Chapter 14

Curvature in Riemannian Manifolds

14.1 The Curvature Tensor

Since the notion of curvature can be defined for curves and surfaces, it is natural to wonder whether it can be generalized to manifolds of dimension $n \geq 3$.

Such a generalization does exist and was first proposed by Riemann.

However, Riemann's seminal paper published in 1868 two years after his death only introduced the sectional curvature, and did not contain any proofs or any general methods for computing the sectional curvature.

Fifty years or so later, the idea emerged that the *curvature* of a Riemannian manifold M should be viewed as a measure $R(X,Y)Z$ of the extent to which the operator $(X,Y) \mapsto \nabla_X \nabla_Y Z$ is symmetric, where ∇ is a connection on M (where X, Y, Z are vector fields, with Z fixed).

It turns out that the operator $R(X,Y)Z$ is $C^\infty(M)$-linear in all of its three arguments, so for all $p \in M$, it defines a trilinear map

$$R_p \colon T_pM \times T_pM \times T_pM \longrightarrow T_pM.$$

The curvature operator R is a rather complicated object, so it is natural to seek a simpler object.

Fortunately, there is a simpler object, namely the *sectional curvature* $K(u, v)$, which arises from R through the formula

$$K(u, v) = \langle R(u, v)u, v\rangle,$$

for linearly independent unit vectors u, v.

When ∇ is the Levi-Civita connection induced by a Riemannian metric on M, it turns out that the curvature operator R can be recovered from the sectional curvature.

Another important notion of curvature is the *Ricci curvature*, $\mathrm{Ric}(x, y)$, which arises as the trace of the linear map $v \mapsto R(x, v)y$.

As we said above, if M is a Riemannian manifold and if ∇ is a connection on M, the Riemannian curvature $R(X, Y)Z$ measures the extent to which the operator $(X, Y) \mapsto \nabla_X \nabla_Y Z$ is symmetric (for any fixed Z).

If $(M, \langle -, - \rangle)$ is a Riemannian manifold of dimension n, and if the connection ∇ on M is the *flat connection*, which means that

$$\nabla_X \left(\frac{\partial}{\partial x_i} \right) = 0, \quad i = 1, \ldots, n,$$

for every chart (U, φ) and all $X \in \mathfrak{X}(U)$, since every vector field Y on U can be written uniquely as

$$Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}$$

for some smooth functions Y_i on U, for every other vector field X on U, because the connection is flat and by the Leibniz property of connections, we have

$$\nabla_X \left(Y_i \frac{\partial}{\partial x_i} \right) = X(Y_i) \frac{\partial}{\partial x_i} + Y_i \nabla_X \left(\frac{\partial}{\partial x_i} \right) = X(Y_i) \frac{\partial}{\partial x_i}.$$

Then it is easy to check that the above implies that

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Consequently, it is natural to define the deviation of a connection from the flat connection by the quantity

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition 14.1. Let (M, g) be a Riemannian manifold, and let ∇ be any connection on M. The formula

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for $X, Y, Z \in \mathfrak{X}(M)$, defines a function

$$R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

called the *Riemannian curvature* of M.

Proposition 14.1. *Let M be a manifold with any connection ∇. The function*

$$R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is $C^\infty(M)$-linear in X, Y, Z, and skew-symmetric in X and Y. As a consequence, for any $p \in M$, $(R(X,Y)Z)_p$ depends only on $X(p), Y(p), Z(p)$.

It follows that R defines for every $p \in M$ a trilinear map

$$R_p \colon T_pM \times T_pM \times T_pM \longrightarrow T_pM.$$

Experience shows that it is useful to consider the family of quadrilinear forms (unfortunately!) also denoted R, given by

$$R_p(x, y, z, w) = \langle R_p(x, y)z, w\rangle_p,$$

as well as the expression $R(x, y, y, x)$, which, for an orthonormal pair of vectors (x, y), is known as the *sectional curvature* $K(x, y)$.

This last expression brings up a dilemma regarding the choice for the sign of R.

With our present choice, the sectional curvature $K(x, y)$ is given by $K(x, y) = R(x, y, y, x)$, but many authors define K as $K(x, y) = R(x, y, x, y)$.

