to the representation $\mathrm{Ad}^G\colon H \to \mathbf{GL}(\mathfrak{m})$ (where Ad_h is restricted to $\mathfrak{m}$ for every $h \in H$): this means that for every $h \in H$, we have the commutative diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\mathrm{Ad}_h^G"] \arrow[d, "d\pi_1"'] & \mathfrak{m} \arrow[d, "d\pi_1"] \\ T_o(G/H) \arrow[r, "(d\tau_h)_o"'] & T_o(G/H), \end{tikzcd}$$

where $\pi\colon G \to G/H$ is the canonical projection.

(2) By making $d\pi_1$ *an isometry between* $\mathfrak{m}$ *and* $T_o(G/H)$ *(as explained above), there is a one-to-one correspondence between* G*-invariant metrics on* G/H *and* $\mathrm{Ad}(H)$*-invariant inner products on* $\mathfrak{m}$ *(inner products* $\langle -, - \rangle_{\mathfrak{m}}$ *such that*

$$\langle u, v\rangle_{\mathfrak{m}} = \langle \mathrm{Ad}_h(u), \mathrm{Ad}_h(v)\rangle_{\mathfrak{m}}, \quad \text{for all } h \in H \text{ and all } u, v \in \mathfrak{m}).$$

(3) The homogeneous space G/H *has some* G*-invariant metric iff the closure of* $\mathrm{Ad}^G(H)$ *is compact in* $\mathbf{GL}(\mathfrak{m})$*. Furthermore, if the representation* $\mathrm{Ad}^G\colon H \to \mathbf{GL}(\mathfrak{m})$ *is irreducible, then such a metric is unique up to a scalar. In particular, if* H *is compact, then a* G*-invariant metric on* G/H *always exists.*

At this stage we have a mechanism to equip G/H with a Riemannian metric from an inner product $\mathfrak{m}$ which has the special property of being $\mathrm{Ad}(H)$-invariant, but this mechanism does *not* provide a Riemannian metric on G.

The construction of a Riemannian metric on G can be done by extending the $\mathrm{Ad}(H)$-invariant metric on $\mathfrak{m}$ to all of $\mathfrak{g}$, and using the bijective correspondence between left-invariant metrics on a Lie group G, and inner products on its Lie algebra $\mathfrak{g}$ given by Proposition 18.1.

Proposition 20.22. *Let (G, H) be a pair of Lie groups defining a reductive homogeneous space $M = G/H$, with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. If $\mathfrak{m}$ has some $\mathrm{Ad}(H)$-invariant inner product $\langle -, - \rangle_{\mathfrak{m}}$, for any inner product $\langle -, - \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ extending $\langle -, - \rangle_{\mathfrak{m}}$ such that $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal, if we give G the left-invariant metric induced by $\langle -, - \rangle_{\mathfrak{g}}$, then the map $\pi\colon G \to G/H$ is a Riemannian submersion.*

By Proposition 15.8, a Riemannian submersion carries horizontal geodesics to geodesics.

20.5 Examples of Reductive Homogeneous Spaces

We now apply the theory of Propositions 20.21 and 20.22 to construct a family of reductive homogeneous spaces, the Stiefel manifolds $S(k, n)$.

For any $n \geq 1$ and any k with $1 \leq k \leq n$, let $S(k, n)$ be the set of all orthonormal k-frames, where an orthonormal k-frame is a k-tuples of orthonormal vectors $(u_1, \ldots, u_k)$ with $u_i \in \mathbb{R}^n$.

Recall that $\mathbf{SO}(n)$ acts transitively on $S(k, n)$ via the action $\cdot \colon \mathbf{SO}(n) \times S(k, n) \to S(k, n)$

$$R \cdot (u_1, \ldots, u_k) = (Ru_1, \ldots, Ru_k).$$

and that the stabilizer of this action is

$$H = \left\{ \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \;\middle|\; R \in \mathbf{SO}(n-k) \right\}.$$

Theorem 20.12 implies that $S(k,n) \cong G/H$, with $G = \mathbf{SO}(n)$ and $H \cong \mathbf{SO}(n-k)$.

Observe that the points of $G/H \cong S(k,n)$ are the cosets QH, with $Q \in \mathbf{SO}(n)$; that is, the equivalence classes $[Q]$, with the equivalence relation on $\mathbf{SO}(n)$ given by

$$Q_1 \equiv Q_2 \quad \text{iff} \quad Q_2 = Q_1 \widetilde{R}, \text{ for some } \widetilde{R} \in H.$$

If we write $Q = [Y\ Y_{\perp}]$, where Y consists of the first k columns of Q and $Y_{\perp}$ consists of the last $n-k$ columns of Q, it is clear that $[Q]$ is uniquely determined by Y.

In fact, if $P_{n,k}$ denotes the projection matrix consisting of the first k columns of the identity matrix I_n,

$$P_{n,k} = \begin{pmatrix} I_k \\ 0_{n-k,k} \end{pmatrix},$$

for any $Q = [Y\ Y_\perp]$, the unique representative Y of the equivalence class $[Q]$ is given by

$$Y = QP_{n,k}.$$

Furthermore $Y_\perp$ is characterized by the fact that $Q = [Y\ Y_\perp]$ is orthogonal, namely, $YY^\top + Y_\perp Y_\perp^\top = I$.

Define

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(n-k) \right\},$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} T & -A^\top \\ A & 0 \end{pmatrix} \;\middle|\; T \in \mathfrak{so}(k),\ A \in \mathrm{M}_{n-k,k}(\mathbb{R}) \right\}.$$

Clearly $\mathfrak{g} = \mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{m}$.

It is easy to check that $\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m}$.

Therefore Definition 20.8 shows that $S(k, n) \cong G/H$ is a reductive homogeneous manifold with $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$.

Since $H \cong \mathbf{SO}(n-k)$ is compact, Proposition 20.21 guarantees the existence of a G-invariant metric on G/H, which in turn ensures the existence of an $\mathrm{Ad}(H)$-invariant metric on $\mathfrak{m}$.

Theorem 18.26 implies that we may construct such a metric by using the Killing form on $\mathfrak{so}(n)$.

Proposition 20.23. *If $X, Y \in \mathfrak{m}$, with*

$$X = \begin{pmatrix} S & -A^\top \\ A & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} T & -B^\top \\ B & 0 \end{pmatrix},$$

then the fomula

$$\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr}(XY) = \frac{1}{2}\mathrm{tr}(S^\top T) + \mathrm{tr}(A^\top B)$$

defines an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$. If we give $\mathfrak{h}$ the same inner product so that $\mathfrak{g}$ also has the inner product $\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr}(XY)$, then $\mathfrak{m}$ and $\mathfrak{h}$ are orthogonal.

In order to describe the geodesics of $S(k, n) \cong G/H$, we will need the additional requirement of naturally reductiveness which is defined in the next section.

20.6 Naturally Reductive Homogeneous Spaces

When $M = G/H$ is a reductive homogeneous space that has a G-invariant metric, it is possible to give an expression for $(\nabla_{X^*}Y^*)_o$ (where X^* and Y^* are the vector fields corresponding to $X, Y \in \mathfrak{m}$).

If X^*, Y^*, Z^* are the Killing vector fields associated with $X, Y, Z \in \mathfrak{m}$, then it can be shown that

$$\begin{aligned} 2\langle \nabla_{X^*}Y^*, Z^*\rangle = -\langle [X,Y]^*, Z^*\rangle - \langle [X,Z]^*, Y^*\rangle \\ - \langle [Y,Z]^*, X^*\rangle. \end{aligned}$$

The problem is that the vector field $\nabla_{X^*}Y^*$ is not necessarily of the form W^* for some $W \in \mathfrak{g}$. However, we can find its value at o.

By evaluating at o and using the fact that $X^*_o = (X^*_{\mathfrak{m}})_o$ for any $X \in \mathfrak{g}$, we obtain

$$\begin{aligned}2\langle (\nabla_{X^*} Y^*)_o, Z^*_o\rangle &+ \langle ([X, Y]^*_{\mathfrak{m}})_o, Z^*_o\rangle \\ &= \langle ([Z, X]^*_{\mathfrak{m}})_o, Y^*_o\rangle + \langle ([Z, Y]^*_{\mathfrak{m}})_o, X^*_o\rangle.\end{aligned}$$

Consequently,

$$(\nabla_{X^*} Y^*)_o = -\frac{1}{2}([X, Y]^*_{\mathfrak{m}})_o + U(X, Y)^*_o,$$

where $[X, Y]_{\mathfrak{m}}$ is the component of $[X, Y]$ on $\mathfrak{m}$ and $U(X, Y)$ is determined by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{m}}\rangle,$$

for all $Z \in \mathfrak{m}$.

Here, we are using the isomorphism $X \mapsto X_0^*$ between $\mathfrak{m}$ and $T_o(G/H)$ and the fact that the inner product on $\mathfrak{m}$ is chosen so that $\mathfrak{m}$ and $T_o(G/H)$ are isometric

Since the term $U(X, Y)$ clearly complicates matters, it is natural to make the following definition, which is equivalent to requiring that $U(X, Y) = 0$ for all $X, Y \in \mathfrak{m}$.

Definition 20.9. A homogeneous space G/H is *naturally reductive* if it is reductive with some reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, if it has a G-invariant metric, and if

$$\langle [X, Z]_{\mathfrak{m}}, Y \rangle = \langle X, [Z, Y]_{\mathfrak{m}} \rangle, \quad \text{for all } X, Y, Z \in \mathfrak{m}.$$

Note that one of the requirements of Definition 20.9 is that G/H must have a G-invariant metric.

The above computation yield the following result.

Proposition 20.24. *If G/H is naturally reductive, then the Levi-Civita connection associated with the G-invariant metric on G/H is given by*

$$(\nabla_{X^*}Y^*)_o = -\frac{1}{2}([X,Y]^*_{\mathfrak{m}})_o = -\frac{1}{2}[X,Y]_{\mathfrak{m}},$$

for all $X, Y \in \mathfrak{m}$.

We can now find the geodesics on a naturally reductive homogenous space.

Indeed, if $M = (G, H)$ is a reductive homogeneous space and M has a G-invariant metric, then there is an $\mathrm{Ad}(H)$-invariant inner product $\langle -, - \rangle_{\mathfrak{m}}$ on $\mathfrak{m}$.

Pick any inner product $\langle -, - \rangle_{\mathfrak{h}}$ on $\mathfrak{h}$, and define an inner product on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ by setting $\mathfrak{h}$ and $\mathfrak{m}$ to be orthogonal.

Then, we get a left-invariant metric on G for which the elements of $\mathfrak{h}$ are vertical vectors and the elements of $\mathfrak{m}$ are horizontal vectors.

Observe that in this situation, the condition for being naturally reductive extends to left-invariant vector fields on G induced by vectors in $\mathfrak{m}$.

Since $(d\tau_g)_1 \colon \mathfrak{g} \to T_gG$ is a linear isomorphism for all $g \in G$, the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ yields a direct sum decomposition $T_gG = (d\tau_g)_1(\mathfrak{h}) \oplus (d\tau_g)_1(\mathfrak{m})$.

Given a left-invariant vector field X^L induced by a vector $X \in \mathfrak{g}$, if $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$ is the decomposition of X onto $\mathfrak{h} \oplus \mathfrak{m}$, we obtain a decomposition

$$X^L = X^L_{\mathfrak{h}} + X^L_{\mathfrak{m}},$$

into a left-invariant vector field $X^L_{\mathfrak{h}} \in \mathfrak{h}^L$ and a left-invariant vector field $X^L_{\mathfrak{m}} \in \mathfrak{m}^L$, with

$$X^L_{\mathfrak{h}}(g) = (d\tau_g)_1(X_{\mathfrak{h}}), \quad X^L_{\mathfrak{m}} = (d\tau_g)_1(X_{\mathfrak{m}}).$$

Since the $(d\tau_g)_1$ are isometries, if $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal, so are $(d\tau_g)_1(\mathfrak{h})$ and $(d\tau_g)_1(\mathfrak{m})$, and so $X^L_{\mathfrak{h}}$ and $X^L_{\mathfrak{m}}$ are orthogonal vector fields.

Proposition 20.25. *If the condition for being naturally reductive holds, namely*

$$\langle [X, Z]_{\mathfrak{m}}, Y\rangle = \langle X, [Z, Y]_{\mathfrak{m}}\rangle, \quad \textit{for all } X, Y, Z \in \mathfrak{m},$$

then a similar condition holds for left-invariant vector fields:

$$\langle [X^L, Z^L]_{\mathfrak{m}}, Y^L\rangle = \langle X^L, [Z^L, Y^L]_{\mathfrak{m}}\rangle,$$

for all $X^L, Y^L, Z^L \in \mathfrak{m}^L$.

Recall that the left action of G on G/H is given by $g_1 \cdot g_2H = g_1g_2H$, and that o denotes the coset $1H$.

Proposition 20.26. *If $M = G/H$ is a naturally reductive homogeneous space, for every G-invariant metric on G/H, for every $X \in \mathfrak{m}$, the geodesic $\gamma_{d\pi_1(X)}$ through o is given by*

$$\gamma_{d\pi_1(X)}(t) = \pi \circ \exp(tX) = \exp(tX) \cdot o, \quad \textit{for all } t \in \mathbb{R}.$$

Proposition 20.26 shows that the geodesics in G/H are given by the obits of the one-parameter groups $(t \mapsto \exp tX)$ generated by the members of $\mathfrak{m}$.

We can also obtain a formula for the geodesic through every point $p = gH \in G/H$.

Proposition 20.27. *If $M = G/H$ is a naturally reductive homogeneous space, for every $X \in \mathfrak{m}$, the geodesic through $p = gH$ with initial velocity $(\mathrm{Ad}_g X)^*_p = (\tau_g)_* X^*_p$ is given*

$$t \mapsto g \exp(tX) \cdot o.$$

An important corollary of Proposition 20.26 is that naturally reductive homogeneous spaces are complete.

Indeed, the one-parameter group $t \mapsto \exp(tX)$ is defined for all $t \in \mathbb{R}$.

One can also figure out a formula for the sectional curvature (see (O'Neill [38], Chapter 11, Proposition 26).

Under the identification of $\mathfrak{m}$ and $T_o(G/H)$ given by the restriction of $d\pi_1$ to $\mathfrak{m}$, we have

$$\begin{aligned}\langle R(X,Y)X, Y\rangle &= \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle \\ &\quad + \langle [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y\rangle, \text{ for all } X, Y \in \mathfrak{m}.\end{aligned}$$

Conditions on a homogeneous space that ensure that such a space is naturally reductive are obviously of interest. Here is such a condition.

Proposition 20.28. *Let $M = G/H$ be a homogeneous space with G a connected Lie group, assume that $\mathfrak{g}$ admits an $\mathrm{Ad}(G)$-invariant inner product $\langle -, - \rangle$, and let $\mathfrak{m} = \mathfrak{h}^{\perp}$ be the orthogonal complement of $\mathfrak{h}$ with respect to $\langle -, - \rangle$. Then, the following properties hold:*

(1) The space G/H is reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

(2) Under the G-invariant metric induced by $\langle -, - \rangle$, the homogeneous space G/H is naturally reductive.

(3) The sectional curvature is determined by

$$\begin{aligned}\langle R(X,Y)X, Y\rangle = \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle \\ + \langle [X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}}\rangle.\end{aligned}$$

Recall a Lie group G is said to be *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple.

From Theorem 18.25, a Lie algebra $\mathfrak{g}$ is semisimple iff its Killing form B is nondegenerate, and from Theorem 18.26, a connected Lie group G is compact and semisimple iff its Killing form B is negative definite.

By Proposition 18.24, the Killing form is $\mathrm{Ad}(G)$-invariant.

Thus, for any connected compact semisimple Lie group G, for any constant $c > 0$, the bilinear form $-cB$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$.

Then, as a corollary of Proposition 20.28, we obtain the following result.

Proposition 20.29. *Let $M = G/H$ be a homogeneous space such that G is a connected compact semisimple group. Then, under any inner product $\langle -, - \rangle$ on $\mathfrak{g}$ given by $-cB$, where B is the Killing form of $\mathfrak{g}$ and $c > 0$ is any positive real, the space G/H is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{h}^{\perp}$ be the orthogonal complement of $\mathfrak{h}$ with respect to $\langle -, - \rangle$. The sectional curvature is non-negative.*

A homogeneous space as in Proposition 20.29 is called a *normal homogeneous space*.

20.7 Examples of Naturally Reductive Homogeneous Spaces

Since $\mathbf{SO}(n)$ is semisimple and compact for $n \geq 3$, the Stiefel manifolds $S(k,n)$ and the Grassmannian manifolds $G(k,n)$ are examples of homogeneous spaces satisying the assumptions of Proposition 20.29 (with an inner product induced by a scalar factor of the Killing form on $\mathbf{SO}(n)$).

Therefore, Stiefel manifolds $S(k,n)$ and Grassmannian manifolds $G(k,n)$ are naturally reductive homogeneous spaces for $n \geq 3$ (under the reduction $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ induced by the Killing form).

If $n = 2$, then $\mathbf{SO}(2)$ is an abelian group, and thus not semisimple. However, in this case, $G(1,2) = \mathbb{RP}(1) \cong \mathbf{SO}(2)/S(\mathbf{O}(1) \times \mathbf{O}(1)) \cong \mathbf{SO}(2)/\mathbf{O}(1)$, and $S(1,2) = S^1 \cong \mathbf{SO}(2)/\mathbf{SO}(1) \cong \mathbf{SO}(2)$.

These are special cases of symmetric cases discussed in Section 20.9.

In the first case, $H = S(\mathbf{O}(1) \times \mathbf{O}(1))$, and in the second case, $H = \mathbf{SO}(1)$. In both cases,

$$\mathfrak{h} = (0),$$

and we can pick

$$\mathfrak{m} = \mathfrak{so}(2),$$

which is trivially $\mathrm{Ad}(H)$-invariant.

In Section 20.9, we show that the inner product on $\mathfrak{so}(2)$ given by

$$\langle X, Y \rangle = \mathrm{tr}(X^\top Y)$$

is $\mathrm{Ad}(H)$-invariant, and with the induced metric, $\mathbb{RP}(1)$ and $S^1 \cong \mathbf{SO}(2)$ are naturally reductive manifolds.

For $n \geq 3$, we have $S(1, n) = S^{n-1}$ and $S(n-1, n) = \mathbf{SO}(n)$, which are symmetric spaces.

On the other hand, $S(k, n)$ it is not a symmetric space if $2 \leq k \leq n-2$. A justification is given in Section 20.10.

Since the Grassmannian manifolds $G(k, n)$ have more structure (they are symmetric spaces), let us first consider the Stiefel manifolds $S(k, n)$ in more detail.

Readers may find material from Absil, Mahony and Sepulchre [1], especially Chapters 1 and 2, a good complement to our presentation, which uses more advanced concepts (reductive homogeneous spaces).

By Proposition 20.26, the geodesic through o with initial velocity

$$X = \begin{pmatrix} S & -A^\top \\ A & 0 \end{pmatrix}$$

is given by

$$\gamma(t) = \exp\left(t \begin{pmatrix} S & -A^\top \\ A & 0 \end{pmatrix}\right) P_{n,k}.$$

This is not a very explicit formula. It is possible to do better, see Edelman, Arias and Smith [16] for details.

Let us consider the *case where $k = n-1$, which is simpler*.

If $k = n-1$, then $n-k = 1$, so $S(n-1, n) = \mathbf{SO}(n)$, $H \cong \mathbf{SO}(1) = \{1\}$, $\mathfrak{h} = (0)$ and $\mathfrak{m} = \mathfrak{so}(n)$.

The inner product on $\mathfrak{so}(n)$ is given by

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}(XY) = \frac{1}{2}\mathrm{tr}(X^\top Y), \qquad X, Y \in \mathfrak{so}(n).$$

Every matrix $X \in \mathfrak{so}(n)$ is a skew-symmetric matrix, and we know that every such matrix can be written as $X = P^\top D P$, where P is orthogonal and where D is a block diagonal matrix whose blocks are either a 1-dimensional block consisting of a zero, of a 2×2 matrix of the form

$$D_j = \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix},$$

with $\theta_j > 0$.