Since $R(x, y)$ is skew-symmetric in x, y, the latter choice corresponds to using $-R(x, y)$ instead of $R(x, y)$, that is, to define $R(X, Y)Z$ by

$$R(X, Y)Z = \nabla_{[X,Y]}Z + \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z.$$

As pointed out by Milnor [33] (Chapter II, Section 9), the latter choice for the sign of R has the advantage that, in coordinates, the quantity $\langle R(\partial/\partial x_h, \partial/\partial x_i)\partial/\partial x_j, \partial/\partial x_k\rangle$ coincides with the classical Ricci notation, R_{hijk}.

Gallot, Hulin and Lafontaine [19] (Chapter 3, Section A.1) give other reasons supporting this choice of sign.

Clearly, the choice for the sign of R is mostly a matter of taste and we apologize to those readers who prefer the first choice but we will adopt the second choice advocated by Milnor and others.

Therefore, we make the following formal definition:

Definition 14.2. Let $(M, \langle -, - \rangle)$ be a Riemannian manifold equipped with the Levi-Civita connection. The *curvature tensor* is the family of trilinear functions $R_p\colon T_pM \times T_pM \times T_pM \to T_pM$ defined by

$$R_p(x, y)z = \nabla_{[X,Y]}Z + \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z,$$

for every $p \in M$ and for any vector fields $X, Y, Z \in \mathfrak{X}(M)$ such that $x = X(p)$, $y = Y(p)$, and $z = Z(p)$.

The family of quadrilinear forms associated with R, also denoted R, is given by

$$R_p(x, y, z, w) = \langle (R_p(x, y)z, w\rangle,$$

for all $p \in M$ and all $x, y, z, w \in T_pM$.

Locally in a chart, we write

$$R\left(\frac{\partial}{\partial x_h}, \frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_j} = \sum_l R^l_{jhi}\frac{\partial}{\partial x_l}$$

and

$$R_{hijk} = \left\langle R\left(\frac{\partial}{\partial x_h}, \frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle = \sum_l g_{lk}R^l_{jhi}.$$

The coefficients R^l_{jhi} can be expressed in terms of the Christoffel symbols Γ^k_{ij}, by a rather unfriendly formula (see Gallot, Hulin and Lafontaine [19] (Chapter 3, Section 3.A.3) or O'Neill [38] (Chapter III, Lemma 38).

Since we have adopted O'Neill's conventions for the order of the subscripts in R^l_{jhi}, here is the formula from O'Neill:

$$R^l_{jhi} = \partial_i \Gamma^l_{hj} - \partial_h \Gamma^l_{ij} + \sum_m \Gamma^l_{im}\Gamma^m_{hj} - \sum_m \Gamma^l_{hm}\Gamma^m_{ij}.$$

There is another way of defining the curvature tensor which is useful for comparing second covariant derivatives of one-forms.

For any fixed vector field Z, the map $Y \mapsto \nabla_Y Z$ from $\mathfrak{X}(M)$ to $\mathcal{X}(M)$ is a $C^\infty(M)$-linear map that we will denote $\nabla_{-} Z$ (this is a $(1, 1)$ tensor).

The covariant derivative $\nabla_X \nabla_{-} Z$ of $\nabla_{-} Z$ is defined by

$$(\nabla_X(\nabla_{-}Z))(Y) = \nabla_X(\nabla_Y Z) - (\nabla_{\nabla_X Y})Z.$$

Usually, $(\nabla_X(\nabla_{-}Z))(Y)$ is denoted by $\nabla^2_{X,Y} Z$, and

$$\nabla^2_{X,Y} Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z$$

is called the *second covariant derivative* of Z with respect to X and Y.

Proposition 14.2. *Given a Riemanniain manifold (M, g), if ∇ is the Levi-Civita connection induced by g, then the curvature tensor is given by*

$$R(X, Y)Z = \nabla^2_{Y,X} Z - \nabla^2_{X,Y} Z.$$

We already know that the curvature tensor has some symmetry properties, for example $R(y, x)z = -R(x, y)z$, but when it is induced by the Levi-Civita connection, it has more remarkable properties stated in the next proposition.

Proposition 14.3. *For a Riemannian manifold $(M, \langle -, - \rangle)$ equipped with the Levi-Civita connection, the curvature tensor satisfies the following properties:*

(1) $R(x, y)z = -R(y, x)z$

(2) (First Bianchi Identity)
$$R(x, y)z + R(y, z)x + R(z, x)y = 0$$

(3) $R(x, y, z, w) = -R(x, y, w, z)$

(4) $R(x, y, z, w) = R(z, w, x, y)$.