Then, $e^X = P^\top e^D P = P^\top \Sigma P$, where Σ is a block diagonal matrix whose blocks are either a 1-dimensional block consisting of a 1, of a 2×2 matrix of the form

$$D_j = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}.$$

We also know that every matrix $R \in \mathbf{SO}(n)$ can be written as

$$R = e^X,$$

for some matrix $X \in \mathfrak{so}(n)$ as above, with $0 < \theta_j \leq \pi$.

Then, we can give a formula for the distance $d(I, Q)$ between the identity matrix and any matrix $Q \in \mathrm{SO}(n)$.

Since the geodesics from I through Q are of the fom

$$\gamma(t) = e^{tX} \quad \text{with} \quad e^X = Q,$$

and since the length $L(\gamma)$ of the geodesic from I to e^X is

$$L(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}} dt.$$

We find that

$$d(I, Q) = (\theta_1^2 + \cdots + \theta_m^2)^{\frac{1}{2}},$$

where $\theta_1, \ldots, \theta_m$ are the angles associated with the eigenvalues $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_m}$ of Q distinct from 1, and with $0 < \theta_j \leq \pi$.

If $Q, R \in \mathbf{SO}(n)$, then

$$d(Q, R) = (\theta_1^2 + \cdots + \theta_m^2)^{\frac{1}{2}},$$

where $\theta_1, \ldots, \theta_m$ are the angles associated with the eigenvalues $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_m}$ of $Q^{-1}R = Q^\top R$ distinct from 1, and with $0 < \theta_j \leq \pi$.

Remark: Since $X^\top = -X$, the square distance $d(I, Q)^2$ can also be expressed as

$$d(I, Q)^2 = -\frac{1}{2} \min_{X \mid e^X = Q} \operatorname{tr}(X^2),$$

or even (with some abuse of notation, since log is multivalued) as

$$d(I, Q)^2 = -\frac{1}{2} \min \operatorname{tr}((\log Q)^2).$$

In the *other special case where $k = 1$*, we have $S(1, n) = S^{n-1}$, $H \cong \mathbf{SO}(n-1)$,

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(n-1) \right\},$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -u^\top \\ u & 0 \end{pmatrix} \;\middle|\; u \in \mathbb{R}^{n-1} \right\}.$$

Therefore, there is a one-to-one correspondence between $\mathfrak{m}$ and $\mathbb{R}^{n-1}$.

Given any $Q \in \mathbf{SO}(n)$, the equivalence class $[Q]$ of Q is uniquely determined by the first column of Q, and we view it as a point on S^{n-1}.

If we let $\|u\| = \sqrt{u^\top u}$, we leave it as an exercise to prove that for any

$$X = \begin{pmatrix} 0 & -u^\top \\ u & 0 \end{pmatrix},$$

we have

$$e^{tX} = \begin{pmatrix} \cos(\|u\|\, t) & -\sin(\|u\|\, t)\frac{u^\top}{\|u\|} \\ \sin(\|u\|\, t)\frac{u}{\|u\|} & I + (\cos(\|u\|\, t) - 1)\frac{uu^\top}{\|u\|^2} \end{pmatrix}.$$

Consequently (under the identification of S^{n-1} with the first column of matrices $Q \in \mathbf{SO}(n)$), the geodesic γ through e_1 (the column vector corresponding to the point $o \in S^{n-1}$) with initial tangent vector u is given by

$$\gamma(t) = \begin{pmatrix} \cos(\|u\|\, t) \\ \sin(\|u\|\, t)\frac{u}{\|u\|} \end{pmatrix} = \cos(\|u\|\, t)e_1 + \sin(\|u\|\, t)\frac{u}{\|u\|},$$

where $u \in \mathbb{R}^{n-1}$ is viewed as the vector in $\mathbb{R}^n$ whose first component is 0.

Then, we have

$$\gamma'(t) = \|u\| \left(-\sin(\|u\|\, t)e_1 + \cos(\|u\|\, t)\frac{u}{\|u\|} \right),$$

and we find the that the length $L(\gamma)(\theta)$ of the geodesic from e_1 to the point

$$p(\theta) = \gamma(\theta) = \cos(\|u\|\, \theta)e_1 + \sin(\|u\|\, \theta)\frac{u}{\|u\|}$$

is given by

$$L(\gamma)(\theta) = \int_0^\theta \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}}\, dt = \theta \, \|u\| \, .$$

Since

$$\langle e_1, p(\theta) \rangle = \cos(\theta \, \|u\|),$$

we see that for a unit vector u and for any angle θ such that $0 \leq \theta \leq \pi$, the length of the geodesic from e_1 to $p(\theta)$ can be expressed as

$$L(\gamma)(\theta) = \theta = \arccos(\langle e_1, p \rangle);$$

that is, the angle between the unit vectors e_1 and p. This is a generalization of the distance between two points on a circle.

Geodesics can also be determined in the general case where $2 \leq k \leq n-2$; we follow Edelman, Arias and Smith [16], with one change because some point in that paper requires some justification which is not provided.

Given a point $[Y\, Y_\perp] \in S(k,n)$, and given and any tangent vector $X = YS + Y_\perp A$, we need to compute

$$\gamma(t) = [Y\, Y_\perp] \exp\left(t\begin{pmatrix} S & -A^\top \\ A & 0 \end{pmatrix}\right) P_{n,k}.$$

We can compute this exponential if we replace the matrix by a more "regular matrix," and for this, we use a QR-decomposition of A. Let

$$A = U\begin{pmatrix} R \\ 0 \end{pmatrix}$$

be a QR-decomposition of A, with U an orthogonal $(n-k) \times (n-k)$ matrix and R an upper triangular $k \times k$ matrix.

We can write $U = [U_1\, U_2]$, where U_1 consists of the first k columns on U and U_2 of the last $n-2k$ columns of U (if $2k \leq n$).

We have

$$A = U_1 R,$$

and we can write

$$\begin{pmatrix} S & -A^\top \\ A & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} S & -R^\top \\ R & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & U_1^\top \end{pmatrix}.$$

Then, we find

$$\gamma(t) = [Y\; Y_\perp U_1] \exp t \begin{pmatrix} S & -R^\top \\ R & 0 \end{pmatrix} \begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

This is essentially the formula given in Section 2.4.2 of Edelman, Arias and Smith [16], except for the term $Y_\perp U_1$.

We can easily compute the length $L(\gamma)(s)$ of the geodesic γ from o to $p = e^{sX} \cdot o$, for any $X \in \mathfrak{m}$.

Indeed, for any

$$X = \begin{pmatrix} S & -A^\top \\ A & 0 \end{pmatrix} \in \mathfrak{m},$$

we know that the geodesic from o with initial velocity X is $\gamma(t) = e^{tX} \cdot o$, so we have

$$L(\gamma)(s) = \int_0^s \langle (e^{tX})', (e^{tX})' \rangle^{\frac{1}{2}} dt,$$

but we already did this computation and found that

$$\begin{aligned}(L(\gamma)(s))^2 &= s^2 \left(\frac{1}{2} \mathrm{tr}(X^\top X) \right) \\ &= s^2 \left(\frac{1}{2} \mathrm{tr}(S^\top S) + \mathrm{tr}(A^\top A) \right).\end{aligned}$$

We can compute these traces using the eigenvalues of S and the singular values of A.

If $\pm i\theta_1, \ldots, \pm i\theta_m$ are the nonzero eigenvalues of S and $\sigma_1, \ldots, \sigma_k$ are the singular values of A, then

$$L(\gamma)(s) = s(\theta_1^2 + \cdots + \theta_m^2 + \sigma_1^2 + \cdots + \sigma_k^2)^{\frac{1}{2}}.$$

We conclude this section with a proposition that shows that under certain conditions, G is determined by $\mathfrak{m}$ and H.

A point $p \in M = G/H$ is called a *pole* if the exponential map at p is a diffeomorphism. The following proposition is proved in O'Neill [38] (Chapter 11, Lemma 27).

Proposition 20.30. *If $M = G/H$ is a naturally reductive homogeneous space, then for any pole $o \in M$, there is a diffeomorphism $\mathfrak{m} \times H \cong G$ given by the map $(X, h) \mapsto (\exp(X))h$.*

Next, we will see that there exists a large supply of naturally reductive homogeneous spaces: symmetric spaces.

20.8 A Glimpse at Symmetric Spaces

There is an extensive theory of symmetric spaces and our goal is simply to show that the additional structure afforded by an involutive automorphism of G yields spaces that are naturally reductive.

The theory of symmetric spaces was entirely created by one person, Élie Cartan, who accomplished the tour de force of giving a complete classification of these spaces using the classification of semisimple Lie algebras that he had obtained earlier.

One of the most complete exposition is given in Helgason [21]. O'Neill [38], Petersen [39], Sakai [43] and Jost [24] have nice and more concise presentations. Ziller [48] is also an excellent introduction.

Given a homogeneous space G/K, the new ingredient is that we have an involutive automorphism σ of G.

Definition 20.10. Given a Lie group G, an automorphism σ of G such that $\sigma \neq \mathrm{id}$ and $\sigma^2 = \mathrm{id}$ called an *involutive automorphism* of G. Let G^σ be the set of fixed points of σ, the subgroup of G given by

$$G^\sigma = \{g \in G \mid \sigma(g) = g\},$$

and let G_0^σ be the identity component of G^σ (the connected component of G^σ containing 1).

If we have an involutive automorphism $\sigma\colon G \to G$, then we can consider the $+1$ and -1 eigenspaces of $d\sigma_1\colon \mathfrak{g} \to \mathfrak{g}$, given by

$$\begin{aligned}\mathfrak{k} &= \{X \in \mathfrak{g} \mid d\sigma_1(X) = X\}\\ \mathfrak{m} &= \{X \in \mathfrak{g} \mid d\sigma_1(X) = -X\}.\end{aligned}$$

Definition 20.11. An involutive automorphism of G satisfying $G_0^\sigma \subseteq K \subseteq G^\sigma$ is called a *Cartan involution*. The map $d\sigma_1$ is often denoted by θ.

The following proposition will be needed later.

Proposition 20.31. *Let σ be an involutive automorphism of G and let $\mathfrak{k}$ and $\mathfrak{m}$ be the $+1$ and -1 eigenspaces of $d\sigma_1\colon \mathfrak{g} \to \mathfrak{g}$. Then for all $X \in \mathfrak{m}$ and all $Y \in \mathfrak{k}$, we have*

$$B(X, Y) = 0,$$

where B is the Killing form of $\mathfrak{g}$.