The next proposition will be needed in the proof of the second variation formula. Recall the notion of a vector field along a surface given in Definition 13.9.

Proposition 14.4. *For a Riemannian manifold* $(M, \langle -, - \rangle)$ *equipped with the Levi-Civita connection, for every parametrized surface* $\alpha\colon \mathbb{R}^2 \to M$*, for every vector field* $V \in \mathfrak{X}(M)$ *along* α*, we have*

$$\frac{D}{\partial y}\frac{D}{\partial x}V - \frac{D}{\partial x}\frac{D}{\partial y}V = R\left(\frac{\partial\alpha}{\partial x}, \frac{\partial\alpha}{\partial y}\right)V.$$

Remark: Since the Levi-Civita connection is torsion-free, it is easy to check that

$$\frac{D}{\partial x}\frac{\partial\alpha}{\partial y} = \frac{D}{\partial y}\frac{\partial\alpha}{\partial x}.$$

The curvature tensor is a rather complicated object.

Thus, it is quite natural to seek simpler notions of curvature.

The sectional curvature is indeed a simpler object, and it turns out that the curvature tensor can be recovered from it.

14.2 Sectional Curvature

Basically, the sectional curvature is the curvature of two-dimensional sections of our manifold.

Given any two vectors $u, v \in T_pM$, recall by Cauchy-Schwarz that

$$\langle u, v\rangle_p^2 \leq \langle u, u\rangle_p \langle v, v\rangle_p,$$

with equality iff u and v are linearly dependent.

Consequently, if u and v are linearly independent, we have

$$\langle u, u\rangle_p \langle v, v\rangle_p - \langle u, v\rangle_p^2 \neq 0.$$

In this case, we claim that the ratio

$$K(u, v) = \frac{R_p(u, v, u, v)}{\langle u, u\rangle_p \langle v, v\rangle_p - \langle u, v\rangle_p^2}$$

is independent of the plane Π spanned by u and v.

Definition 14.3. Let $(M, \langle -, - \rangle)$ be any Riemannian manifold equipped with the Levi-Civita connection. For every $p \in T_pM$, for every 2-plane $\Pi \subseteq T_pM$, the *sectional curvature* $K(\Pi)$ of Π is given by

$$K(\Pi) = K(x, y) = \frac{R_p(x, y, x, y)}{\langle x, x \rangle_p \langle y, y \rangle_p - \langle x, y \rangle_p^2},$$

for any basis (x, y) of Π.

Observe that if (x, y) is an orthonormal basis, then the denominator is equal to 1.

The expression $R_p(x, y, x, y)$ is often denoted $\kappa_p(x, y)$.

Remarkably, κ_p determines R_p. We denote the function $p \mapsto \kappa_p$ by κ.

Proposition 14.5. *Let $(M, \langle -, - \rangle)$ be any Riemannian manifold equipped with the Levi-Civita connection. The function κ determines the curvature tensor R. Thus, the knowledge of all the sectional curvatures determines the curvature tensor. Moreover, we have*

$$
\begin{aligned}
6\langle R(x,y)z, w\rangle \;=\; & \kappa(x+w, y+z) - \kappa(x, y+z) \\
& -\kappa(w, y+z) - \kappa(y+w, x+z) \\
& +\kappa(y, x+z) + \kappa(w, x+z) \\
& - \kappa(x+w, y) + \kappa(x, y) + \kappa(w, y) \\
& - \kappa(x+w, z) + \kappa(x, z) + \kappa(w, z) \\
& + \kappa(y+w, x) - \kappa(y, x) - \kappa(w, x) \\
& + \kappa(y+w, z) - \kappa(y, z) - \kappa(w, z).
\end{aligned}
$$

For a proof of this formidable equation, see Kuhnel [27] (Chapter 6, Theorem 6.5).

A different proof of the above proposition (without an explicit formula) is also given in O'Neill [38] (Chapter III, Corollary 42).

Let

$$R_1(x, y)z = \langle x, z\rangle y - \langle y, z\rangle x.$$

Observe that

$$\langle R_1(x, y)x, y\rangle = \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2.$$

As a corollary of Proposition 14.5, we get:

Proposition 14.6. *Let $(M, \langle -, -\rangle)$ be any Riemannian manifold equipped with the Levi-Civita connection. If the sectional curvature $K(\Pi)$ does not depend on the plane Π but only on $p \in M$, in the sense that K is a scalar function $K\colon M \to \mathbb{R}$, then*

$$R = KR_1.$$

In particular, in dimension $n = 2$, the assumption of Proposition 14.6 holds and K is the well-known *Gaussian curvature* for surfaces.

Definition 14.4. A Riemannian manifold $(M, \langle -, - \rangle)$ is said to have *constant (resp. negative, resp. positive) curvature* iff its sectional curvature is constant (resp. negative, resp. positive).

In dimension $n \geq 3$, we have the following somewhat surprising theorem due to F. Schur:

Proposition 14.7. *(F. Schur, 1886) Let $(M, \langle -, - \rangle)$ be a connected Riemannian manifold. If $\dim(M) \geq 3$ and if the sectional curvature $K(\Pi)$ does not depend on the plane $\Pi \subseteq T_pM$ but only on the point $p \in M$, then K is constant (i.e., does not depend on p).*

The proof, which is quite beautiful, can be found in Kuhnel [27] (Chapter 6, Theorem 6.7).

If we replace the metric $g = \langle -, - \rangle$ by the metric $\widetilde{g} = \lambda\langle -, - \rangle$ where $\lambda > 0$ is a constant, some simple calculations show that the Christoffel symbols and the Levi-Civita connection are unchanged, as well as the curvature tensor, but the sectional curvature is changed, with

$$\widetilde{K} = \lambda^{-1}K.$$

As a consequence, if M is a Riemannian manifold of constant curvature, by rescaling the metric, we may assume that either $K = -1$, or $K = 0$, or $K = +1$.

Here are standard examples of spaces with constant curvature.

(1) The *sphere* $S^n \subseteq \mathbb{R}^{n+1}$ with the metric induced by $\mathbb{R}^{n+1}$, where

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

The sphere S^n has constant sectional curvature $K = +1$.

(2) *Euclidean space* $\mathbb{R}^{n+1}$ with its natural Euclidean metric. Of course, $K = 0$.

(3) The *hyperbolic space* $\mathcal{H}_n^+(1)$ from Definition 6.1. Recall that this space is defined in terms of the *Lorentz innner product* $\langle -, - \rangle_1$ on $\mathbb{R}^{n+1}$, given by

$$\langle (x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}) \rangle_1 = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i.$$

By definition, $\mathcal{H}_n^+(1)$, written simply H^n, is given by

$$H^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \\ \mid \langle x, x \rangle_1 = -1,\ x_1 > 0\}.$$

It can be shown that the restriction of $\langle -, - \rangle_1$ to H^n is positive, definite, which means that it is a metric on $T_p H^n$.

The space H^n equipped with this metric g_H is called *hyperbolic space* and it has constant curvature $K = -1$.

There are other isometric models of H^n that are perhaps intuitively easier to grasp but for which the metric is more complicated.

For example, there is a map $\mathrm{PD}\colon B^n \to H^n$ where $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ is the open unit ball in $\mathbb{R}^n$, given by

$$\mathrm{PD}(x) = \left(\frac{1 + \|x\|^2}{1 - \|x\|^2}, \frac{2x}{1 - \|x\|^2} \right).$$

It is easy to check that $\langle \mathrm{PD}(x), \mathrm{PD}(x) \rangle_1 = -1$ and that PD is bijective and an isometry.

One also checks that the pull-back metric $g_{\mathrm{PD}} = \mathrm{PD}^* g_H$ on B^n is given by

$$g_{\mathrm{PD}} = \frac{4}{(1 - \|x\|^2)^2}\,(dx_1^2 + \cdots + dx_n^2).$$

The metric g_{PD} is called the *conformal disc metric*, and the Riemannian manifold (B^n, g_{PD}) is called the *Poincaré disc model* or *conformal disc model*.

The metric g_{PD} is proportional to the Euclidean metric, and thus angles are preserved under the map PD.

Another model is the *Poincaré half-plane model* $\{x \in \mathbb{R}^n \mid x_1 > 0\}$, with the metric

$$g_{\mathrm{PH}} = \frac{1}{x_1^2}\,(dx_1^2 + \cdots + dx_n^2).$$

We already encountered this space for $n = 2$.