Remarkably, $\mathfrak{k}$ and $\mathfrak{m}$ yield a reductive decomposition of G/K.

Proposition 20.32. *Given a homogeneous space G/K with a Cartan involution σ ($G_0^\sigma \subseteq K \subseteq G^\sigma$), if $\mathfrak{k}$ and $\mathfrak{m}$ are defined as above, then*

(1) $\mathfrak{k}$ is indeed the Lie algebra of K.

(2) We have a direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

(3) $\mathrm{Ad}(K)(\mathfrak{m}) \subseteq \mathfrak{m}$; in particular, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$.

(4) We have

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \quad \textit{and} \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}.$$

In particular, the pair (G, K) is a reductive homogeneous space (as in Definition 20.8), with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

If we also assume that G is connected and that G_0^σ is compact, then we obtain the following remarkable result proved in O'Neill [38] (Chapter 11) and Ziller [48] (Chapter 6).

Theorem 20.33. *Let G be a connected Lie group and let $\sigma\colon G \to G$ be an automorphism such that $\sigma^2 = \mathrm{id}$, $\sigma \neq \mathrm{id}$ (an involutive automorphism), and G_0^σ is compact. For every compact subgroup K of G, if $G_0^\sigma \subseteq K \subseteq G^\sigma$, then G/K has G-invariant metrics, and for every such metric, G/K is naturally reductive. The reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is given by the $+1$ and -1 eigenspaces of $d\sigma_1$. Furthermore, for every $p \in G/K$, there is an isometry $s_p\colon G/K \to G/K$ such that $s_p(p) = p$, $d(s_p)_p = -\mathrm{id}$, and*

$$s_p \circ \pi = \pi \circ \sigma,$$

as illustrated in the diagram below:

$$\begin{tikzcd}
G \arrow[r, "\sigma"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\
G/K \arrow[r, "s_p"'] & G/K.
\end{tikzcd}$$

Definition 20.12. A triple (G, K, σ) satisfying the assumptions of Theorem 20.33 is called a *symmetric pair*.[2]

A triple (G, K, σ) as above defines a special kind of naturally homogeneous space G/K known as a symmetric space.

Definition 20.13. If M is a connected Riemannian manifold, for any $p \in M$, an isometry s_p such that $s_p(p) = p$ and $d(s_p)_p = -\mathrm{id}$ is a called a *global symmetry at* p. A connected Riemannian manifold M for which there is a global symmetry for every point of M is called a *symmetric space*.

Theorem 20.33 implies that the naturally reductive homogeneous space G/K defined by a symmetric pair (G, K, σ) is a symmetric space.

[2]Once again we fall victims of tradition. A symmetric pair is actually a triple!

It can be shown that a global symmetry s_p reverses geodesics at p and that $s_p^2 = \mathrm{id}$, so s_p is an involution.

It should be noted that although $s_p \in \mathrm{Isom}(M)$, the isometry s_p does not necessarily lie in $\mathrm{Isom}(M)_0$.

The following facts are proved in O'Neill [38] (Chapters 9 and 11), Ziller [48] (Chapter 6), and Sakai [43] (Chapter IV).

1. Every symmetric space M is complete, and $\mathrm{Isom}(M)$ acts transitively on M. In fact the identity component $\mathrm{Isom}(M)_0$ acts transitively on M.

2. Thus, every symmetric space M is a homogeneous space of the form $\mathrm{Isom}(M)_0/K$, where K is the isotropy group of any chosen point $p \in M$ (it turns out that K is compact).

3. The symmetry s_p gives rise to a Cartan involution σ of $G = \mathrm{Isom}(M)_0$ defined so that

$$\sigma(g) = s_p \circ g \circ s_p \quad g \in G.$$

Then we have

$$G_0^\sigma \subseteq K \subseteq G^\sigma.$$

4. Thus, every symmetric space M is presented by a symmetric pair $(\mathrm{Isom}(M)_0, K, \sigma)$.

However, beware that in the presentation of the symmetric space $M = G/K$ given by a symmetric pair (G, K, σ), the group G is not necessarily equal to $\mathrm{Isom}(M)_0$.

Thus, we do not have a one-to-one correspondence between symmetric spaces and homogeneous spaces with a Cartan involution.

From our point of view, this does not matter since we are more interested in getting symmetric spaces from the data (G, K, σ).

By abuse of terminology (and notation), we refer to the homogeneous space G/K defined by a symmetric pair (G, K, σ) as the *symmetric space* (G, K, σ).

Remark: The reader may have noticed that in this section on symmetric spaces, the closed subgroup of G is denoted by K rather than H. This is in accordance with the convention that G/K usually refers to a symmetric space rather than just a homogeneous space.

Since the homogeneous space G/H defined by a symmetric pair (G, K, σ) is naturally reductive and has a G-invariant metric, by Proposition 20.26, its geodesics coincide with the one-parameter groups (they are given by the Lie group exponential).

The Levi-Civita connection on a symmetric space depends only on the Lie bracket on $\mathfrak{g}$. Indeed, we have the following formula proved in Ziller [48] (Chapter 6).

Proposition 20.34. *Given any symmetric space M defined by the triple (G, K, σ), for any $X \in \mathfrak{m}$ and and vector field Y on $M \cong G/K$, we have*

$$(\nabla_{X^*}Y)_o = [X^*, Y]_o.$$

Another nice property of symmetric space is that the curvature formulae are quite simple.

If we use the isomorphism between $\mathfrak{m}$ and $T_0(G/K)$ induced by the restriction of $d\pi_1$ to $\mathfrak{m}$, then for all $X, Y, Z \in \mathfrak{m}$ we have:

1. The curvature at o is given by
$$R(X, Y)Z = [[X, Y]_{\mathfrak{h}}, Z],$$
or more precisely by
$$R(d\pi_1(X), d\pi_1(Y))d\pi_1(Z) = d\pi_1([[X, Y]_{\mathfrak{h}}, Z]).$$
In terms of the vector fields X^*, Y^*, Z^*, we have
$$R(X^*, Y^*)Z^* = [[X, Y], Z]^* = [[X^*, Y^*], Z^*].$$

2. The sectional curvature $K(X^*, Y^*)$ at o is determined by
$$\langle R(X^*, Y^*)X^*, Y^*\rangle = \langle [[X, Y]_{\mathfrak{h}}, X], Y\rangle.$$

3. The Ricci curvature at o is given by

$$\mathrm{Ric}(X^*, X^*) = -\frac{1}{2}B(X, X),$$

where B is the Killing form associated with $\mathfrak{g}$.

Proof of the above formulae can be found in O'Neill [38] (Chapter 11), Ziller [48] (Chapter 6), Sakai [43] (Chapter IV) and Helgason [21] (Chapter IV, Section 4).

However, beware that Ziller, Sakai and Helgason use the opposite of the sign convention that we are using for the curvature tensor (which is the convention used by O'Neill [38], Gallot, Hulin, Lafontaine [19], Milnor [33], and Arvanitoyeorgos [2]).

Recall that we define the Riemann tensor by

$$R(X,Y) = \nabla_{[X,Y]} + \nabla_Y \circ \nabla_X - \nabla_X \circ \nabla_Y,$$

whereas Ziller, Sakai and Helgason use

$$R(X,Y) = -\nabla_{[X,Y]} - \nabla_Y \circ \nabla_X + \nabla_X \circ \nabla_Y.$$

With our convention, the sectional curvature $K(x,y)$ is determined by $\langle R(x,y)x, y\rangle$, and the Ricci curvature $\mathrm{Ric}(x,y)$ as the trace of the map $v \mapsto R(x,v)y$.

With the opposite sign convention, the sectional curvature $K(x, y)$ is determined by $\langle R(x, y)y, x\rangle$, and the Ricci curvature $\mathrm{Ric}(x, y)$ as the trace of the map $v \mapsto R(v, x)y$.

Therefore, the sectional curvature and the Ricci curvature are identical under both conventions (as they should!).

In Ziller, Sakai and Helgason, the curvature formula is

$$R(X^*, Y^*)Z^* = -[[X, Y], Z]^*.$$

We are now going to see that basically all of the familiar spaces are symmetric spaces.

20.9 Examples of Symmetric Spaces

Recall that the set of all k-dimensional spaces of $\mathbb{R}^n$ is a homogeneous space $G(k, n)$, called a *Grassmannian*, and that $G(k, n) \cong \mathbf{SO}(n)/S(\mathbf{O}(k) \times \mathbf{O}(n-k))$.

We can also consider the set of k-dimensional *oriented subspaces* of $\mathbb{R}^n$.

An oriented k-subspace is a k-dimensional subspace W together with the choice of a basis $(u_1, \ldots, u_k)$ determining the orientation of W.

Another basis $(v_1, \ldots, v_k)$ of W is *positively oriented* if $\det(f) > 0$, where f is the unique linear map f such that $f(u_i) = v_i$, $i = 1, \ldots, k$.

The set of of k-dimensional oriented subspaces of $\mathbb{R}^n$ is denoted by $G^0(k, n)$.

The group $\mathbf{SO}(n)$ acts transitively on $G^0(k,n)$, and using a reasoning similar to the one used in the case where $\mathbf{SO}(n)$ acts on $G(k,n)$, we find that the stabilizer of the oriented subspace $(e_1,\ldots,e_k)$ is the set of orthogonal matrices of the form

$$\begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix},$$

where $Q \in \mathbf{SO}(k)$ and $R \in \mathbf{SO}(n-k)$, because this time, Q has to preserve the orientation of the subspace spanned by $(e_1,\ldots,e_k)$.

Thus, the isotropy group is isomorphic to

$$\mathbf{SO}(k) \times \mathbf{SO}(n-k).$$

It follows that

$$G^0(k, n) \cong \mathbf{SO}(n)/\mathbf{SO}(k) \times \mathbf{SO}(n-k).$$

Let us see how both $G^0(k, n)$ and $G(k, n)$ are presented as symmetric spaces.

Again, readers may find material from Absil, Mahony and Sepulchre [1], especially Chapters 1 and 2, a good complement to our presentation, which uses more advanced concepts (symmetric spaces).

1. Grassmannians as Symmetric Spaces

Let $G = \mathbf{SO}(n)$ (with $n \geq 2$), let

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$-identity matrix, and let σ be given by

$$\sigma(P) = I_{k,n-k} P I_{k,n-k}, \quad P \in \mathbf{SO}(n).$$

It is clear that σ is an involutive automorphism of G.