In general, it is practically impossible to find an explicit formula for the sectional curvature of a Riemannian manifold.

The spaces S^n, $\mathbb{R}^{n+1}$, and H^n are exceptions.

Nice formulae can be given for Lie groups with bi-invariant metrics (see Chapter 18) and for certain kinds of reductive homogeneous manifolds (see Chapter 20).

14.3 Ricci Curvature

The Ricci tensor is another important notion of curvature.

It is mathematically simpler than the sectional curvature (since it is symmetric) but it plays an important role in the theory of gravitation as it occurs in the Einstein field equations.

Recall that if $f \colon E \to E$ is a linear map from a finite-dimensional Euclidean vector space to itself, given any orthonormal basis $(e_1, \ldots, e_n)$, we have

$$\mathrm{tr}(f) = \sum_{i=1}^{n} \langle f(e_i), e_i \rangle.$$

Definition 14.5. Let $(M, \langle -, - \rangle)$ be a Riemannian manifold (equipped with the Levi-Civita connection). The *Ricci curvature* Ric of M is defined as follows: For every $p \in M$, for all $x, y \in T_pM$, set $\mathrm{Ric}_p(x, y)$ to be the trace of the endomorphism $v \mapsto R_p(x, v)y$. With respect to any orthonormal basis $(e_1, \ldots, e_n)$ of T_pM, we have

$$\mathrm{Ric}_p(x, y) = \sum_{j=1}^{n} \langle R_p(x, e_j)y, e_j \rangle_p = \sum_{j=1}^{n} R_p(x, e_j, y, e_j).$$

The *scalar curvature* S of M is the trace of the Ricci curvature; that is, for every $p \in M$,

$$S(p) = \sum_{i \neq j} R(e_i, e_j, e_i, e_j) = \sum_{i \neq j} K(e_i, e_j),$$

where $K(e_i, e_j)$ denotes the sectional curvature of the plane spanned by e_i, e_j.

In a chart the Ricci curvature is given by

$$R_{ij} = \mathrm{Ric}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_m R^m_{ijm},$$

and the scalar curvature is given by

$$S(p) = \sum_{i,j} g^{ij} R_{ij},$$

where (g^{ij}) is the inverse of the Riemann metric matrix (g_{ij}). See O'Neill, pp. 87-88 [38]

If M is equipped with the Levi-Civita connection, in view of Proposition 14.3 (4), the Ricci curvature is symmetric.

The tensor Ric is a $(0, 2)$-tensor but it can be interpreted as a $(1, 1)$-tensor as follows.

Definition 14.6. Let $(M, \langle -, - \rangle)$ be a Riemannian manifold (equipped with any connection). The $(1, 1)$-tensor $\mathrm{Ric}_p^{\#}$ is defined to be

$$\langle \mathrm{Ric}_p^{\#} u, v \rangle_p = \mathrm{Ric}_p(u, v),$$

for all $u, v \in T_pM$.

Proposition 14.8. *Let (M, g) be a Riemannian manifold and let ∇ be any connection on M. If $(e_1, \ldots, e_n)$ is any orthonormal basis of T_pM, we have*

$$\mathrm{Ric}_p^{\#}(u) = \sum_{j=1}^{n} R_p(e_j, u)e_j.$$

Then it is easy to see that

$$S(p) = \mathrm{tr}(\mathrm{Ric}_p^{\#}).$$

This is why we said (by abuse of language) that S is the trace of Ric.

Observe that in dimension $n = 2$, we get $S(p) = 2K(p)$. Therefore, in dimension 2, the scalar curvature determines the curvature tensor.

In dimension $n = 3$, it turns out that the Ricci tensor completely determines the curvature tensor, although this is not obvious.

Since $\mathrm{Ric}(x, y)$ is symmetric, $\mathrm{Ric}(x, x)$ determines $\mathrm{Ric}(x, y)$ completely.

Observe that for any orthonormal frame $(e_1, \ldots, e_n)$ of T_pM, using the definition of the sectional curvature K, we have

$$\mathrm{Ric}(e_1, e_1) = \sum_{i=1}^{n} \langle (R(e_1, e_i)e_1, e_i \rangle = \sum_{i=2}^{n} K(e_1, e_i).$$

Thus, $\mathrm{Ric}(e_1, e_1)$ is the sum of the sectional curvatures of any $n-1$ orthogonal planes orthogonal to e_1 (a unit vector).