The set $F = G^\sigma$ of fixed points of σ is given by

$$F = G^\sigma = S(\mathbf{O}(k) \times \mathbf{O}(n-k)),$$

and

$$G_0^\sigma = \mathbf{SO}(k) \times \mathbf{SO}(n-k).$$

Therefore, there are two choices for K:

1. $K = \mathbf{SO}(k) \times \mathbf{SO}(n-k)$, in which case we get the Grassmannian $G^0(k,n)$ of oriented k-subspaces.
2. $K = S(\mathbf{O}(k) \times \mathbf{O}(n-k))$, in which case we get the Grassmannian $G(k,n)$ of k-subspaces.

As in the case of Stiefel manifolds, given any $Q \in \mathbf{SO}(n)$, the first k columns Y of Q constitute a representative of the equivalence class $[Q]$, but these representatives are not unique; there is a further equivalence relation given by

$$Y_1 \equiv Y_2 \quad \text{iff} \quad Y_2 = Y_1 R \quad \text{for some } \; R \in \mathbf{O}(k).$$

Nevertheless, it is useful to consider the first k columns of Q, given by $QP_{n,k}$, as representative of $[Q] \in G(k,n)$.

Because σ is a linear map, its derivative $d\sigma$ is equal to σ, and since $\mathfrak{so}(n)$ consists of all skew-symmetric $n \times n$ matrices, the $+1$-eigenspace is given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \,\middle|\, S \in \mathfrak{so}(k),\ T \in \mathfrak{so}(n-k) \right\},$$

and the -1-eigenspace by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix} \,\middle|\, A \in \mathrm{M}_{n-k,k}(\mathbb{R}) \right\}.$$

Thus, $\mathfrak{m}$ is isomorphic to $M_{n-k,k}(\mathbb{R}) \cong \mathbb{R}^{(n-k)k}$.

It is also easy to show that the isotropy representation is given by

$$\mathrm{Ad}((Q,R))A = QAR^\top,$$

where (Q, R) represents an element of $S(\mathbf{O}(k) \times \mathbf{O}(n-k))$, and A represents an element of $\mathfrak{m}$.

It can be shown that this representation is irreducible iff $(k, n) \neq (2, 4)$.

It can also be shown that if $n \geq 3$, then $G^0(k, n)$ is simply connected, $\pi_1(G(k, n)) = \mathbb{Z}_2$, and $G^0(k, n)$ is a double cover of $G(n, k)$.

An $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ is given by

$$\begin{aligned}\left\langle \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix}, \begin{pmatrix} 0 & -B^\top \\ B & 0 \end{pmatrix} \right\rangle \\ = -\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & -B^\top \\ B & 0 \end{pmatrix} \right) = \mathrm{tr}(A^\top B).\end{aligned}$$

We also give $\mathfrak{g}$ the same inner product. Then, we immediately check that $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal.

In the special case where $k = 1$, we have $G^0(1, n) = S^{n-1}$ and $G(1, n) = \mathbb{RP}^{n-1}$, and then the $\mathbf{SO}(n)$-invariant metric on S^{n-1} (resp. $\mathbb{RP}^{n-1}$) is the canonical one.

Skip upon first reading ⊛

For any point $[Q] \in G(k, n)$ with $Q \in \mathbf{SO}(n)$, if we write $Q = [Y\ \ Y_\perp]$, where Y denotes the first k columns of Q and $Y_\perp$ denotes the last $n - k$ columns of Q, the tangent vectors $X \in T_{[Q]}G(k, n)$ are of the form

$$X = [Y\ \ Y_\perp] \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix} = [Y_\perp A\ \ -YA^\top], \quad A \in M_{n-k,k}(\mathbb{R}).$$

Consequently, there is a one-to-one correspondence between matrices X as above and $n \times k$ matrices of the form $X' = Y_\perp A$, for any matrix $A \in M_{n-k,k}(\mathbb{R})$.

As noted in Edelman, Arias and Smith [16], because the spaces spanned by Y and $Y_\perp$ form an orthogonal direct sum in $\mathbb{R}^n$, there is a one-to-one correspondence between $n \times k$ matrices of the form $Y_\perp A$ for any matrix $A \in M_{n-k,k}(\mathbb{R})$, and matrices $X' \in M_{n,k}(\mathbb{R})$ such that

$$Y^\top X' = 0.$$

This second description of tangent vectors to $G(k, n)$ at $[Y]$ is sometimes more convenient.

The tangent vectors $X' \in M_{n,k}(\mathbb{R})$ to the Stiefel manifold $S(k, n)$ at Y satisfy the weaker condition that $Y^\top X'$ is *skew-symmetric*.

End $\circledast$

Given any $X \in \mathfrak{m}$ of the form

$$X = \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix},$$

the geodesic starting at o is given by

$$\gamma(t) = \exp(tX) \cdot o.$$

Thus, we need to compute

$$\exp(tX) = \exp \begin{pmatrix} 0 & -tA^\top \\ tA & 0 \end{pmatrix}.$$

This can be done using SVD.

Since $G(k, n)$ and $G(n-k, n)$ are isomorphic, without loss of generality, assume that $2k \leq n$. Then, let

$$A = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^\top$$

be an SVD for A, with U a $(n-k) \times (n-k)$ orthogonal matrix, Σ a $k \times k$ matrix, and V a $k \times k$ orthogonal matrix.

Since we assumed that $k \leq n-k$, we can write

$$U = [U_1 \, U_2],$$

with U_1 is a $(n-k) \times k$ matrix and U_2 an $(n-k) \times (n-2k)$ matrix.

We find that

$$\begin{aligned}\exp(tX) &= \exp t \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix} \\ &= \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \cos t\Sigma & -\sin t\Sigma & 0 \\ \sin t\Sigma & \cos t\Sigma & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} V^\top & 0 \\ 0 & U^\top \end{pmatrix}.\end{aligned}$$

Now, $\exp(tX)P_{n,k}$ is certainly a representative of the equivalence class of $[\exp(tX)]$, so as a $n \times k$ matrix, the geodesic through o with initial velocity

$$X = \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix}$$

(with A any $(n-k) \times k$ matrix with $n - k \geq k$) is given by

$$\gamma(t) = \begin{pmatrix} V & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} \cos t\Sigma \\ \sin t\Sigma \end{pmatrix} V^\top,$$

where $A = U_1 \Sigma V^\top$, a compact SVD of A.

Because symmetric spaces are geodesically complete, we get an interesting corollary. Indeed, every equivalence class $[Q] \in G(k,n)$ possesses some representative of the form e^X for some $X \in \mathfrak{m}$, so we conclude that for every orthogonal matrix $Q \in \mathbf{SO}(n)$, there exist some orthogonal matrices $V, \widetilde{V} \in \mathbf{O}(k)$ and $U, \widetilde{U} \in \mathbf{O}(n-k)$, and some diagonal matrix Σ with nonnegatives entries, so that

$$Q = \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \cos\Sigma & -\sin\Sigma & 0 \\ \sin\Sigma & \cos\Sigma & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} (\widetilde{V})^\top & 0 \\ 0 & (\widetilde{U})^\top \end{pmatrix}.$$

This is an instance of the CS-decomposition; see Golub and Van Loan [20].

The matrices $\cos\Sigma$ and $\sin\Sigma$ are actually diagonal matrices of the form

$$\cos\Sigma = \mathrm{diag}(\cos\theta_1, \ldots, \cos\theta_k) \qquad \text{and} \quad \sin\Sigma = \mathrm{diag}(\sin\theta_1, \ldots, \sin\theta_k),$$

so we may assume that $0 \leq \theta_i \leq \pi/2$, because if $\cos\theta_i$ or $\sin\theta_i$ is negative, we can change the sign of the ith row of V (resp. the sign of the i-th row of U) and still obtain orthogonal matrices U' and V' that do the job.

Now, it is known that $(\theta_1, \ldots, \theta_k)$ are the *principal angles* (or *Jordan angles*) between the subspaces spanned the first k columns of I_n and the subspace spanned by the columns of Y (see Golub and van Loan [20]).

Recall that given two k-dimensional subspaces $\mathcal{U}$ and $\mathcal{V}$ determined by two $n \times k$ matrices Y_1 and Y_2 of rank k, the principal angles $\theta_1, \ldots, \theta_k$ between $\mathcal{U}$ and $\mathcal{V}$ are defined recursively as follows: Let $\mathcal{U}_1 = \mathcal{U}$, $\mathcal{V}_1 = \mathcal{V}$, let

$$\cos\theta_1 = \max_{\substack{u \in \mathcal{U}, v \in \mathcal{V} \\ \|u\|_2 = 1, \|v\|_2 = 1}} \langle u, v \rangle,$$

let $u_1 \in \mathcal{U}$ and $v_1 \in \mathcal{V}$ be any two unit vectors such that $\cos\theta_1 = \langle u_1, v_1 \rangle$, and for $i = 2, \ldots, k$, if
$\mathcal{U}_i = \mathcal{U}_{i-1} \cap \{u_{i-1}\}^{\perp}$ and $\mathcal{V}_i = \mathcal{V}_{i-1} \cap \{v_{i-1}\}^{\perp}$, let

$$\cos\theta_i = \max_{\substack{u \in \mathcal{U}_i, v \in \mathcal{V}_i \\ \|u\|_2 = 1, \|v\|_2 = 1}} \langle u, v \rangle,$$

and let $u_i \in \mathcal{U}_i$ and $v_i \in \mathcal{V}_i$ be any two unit vectors such that $\cos\theta_i = \langle u_i, v_i \rangle$.

The vectors u_i and v_i are not unique, but it is shown in Golub and van Loan [20] that $(\cos\theta_1, \ldots, \cos\theta_k)$ are the singular values of $Y_1^\top Y_2$ (with $0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k \leq \pi/2$).