For a Riemannian manifold with constant sectional curvature, we have

$$\mathrm{Ric}(x, x) = (n-1)Kg(x, x), \qquad S = n(n-1)K,$$

where $g = \langle -, - \rangle$ is the metric on M.

Spaces for which the Ricci tensor is proportional to the metric are called *Einstein spaces*.

Definition 14.7. A Riemannian manifold (M, g) is called an *Einstein space* iff the Ricci curvature is proportional to the metric g; that is:

$$\mathrm{Ric}(x, y) = \lambda g(x, y),$$

for some function $\lambda\colon M \to \mathbb{R}$.

If M is an Einstein space, observe that $S = n\lambda$.

Remark: For any Riemanian manifold (M, g), the quantity

$$G = \mathrm{Ric} - \frac{S}{2} g$$

is called the *Einstein tensor* (or *Einstein gravitation tensor* for space-times spaces).

The Einstein tensor plays an important role in the theory of general relativity. For more on this topic, see Kuhnel [27] (Chapters 6 and 8) O'Neill [38] (Chapter 12).

14.4 The Second Variation Formula and the Index Form

In Section 13.6, we discovered that the geodesics are exactly the critical paths of the energy functional (Theorem 13.24).

For this, we derived the First Variation Formula (Theorem 13.23).

It is not too surprising that a deeper understanding is achieved by investigating the second derivative of the energy functional at a critical path (a geodesic).

By analogy with the Hessian of a real-valued function on $\mathbb{R}^n$, it is possible to define a bilinear functional

$$I_\gamma \colon T_\gamma\Omega(p, q) \times T_\gamma\Omega(p, q) \to \mathbb{R}$$

when γ is a critical point of the energy function E (that is, γ is a geodesic).

This bilinear form is usually called the *index form*.

Note that Milnor denotes I_γ by E_{**} and refers to it as the *Hessian* of E, but this is a bit confusing since I_γ is only defined for critical points, whereas the Hessian is defined for all points, critical or not.

Now, if $f\colon M \to \mathbb{R}$ is a real-valued function on a finite-dimensional manifold M and if p is a critical point of f, which means that $df_p = 0$, it is not hard to prove that there is a symmetric bilinear map $I\colon T_pM \times T_pM \to \mathbb{R}$ such that

$$I(X(p), Y(p)) = X_p(Yf) = Y_p(Xf),$$

for all vector fields $X, Y \in \mathfrak{X}(M)$.

Furthermore, $I(u, v)$ can be computed as follows: for any $u, v \in T_pM$, for any smooth map $\alpha\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\alpha(0,0) = p, \quad \frac{\partial \alpha}{\partial x}(0,0) = u, \quad \frac{\partial \alpha}{\partial y}(0,0) = v,$$

we have

$$I(u,v) = \left.\frac{\partial^2 (f \circ \alpha)(x,y)}{\partial x \partial y}\right|_{(0,0)}.$$

The above suggests that in order to define

$$I_\gamma\colon T_\gamma\Omega(p,q) \times T_\gamma\Omega(p,q) \to \mathbb{R},$$

that is to define $I_\gamma(W_1, W_2)$, where $W_1, W_2 \in T_\gamma\Omega(p,q)$ are vector fields along γ (with $W_1(0) = W_2(0) = 0$ and $W_1(1) = W_2(1) = 0$), we consider *2-parameter variations*

$$\alpha\colon U \times [0,1] \to M,$$

where U is an open subset of $\mathbb{R}^2$ with $(0,0) \in U$, such that

$$\alpha(0,0,t) = \gamma(t), \quad \frac{\partial \alpha}{\partial u_1}(0,0,t) = W_1(t), \quad \frac{\partial \alpha}{\partial u_2}(0,0,t) = W_2(t).$$

See Figure 14.1.