We can also determine the length $L(\gamma)(s)$ of the geodesic $\gamma(t)$ from o to $p = e^{sX}$, for any $X \in \mathfrak{m}$, with

$$X = \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix}.$$

The computation from Section 20.4 remains valid and we obtain

$$(L(\gamma)(s))^2 = s^2\left(\frac{1}{2}\mathrm{tr}(X^\top X)\right) = s^2\mathrm{tr}(A^\top A).$$

Then, if $\theta_1, \ldots, \theta_k$ are the singular values of A, we get

$$L(\gamma)(s) = s(\theta_1^2 + \cdots + \theta_k^2)^{\frac{1}{2}}.$$

In view of the above discussion regarding principal angles, we conclude that if Y_1 consists of the first k columns of an orthogonal matrix Q_1 and if Y_2 consists of the first k columns of an orthogonal matrix Q_2 then the distance between the subspaces $[Q_1]$ and $[Q_2]$ is given by

$$d([Q_1], [Q_2]) = (\theta_1^2 + \cdots + \theta_k^2)^{\frac{1}{2}},$$

where $(\cos\theta_1, \ldots, \cos\theta_k)$ are the singular values of $Y_1^\top Y_2$ (with $0 \leq \theta_i \leq \pi/2$); the angles $(\theta_1, \ldots, \theta_k)$ are the principal angles between the spaces $[Q_1]$ and $[Q_2]$.

In Golub and van Loan, a different distance between subspaces is defined, namely

$$d_{p2}([Q_1], [Q_2]) = \left\| Y_1 Y_1^\top - Y_2 Y_2^\top \right\|_2.$$

If we write $\Theta = \mathrm{diag}(\theta_1 \ldots, \theta_k)$, then it is shown that

$$d_{p2}([Q_1], [Q_2]) = \|\sin \Theta\|_\infty = \max_{1 \leq i \leq k} \sin \theta_i.$$

This metric is derived by embedding the Grassmannian in the set of $n \times n$ projection matrices of rank k, and then using the 2-norm.

Other metrics are proposed in Edelman, Arias and Smith [16].

We leave it to the brave readers to compute $\langle [[X, Y], X], Y \rangle$, where

$$X = \begin{pmatrix} 0 & -A^\top \\ A & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -B^\top \\ B & 0 \end{pmatrix},$$

and check that

$$\begin{aligned} \langle [[X, Y], X], Y \rangle = \langle BA^\top - AB^\top, BA^\top - AB^\top \rangle \\ + \langle A^\top B - B^\top A, A^\top B - B^\top A \rangle, \end{aligned}$$

which shows that the sectional curvature is nonnnegative.

When $k = 1$, which corresponds to $\mathbb{RP}^{n-1}$ (or S^{n-1}), we get a metric of constant positive curvature.

2. Symmetric Positive Definite Matrices

Recall that the space $\mathbf{SPD}(n)$ of symmetric positive definite matrices ($n \geq 2$) appears as the homogeneous space $\mathbf{GL}^+(n,\mathbb{R})/\mathbf{SO}(n)$, under the action of $\mathbf{GL}^+(n,\mathbb{R})$ on $\mathbf{SPD}(n)$ given by

$$A \cdot S = ASA^\top.$$

Write $G = \mathbf{GL}^+(n,\mathbb{R})$, $K = \mathbf{SO}(n)$, and choose the Cartan involution σ given by

$$\sigma(S) = (S^\top)^{-1}.$$

It is immediately verified that

$$G^\sigma = \mathbf{SO}(n).$$

Then, we have $\mathfrak{gl}^+(n) = \mathfrak{gl}(n) = \mathrm{M}_n(\mathbb{R})$, $\mathfrak{k} = \mathfrak{so}(n)$, and $\mathfrak{m} = \mathbf{S}(n)$, the vector space of symmetric matrices. We define an $\mathrm{Ad}(\mathbf{SO}(n))$-invariant inner product on $\mathfrak{gl}^+(n)$ by

$$\langle X, Y\rangle = \mathrm{tr}(X^\top Y).$$

If $X \in \mathfrak{m}$ and $Y \in \mathfrak{k} = \mathfrak{so}(n)$, we find that $\langle X, Y\rangle = 0$. Thus, we have

$$\langle X, Y\rangle = \begin{cases} -\mathrm{tr}(XY) & \text{if } X, Y \in \mathfrak{k} \\ \mathrm{tr}(XY) & \text{if } X, Y \in \mathfrak{m} \\ 0 & \text{if } X \in \mathfrak{m},\ Y \in \mathfrak{k}. \end{cases}$$

We leave it as an exercise (see Petersen [39], Chapter 8, Section 2.5) to show that

$$\langle [[X, Y], X], Y\rangle = -\mathrm{tr}([X, Y]^\top [X, Y]), \text{for all } X, Y \in \mathfrak{m}.$$

This shows that the sectional curvature is nonpositive. It can also be shown that the isotropy representation is given by

$$\chi_A(X) = AXA^{-1} = AXA^{\top},$$

for all $A \in \mathbf{SO}(n)$ and all $X \in \mathfrak{m}$.

Recall that the exponential $\exp\colon \mathbf{S}(n) \to \mathbf{SDP}(n)$ is a bijection.

Then, given any $S \in \mathbf{SPD}(n)$, there is a unique $X \in \mathfrak{m}$ such that $S = e^X$, and the unique geodesic from I to S is given by

$$\gamma(t) = e^{tX}.$$

Let us try to find the length $L(\gamma) = d(I, S)$ of this geodesic.

As in Section 20.4, we have

$$L(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}} dt,$$

but this time, $X \in \mathfrak{m}$ is symmetric and the geodesic is unique, so we have

$$L(\gamma) = \int_0^1 \langle (e^{tX})', (e^{tX})'\rangle^{\frac{1}{2}} dt = \int_0^1 (\mathrm{tr}(X^2 e^{2tX}))^{\frac{1}{2}} dt.$$

Since X is a symmetric matrix, we can write

$$X = P^\top \Lambda P,$$

with P orthogonal and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, a real diagonal matrix, and we have

$$\begin{aligned}
\mathrm{tr}(X^2 e^{2tX}) &= \mathrm{tr}(P^\top \Lambda^2 P P^\top e^{2t\Lambda} P) \\
&= \mathrm{tr}(\Lambda^2 e^{2t\Lambda}) \\
&= \lambda_1^2 e^{2t\lambda_1} + \cdots + \lambda_n^2 e^{2t\lambda_n}.
\end{aligned}$$

Therefore,

$$d(I, S) = L(\gamma) = \int_0^1 ((\lambda_1 e^{t\lambda_1})^2 + \cdots + (\lambda_n e^{t\lambda_n})^2)^{\frac{1}{2}} dt.$$

Actually, since $S = e^X$ and S is SPD, $\lambda_1, \ldots, \lambda_n$ are the logarithms of the eigenvalues $\sigma_1, \ldots, \sigma_n$ of X, so we have

$$\begin{aligned} d(I, S) = L(\gamma) = \int_0^1 ((\log \sigma_1 e^{t \log \sigma_1})^2 + \cdots \\ + (\log \sigma_n e^{t \log \sigma_n})^2)^{\frac{1}{2}} dt. \end{aligned}$$

Unfortunately, there doesn't appear to be a closed form formula for this integral.

The symmetric space $\mathbf{SPD}(n)$ contains an interesting submanifold, namely the space of matrices S in $\mathbf{SPD}(n)$ such that $\det(S) = 1$.

This the symmetric space $\mathbf{SL}(n, \mathbb{R})/\mathbf{SO}(n)$, which we suggest denoting by $\mathbf{SSPD}(n)$. For this space, $\mathfrak{g} = \mathfrak{sl}(n)$, and the reductive decomposition is given by

$$\mathfrak{k} = \mathfrak{so}(n), \quad \mathfrak{m} = \mathbf{S}(n) \cap \mathfrak{sl}(n).$$

Now, recall that the Killing form on $\mathfrak{gl}(n)$ is given by

$$B(X, Y) = 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y).$$

On $\mathfrak{sl}(n)$, the Killing form is $B(X, Y) = 2n\mathrm{tr}(XY)$, and it is proportional to the inner product

$$\langle X, Y \rangle = \mathrm{tr}(XY).$$

Therefore, we see that the restriction of the Killing form of $\mathfrak{sl}(n)$ to $\mathfrak{m} = \mathbf{S}(n) \cap \mathfrak{sl}(n)$ is positive definite, whereas it is negative definite on $\mathfrak{k} = \mathfrak{so}(n)$.

The symmetric space $\mathbf{SSPD}(n) \cong \mathbf{SL}(n, \mathbb{R})/\mathbf{SO}(n)$ is an example of a symmetric space of noncompact type.

On the other hand, the Grassmannians are examples of symmetric spaces of compact type (for $n \geq 3$). In the next section, we take a quick look at these special types of symmetric spaces.

3. The Hyperbolic Space $\mathcal{H}_n^+(1)$ $\circledast$

In Section 6.1 we defined the Lorentz group $\mathbf{SO}_0(n,1)$ as follows: if

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

then a matrix $A \in \mathrm{M}_{n+1}(\mathbb{R})$ belongs to $\mathbf{SO}_0(n,1)$ iff

$$A^\top J A = J, \quad \det(A) = +1, \quad a_{n+1 n+1} > 0.$$

In that same section we also defined the hyperbolic space $\mathcal{H}_n^+(1)$ as the sheet of $\mathcal{H}_n(1)$ which contains $(0,\ldots,0,1)$ where

$$\mathcal{H}_n(1) = \{u = (\mathbf{u}, t) \in \mathbb{R}^{n+1} \mid \|\mathbf{u}\|^2 - t^2 = -1\}.$$

We also showed that the action
$\cdot\colon \mathbf{SO}_0(n,1) \times \mathcal{H}_n^+(1) \longrightarrow \mathcal{H}_n^+(1)$ with

$$A \cdot u = Au$$

is a transitive with stabilizer $\mathbf{SO}(n)$ (see Proposition 6.8).

Thus, $\mathcal{H}_n^+(1)$ arises as the homogeneous space $\mathbf{SO}_0(n,1)/\mathbf{SO}(n)$.

Since the inverse of $A \in \mathbf{SO}_0(n,1)$ is $JA^\top J$, the map $\sigma\colon \mathbf{SO}_0(n,1) \to \mathbf{SO}_0(n,1)$ given by

$$\sigma(A) = JAJ = (A^\top)^{-1}$$

is an involutive automorphism of $\mathbf{SO}_0(n,1)$. Write $G = \mathbf{SO}_0(n,1)$, $K = \mathbf{SO}(n)$.