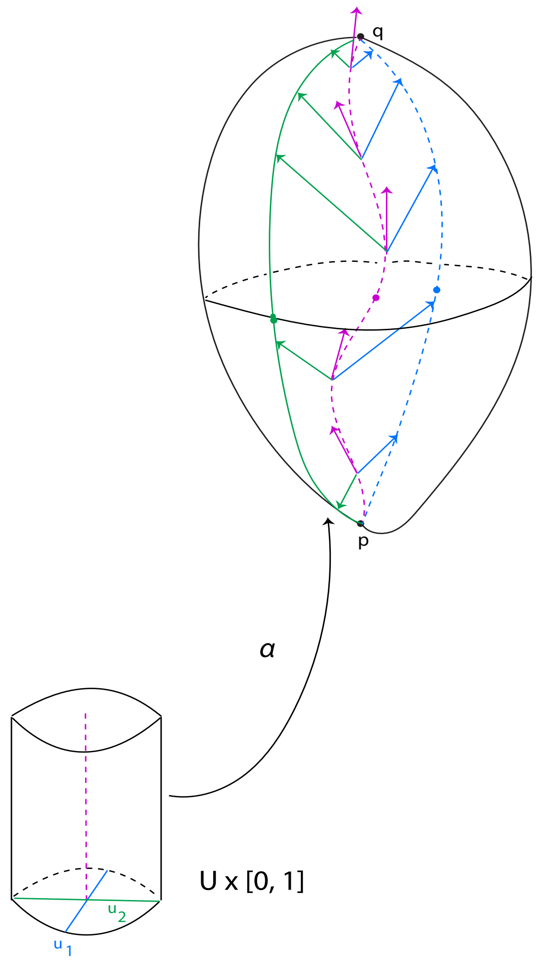

Figure 14.1: A 2-parameter variation α. The pink curve with its associated velocity field is $\alpha(0, 0, t) = \gamma(t)$. The blue vector field is $W_1(t)$ while the green vector field is $W_2(t)$.

Then, we set

$$I_\gamma(W_1, W_2) = \left.\frac{\partial^2(E \circ \widetilde{\alpha})(u_1, u_2)}{\partial u_1 \partial u_2}\right|_{(0,0)},$$

where $\widetilde{\alpha} \in \Omega(p, q)$ is the path given by

$$\widetilde{\alpha}(u_1, u_2)(t) = \alpha(u_1, u_2, t).$$

For simplicity of notation, the above derivative if often written as $\frac{\partial^2 E}{\partial u_1 \partial u_2}(0, 0)$.

To prove that $I_\gamma(W_1, W_2)$ is actually well-defined, we need the following result:

Theorem 14.9. *(Second Variation Formula) Let $\alpha\colon U \times [0,1] \to M$ be a 2-parameter variation of a geodesic $\gamma \in \Omega(p,q)$, with variation vector fields $W_1, W_2 \in T_\gamma\Omega(p,q)$ given by*

$$W_1(t) = \frac{\partial \alpha}{\partial u_1}(0,0,t), \quad W_2(t) = \frac{\partial \alpha}{\partial u_2}(0,0,t).$$

Then, we have the formula

$$\frac{1}{2}\left.\frac{\partial^2 (E \circ \widetilde{\alpha})(u_1,u_2)}{\partial u_1 \partial u_2}\right|_{(0,0)} = -\sum_t \left\langle W_2(t), \Delta_t \frac{dW_1}{dt}\right\rangle - \int_0^1 \left\langle W_2, \frac{D^2 W_1}{dt^2} + R(V, W_1)V \right\rangle dt,$$

where $V(t) = \gamma'(t)$ is the velocity field,

$$\Delta_t \frac{dW_1}{dt} = \frac{dW_1}{dt}(t_+) - \frac{dW_1}{dt}(t_-)$$

is the jump in $\frac{dW_1}{dt}$ at one of its finitely many points of discontinuity in $(0,1)$, and E is the energy function on $\Omega(p,q)$.

Theorem 14.9 shows that the expression

$$\left.\frac{\partial^2(E\circ\widetilde{\alpha})(u_1,u_2)}{\partial u_1\partial u_2}\right|_{(0,0)}$$

only depends on the variation fields W_1 and W_2, and thus $I_\gamma(W_1, W_2)$ is actually well-defined. If no confusion arises, we write $I(W_1, W_2)$ for $I_\gamma(W_1, W_2)$.