It is immediately verified that

$$G^\sigma = \left\{ \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \mid Q \in \mathbf{SO}(n) \right\},$$

so $G^\sigma \cong \mathbf{SO}(n)$. We have

$$\mathfrak{so}(n,1) = \left\{ \begin{pmatrix} B & u \\ u^\top & 0 \end{pmatrix} \mid B \in \mathfrak{so}(n), u \in \mathbb{R}^n \right\},$$

and the derivative $\theta\colon \mathfrak{so}(n,1) \to \mathfrak{so}(n,1)$ of σ at I is given by

$$\theta(X) = JXJ = -X^\top.$$

From this we deduce that the $+1$-eigenspace is given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \mid B \in \mathfrak{so}(n) \right\},$$

and the -1-eigenspace is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix} \mid u \in \mathbb{R}^n \right\},$$

with

$$\mathfrak{so}(n,1) = \mathfrak{k} \oplus \mathfrak{m},$$

a reductive decomposition.

We define an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{so}(n,1)$ by

$$\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}(X^\top Y).$$

In fact, on $\mathfrak{m} \cong \mathbb{R}^n$, we have

$$\begin{aligned}\left\langle \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ v^\top & 0 \end{pmatrix} \right\rangle &= \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix} \begin{pmatrix} 0 & v \\ v^\top & 0 \end{pmatrix} \right) \\ &= \frac{1}{2}\mathrm{tr}(uv^\top + u^\top v) = u^\top v,\end{aligned}$$

the Euclidean product of u and v.

As an exercise, the reader should compute $\langle [[X, Y], X], Y \rangle$, where

$$X = \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & v \\ v^\top & 0 \end{pmatrix},$$

and check that

$$\langle [[X, Y], X], Y \rangle = -\langle uv^\top - vu^\top, uv^\top - vu^\top \rangle,$$

which shows that the sectional curvature is nonpositive. In fact, $\mathcal{H}_n^+(1)$ has constant negative sectional curvature.

We leave it as an exercise to prove that for $n \geq 2$, the Killing form B on $\mathfrak{so}(n, 1)$ is given by

$$B(X, Y) = (n - 1)\mathrm{tr}(XY),$$

for all $X, Y \in \mathfrak{so}(n, 1)$.

If we write

$$X = \begin{pmatrix} B_1 & u \\ u^\top & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} B_2 & v \\ v^\top & 0 \end{pmatrix},$$

then

$$B(X, Y) = (n-1)\mathrm{tr}(B_1 B_2) + 2(n-1)u^\top v.$$

This shows that B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{m}$.

This means that the space $\mathcal{H}_n^+(1)$ is a symmetric space of noncompact type.

The symmetric space $\mathcal{H}_n^+(1) = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$ turns out to be dual, as a symmetric space, to
$S^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$.

4. The Hyperbolic Grassmannian $G^*(q, p+q)$ ⊛

This is the generalization of the hyperbolic space $\mathcal{H}_n^+(1)$ in Example (3).

Recall from Section 6.1 that we define $I_{p,q}$, for $p, q \geq 1$, by

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

If $n = p + q$, the matrix $I_{p,q}$ is associated with the nondegenerate symmetric bilinear form

$$\varphi_{p,q}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{n} x_j y_j$$

with associated quadratic form

$$\Phi_{p,q}((x_1, \ldots, x_n)) = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{n} x_j^2.$$

The group $\mathbf{SO}(p,q)$ is the set of all $n \times n$-matrices (with $n = p+q$)

$$\mathbf{SO}(p,q) = \{A \in \mathbf{GL}(n,\mathbb{R}) \mid A^\top I_{p,q} A = I_{p,q}, \det(A) = 1\}.$$

If we write

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \qquad P \in M_p(\mathbb{R}),\ \ Q \in M_q(\mathbb{R})$$

then it is shown in O'Neill [38] (Chapter 9, Lemma 6) that the connected component $\mathbf{SO}_0(p,q)$ of $\mathbf{SO}(p,q)$ containing I is given by

$$\begin{aligned}\mathbf{SO}_0(p,q) = \{A \in \mathbf{GL}(n,\mathbb{R}) \mid A^\top I_{p,q} A = I_{p,q}, \\ \det(P) > 0,\ \det(S) > 0\}.\end{aligned}$$

For both $\mathbf{SO}(p,q)$ and $\mathbf{SO}_0(p,q)$, the inverse is given by

$$A^{-1} = I_{p,q} A^\top I_{p,q}.$$

This implies that the map $\sigma \colon \mathbf{SO}_0(p,q) \to \mathbf{SO}_0(p,q)$ given by

$$\sigma(A) = I_{p,q} A I_{p,q} = (A^\top)^{-1}$$

is an involution, and its fixed subgroup G^σ is given by

$$G^\sigma = \left\{ \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \mid Q \in \mathbf{SO}(p), R \in \mathbf{SO}(q) \right\}.$$

Thus G^σ is isomorphic to $\mathbf{SO}(p) \times \mathbf{SO}(q)$.

For $p, q \geq 1$, the Lie algebra $\mathfrak{so}(p,q)$ of $\mathbf{SO}_0(p,q)$ (and $\mathbf{SO}(p,q)$ as well) is given by

$$\mathfrak{so}(p,q) = \left\{ \begin{pmatrix} B & A \\ A^\top & C \end{pmatrix} \mid B \in \mathfrak{so}(p), C \in \mathfrak{so}(q), \right.$$
$$\left. A \in \mathrm{M}_{p,q}(\mathbb{R}) \right\}.$$

Since $\theta = d\sigma_I$ is also given by $\theta(X) = I_{p,q} X I_{p,q} = -X^\top$, we find that the $+1$-eigenspace $\mathfrak{k}$ of θ is given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \mid B \in \mathfrak{so}(p), C \in \mathfrak{so}(q) \right\},$$

and the -1-eigenspace $\mathfrak{m}$ of θ is is given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} \mid A \in \mathrm{M}_{p,q}(\mathbb{R}) \right\}.$$

Note that $\mathfrak{k}$ is a subalgebra of $\mathfrak{so}(p,q)$ and $\mathfrak{so}(p,q) = \mathfrak{k} \oplus \mathfrak{m}$.

Write $G = \mathbf{SO}_o(p,q)$ and $K = \mathbf{SO}(p) \times \mathbf{SO}(q)$.

We define an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{so}(p,q)$ by

$$\langle X, Y \rangle = \frac{1}{2}\mathrm{tr}(X^\top Y).$$

Therefore, for $p, q \geq 1$, the coset space $\mathbf{SO}_0(p,q)/(\mathbf{SO}(p) \times \mathbf{SO}(q))$ is a symmetric space.

Observe that on $\mathfrak{m}$, the above inner product is given by

$$\langle X, Y \rangle = \frac{1}{2}\mathrm{tr}(XY).$$

On the other hand, in the case of $\mathbf{SO}(p+q)/(\mathbf{SO}(p) \times \mathbf{SO}(q))$, on $\mathfrak{m}$, the inner product is given by

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}(XY).$$

This space can be described explicitly.

Indeed, let $G^*(q, p+q)$ be the set of q-dimensional subspaces W of $R^n = R^{p+q}$ such that $\Phi_{p,q}$ is negative definite on W.

Then we have an obvious matrix multiplication action of $\mathbf{SO}_0(p, q)$ on $G^*(q, p+q)$, and it is easy to check that this action is transitive.

It is not hard to show that the stabilizer of the subspace spanned by the last q columns of the $(p+q) \times (p+q)$ identity matrix is $\mathbf{SO}(p) \times \mathbf{SO}(q)$, so the space $G^*(q, p+q)$ is isomorphic to the homogeneous (symmetric) space $\mathbf{SO}_0(p, q)/(\mathbf{SO}(p) \times \mathbf{SO}(q))$.

Definition 20.14. The symmetric space $G^*(q, p+q) \cong \mathbf{SO}_0(p, q)/(\mathbf{SO}(p) \times \mathbf{SO}(q))$ is called the *hyperbolic Grassmannian*.

Assume that $p + q \geq 3, p, q \geq 1$. Then it can be shown that the Killing form on $\mathfrak{so}(p, q)$ is given by

$$B(X, Y) = (p + q - 2)\mathrm{tr}(XY),$$

so $\mathfrak{so}(p, q)$ is semisimple.

If we write

$$X = \begin{pmatrix} B_1 & A_1 \\ A_1^\top & C_1 \end{pmatrix}, \quad Y = \begin{pmatrix} B_2 & A_2 \\ A_2^\top & C_2 \end{pmatrix},$$

then

$$\begin{aligned} B(X,Y) = (p+q-2)(\mathrm{tr}(B_1B_2) + \mathrm{tr}(C_1C_2)) \\ + 2(p+q-2)A_1^\top A_2. \end{aligned}$$

Consequently, B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{m}$, so $G^*(q, p+q) = \mathbf{SO}_0(p,q)/(\mathbf{SO}(p)\times\mathbf{SO}(q))$ is another example of a symmetric space of noncompact type.

We leave it to the reader to compute $\langle [[X, Y], X], Y\rangle$, where

$$X = \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix},$$

and check that

$$\begin{aligned} \langle [[X, Y], X], Y\rangle = &-\langle BA^\top - AB^\top, BA^\top - AB^\top\rangle \\ &- \langle A^\top B - B^\top A, A^\top B - B^\top A\rangle, \end{aligned}$$

which shows that the sectional curvature is nonpositive.

In fact, the above expression is the negative of the expression that we found for the sectional curvature of $G^0(p, p+q)$.

When $p = 1$ or $q = 1$, we get a space of constant negative curvature.

The above property is one of the consequences of the fact that the space $G^*(q, p+q) = \mathbf{SO}_0(p, q)/(\mathbf{SO}(p)\times\mathbf{SO}(q))$ is the symmetric space dual to $G^0(p, p+q) = \mathbf{SO}(p+q)/(\mathbf{SO}(p) \times \mathbf{SO}(q))$, the Grassmannian of oriented p-planes;

see O'Neill [38] (Chapter 11, Definition 37) or Helgason [21] (Chapter V, Section 2).

5. Compact Lie Groups

If H be a compact Lie group, then $G = H \times H$ acts on H *via*

$$(h_1, h_2) \cdot h = h_1 h h_2^{-1}.$$

The stabilizer of $(1, 1)$ is clearly
$K = \Delta H = \{(h, h) \mid h \in H\}$.

It is easy to see that the map

$$(g_1, g_2)K \mapsto g_1 g_2^{-1}$$

is a diffeomorphism between the coset space G/K and H (see Helgason [21], Chapter IV, Section 6).

A Cartan involution σ is given by

$$\sigma(h_1, h_2) = (h_2, h_1),$$

and obviously $G^\sigma = K = \Delta H$.

Therefore, H appears as the symmetric space G/K, with $G = H \times H$, $K = \Delta H$, and

$$\mathfrak{k} = \{(X, X) \mid X \in \mathfrak{h}\}, \quad \mathfrak{m} = \{(X, -X) \mid X \in \mathfrak{h}\}.$$

For every $(h_1, h_2) \in \mathfrak{g}$, we have

$$(h_1, h_2) = \left(\frac{h_1 + h_2}{2}, \frac{h_1 + h_2}{2}\right) + \left(\frac{h_1 - h_2}{2}, -\frac{h_1 - h_2}{2}\right)$$

which gives the direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

The natural projection $\pi\colon H \times H \to H$ is given by

$$\pi(h_1, h_2) = h_1 h_2^{-1},$$

which yields $d\pi_{(1,1)}(X, Y) = X - Y$ (see Helgason [21], Chapter IV, Section 6).

It follows that the natural isomorphism $\mathfrak{m} \to \mathfrak{h}$ is given by

$$(X, -X) \mapsto 2X.$$

Given any bi-invariant metric $\langle -, - \rangle$ on H, define a metric on $\mathfrak{m}$ by

$$\langle (X, -X), (Y, -Y) \rangle = 4\langle X, Y \rangle.$$

The reader should check that the resulting symmetric space is isometric to H (see Sakai [43], Chapter IV, Exercise 4).

More examples of symmetric spaces are presented in Ziller [48] and Helgason [21].

To close our brief tour of symmetric spaces, we conclude with a short discussion about the type of symmetric spaces.

20.10 Types of Symmetric Spaces

Suppose (G, K, σ) (G connected and K compact) presents a symmetric space with Cartan involution σ, and with

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where $\mathfrak{k}$ (the Lie algebra of K) is the eigenspace of $d\sigma_1$ associated with the eigenvalue $+1$ and $\mathfrak{m}$ is is the eigenspace associated with the eigenvalue -1.

If B is the Killing form of $\mathfrak{g}$, it turns out that the restriction of B to $\mathfrak{k}$ is always negative semidefinite.

This will be shown as the first part of the proof of Proposition 20.36.

However, to guarantee that B is negative definite (that is, $B(Z, Z) = 0$ implies that $Z = 0$) some additional condition is needed.

This condition has to do with the subgroup N of G defined by

$$\begin{aligned} N &= \{g \in G \mid \tau_g = \mathrm{id}\} \\ &= \{g \in G \mid gaK = aK \text{ for all } a \in G\}. \end{aligned}$$

By setting $a = e$, we see that $N \subseteq K$.

Furthermore, since $n \in N$ implies $na^{-1}bK = a^{-1}bK$ for all $a, b \in G$, we can readily show that N is a normal subgroup of both K and G.

It is not hard to show that N is the largest normal subgroup that K and G have in common (see Ziller [48] (Chapter 6, Section 6.2).

We can also describe the subgroup N in a more explicit fashion. We have

$$\begin{aligned} N &= \{g \in G \mid gaK = aK \text{ for all } a \in G\} \\ &= \{g \in G \mid a^{-1}gaK = K \text{ for all } a \in G\} \\ &= \{g \in G \mid a^{-1}ga \in K \text{ for all } a \in G\}. \end{aligned}$$

Definition 20.15. For any Lie group G and any closed subgroup K of G, the subgroup N of G given by

$$N = \{g \in G \mid a^{-1}ga \in K \text{ for all } a \in G\}$$

is called the *ineffective kernel* of the left action of G on G/K.

The left action of G on G/K is said to be *effective* (or *faithful*) if $N = \{1\}$, *almost effective* if N is a discrete subgroup.

If K is compact, which will be assumed from now on, since a discrete subgroup of a compact group is finite, *the action of G on G/K is almost effective if N is finite*.

For example, the action $\cdot\colon \mathbf{SU}(n+1) \times \mathbb{CP}^n \to \mathbb{CP}^n$ of $\mathbf{SU}(n+1)$ on the (complex) projective space $\mathbb{CP}^n$ discussed in Example (e) of Section 5.3 is almost effective but not effective.

It presents $\mathbb{CP}^n$ as the homogeneous manifold

$$\mathbf{SU}(n+1)/S(\mathbf{U}(1) \times \mathbf{U}(n)) \cong \mathbb{CP}^n.$$

We leave it as an exercise to the reader to prove that the ineffective kernel of the above action is the finite group

$$N = \{\lambda I_{n+1} \mid \lambda^{n+1} = 1,\ \lambda \in \mathbb{C}\}.$$

It turns out that the additional requirement needed for the Killing form to be negative definite is that the action of G on G/K is almost effective.

The following technical proposition gives a criterion for the left action of G on G/K to be almost effective in terms of the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. This is Proposition 6.27 from Ziller [48].

Proposition 20.35. *The left action of G on G/K (with K compact) is almost effective iff $\mathfrak{g}$ and $\mathfrak{k}$ have no nontrivial ideal in common.*

Proposition 20.36. *Let (G, K, σ) be a symmetric space (K compact) with Cartan involution σ, and assume that the left action of G on G/K is almost effective. If B is the Killing form of $\mathfrak{g}$ and $\mathfrak{k} \neq (0)$, then the restriction of B to $\mathfrak{k}$ is negative definite.*

In view of Proposition 20.36, it is natural to classify symmetric spaces depending on the behavior of B on $\mathfrak{m}$.

Definition 20.16. Let $M = (G, K, \sigma)$ be a symmetric space (K compact) with Cartan involution σ and Killing form B. The space M is said to be of

(1) *Euclidean type* if $B = 0$ on $\mathfrak{m}$.

(2) *Compact type* if B is negative definite on $\mathfrak{m}$.

(3) *Noncompact type* if B is positive definite on $\mathfrak{m}$.

Proposition 20.37. *Let $M = (G, K, \sigma)$ be a symmetric space (K compact) with Cartan involution σ and Killing form B. The following properties hold:*

(1) M is of Euclidean type iff $[\mathfrak{m}, \mathfrak{m}] = (0)$. In this case, M has zero sectional curvature.

(2) If M is of compact type, then $\mathfrak{g}$ is semisimple and both G and M are compact.

(3) If M is of noncompact type, then $\mathfrak{g}$ is semisimple and both G and M are non-compact.

Symmetric spaces of Euclidean type are not that interesting, since they have zero sectional curvature.

The Grassmannians $G(k,n)$ and $G^0(k,n)$ are symmetric spaces of compact type, and $\mathbf{SL}(n,\mathbb{R})/\mathbf{SO}(n)$, $\mathcal{H}_n^+(1) = \mathbf{SO}_0(n,1)/\mathbf{SO}(n)$, and the hyperbolic Grassmannian $G^*(q,p+q) = \mathbf{SO}_0(p,q)/(\mathbf{SO}(p)\times\mathbf{SO}(q))$ are of noncompact type.

Since $\mathbf{GL}^+(n,\mathbb{R})$ is not semisimple, $\mathbf{SPD}(n) \cong \mathbf{GL}^+(n,\mathbb{R})/\mathbf{SO}(n)$ is not a symmetric space of noncompact type, but it has many similar properties.

For example, it has nonpositive sectional curvature and because it is diffeomorphic to $\mathbf{S}(n) \cong \mathbb{R}^{n(n-1)/2}$, it is simply connected.

Here is a quick summary of the main properties of symmetric spaces of compact and noncompact types. Proofs can be found in O'Neill [38] (Chapter 11) and Ziller [48] (Chapter 6).

Proposition 20.38. *Let $M = (G, K, \sigma)$ be a symmetric space (K compact) with Cartan involution σ and Killing form B. The following properties hold:*

(1) If M is of compact type, then M has nonnegative sectional curvature and positive Ricci curvature. The fundamental group $\pi_1(M)$ of M is a finite abelian group.

(2) If M is of noncompact type, then M is simply connected, and M has nonpositive sectional curvature and negative Ricci curvature. Furthermore, M is diffeomorphic to $\mathbb{R}^n$ (with $n = \dim(M)$) and G is diffeomorphic to $K \times \mathbb{R}^n$.

There is also an interesting duality between symmetric spaces of compact type and noncompact type, but we will not discuss it here.

We conclude this section by explaining what the Stiefel manifolds $S(k,n)$ are not symmetric spaces for $2 \leq k \leq n-2$.

This has to do with the nature of the involutions of $\mathfrak{so}(n)$. Recall that the matrices $I_{p,q}$ and J_n are defined by

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

with $2 \leq p+q$ and $n \geq 1$. Observe that $I_{p,q}^2 = I_{p+q}$ and $J_n^2 = -I_{2n}$.

It is shown in Helgason [21] (Chapter X, Section 2 and Section 5) that, up to conjugation, the only involutive automorphisms of $\mathfrak{so}(n)$ are given by

1. $\theta(X) = I_{p,q} X I_{p,q}$, in which case the eigenspace $\mathfrak{k}$ of θ associated with the eigenvalue $+1$ is
$$\mathfrak{k}_1 = \left\{ \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(k),\ T \in \mathfrak{so}(n-k) \right\}.$$

2. $\theta(X) = -J_n X J_n$, in which case the eigenspace $\mathfrak{k}$ of θ associated with the eigenvalue $+1$ is
$$\mathfrak{k}_2 = \left\{ \begin{pmatrix} S & -T \\ T & S \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(n),\ T \in \mathbf{S}(n) \right\}.$$

However, in the case of the Stiefel manifold $S(k,n)$, the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{so}(n)$ associated with $\mathbf{SO}(n-k)$ is

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(n-k) \right\},$$

and if $2 \leq k \leq n-2$, then $\mathfrak{k} \neq \mathfrak{k}_1$ and $\mathfrak{k} \neq \mathfrak{k}_2$.

Therefore, the Stiefel manifold $S(k,n)$ is not a symmetric space if $2 \leq k \leq n-2$.

This also has to do with the fact that in this case, $\mathbf{SO}(n-k)$ is not a maximal subgroup of $\mathbf{SO}(n)$.

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