Proposition 14.10. *Given any geodesic $\gamma \in \Omega(p, q)$, the map $I\colon T_\gamma\Omega(p, q) \times T_\gamma\Omega(p, q) \to \mathbb{R}$ defined so that for all $W_1, W_2 \in T_\gamma\Omega(p, q)$,*

$$I(W_1,W_2) = \left.\frac{\partial^2(E\circ\widetilde{\alpha})(u_1,u_2)}{\partial u_1\partial u_2}\right|_{(0,0)},$$

only depends on W_1 and W_2 and is bilinear and symmetric, where $\alpha\colon U \times [0, 1] \to M$ is any 2-parameter variation, with

$$\alpha(0,0,t)=\gamma(t),\quad \frac{\partial\alpha}{\partial u_1}(0,0,t)=W_1(t),\quad \frac{\partial\alpha}{\partial u_2}(0,0,t)=W_2(t).$$

On the diagonal, $I(W, W)$ can be described in terms of a 1-parameter variation of γ. In fact,

$$I(W, W) = \frac{d^2 E(\widetilde{\alpha})}{du^2}(0),$$

where $\widetilde{\alpha}\colon (-\epsilon, \epsilon) \to \Omega(p, q)$ denotes any variation of γ with variation vector field $\frac{d\widetilde{\alpha}}{du}(0)$ equal to W.

Proposition 14.11. *If $\gamma \in \Omega(p, q)$ is a minimal geodesic, then the bilinear index form I is positive semi-definite, which means that $I(W, W) \geq 0$ for all $W \in T_\gamma \Omega(p, q)$.*

14.5 Jacobi Fields and Conjugate Points

Jacobi fields arise naturally when considering the expression involved under the integral sign in the Second Variation Formula and also when considering the derivative of the exponential.

If $B\colon E \times E \to \mathbb{R}$ is a symmetric bilinear form defined on some vector space E (possibly infinite dimentional), recall that the *nullspace* of B is the subset $\mathrm{null}(B)$ of E given by

$$\mathrm{null}(B) = \{u \in E \mid B(u, v) = 0, \quad \text{for all } v \in E\}.$$

The *nullity* ν of B is the dimension of its nullspace.

The bilinear form B is *nondegenerate* iff $\mathrm{null}(B) = (0)$ iff $\nu = 0$.

If U is a subset of E, we say that B is *positive definite* (resp. *negative definite*) *on* U iff $B(u, u) > 0$ (resp. $B(u, u) < 0$) for all $u \in U$, with $u \neq 0$.

The *index* of B is the maximum dimension of a subspace of E on which B is negative definite.

We will determine the nullspace of the symmetric bilinear form

$$I \colon T_\gamma\Omega(p, q) \times T_\gamma\Omega(p, q) \to \mathbb{R},$$

where γ is a geodesic from p to q in some Riemannian manifold M.

Now, if W is a vector field in $T_\gamma\Omega(p, q)$ and W satisfies the equation

$$\frac{D^2W}{dt^2} + R(V, W)V = 0, \qquad (*)$$

where $V(t) = \gamma'(t)$ is the velocity field of the geodesic γ, since W is smooth along γ, it is obvious from the Second Variation Formula that

$$I(W, W_2) = 0, \quad \text{for all } W_2 \in T_\gamma\Omega(p, q).$$

Therefore, any vector field in the nullspace of I must satisfy equation $(*)$. Such vector fields are called *Jacobi fields*.

Definition 14.8. Given a geodesic $\gamma \in \Omega(p, q)$, a vector field J along γ is a *Jacobi field* iff it satisfies the *Jacobi differential equation*

$$\frac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0.$$

The equation of Definition 14.8 is a linear second-order differential equation that can be transformed into a more familiar form by picking some orthonormal parallel vector fields $X_1, \ldots, X_n$ along γ.

To do this, pick any orthonormal basis $(e_1, \ldots, e_n)$ in T_pM, with $e_1 = \gamma'(0)/\left\|\gamma'(0)\right\|$, and use parallel transport along γ to get $X_1, \ldots, X_n$.

Then, we can write $J = \sum_{i=1}^{n} y_i X_i$, for some smooth functions y_i, and the Jacobi equation becomes the system of second-order linear ODE's

$$\frac{d^2 y_i}{dt^2} + \sum_{j=1}^{n} R(\gamma', E_j, \gamma', E_i) y_j = 0, \qquad 1 \leq i \leq n.$$

By the existence and uniqueness theorem for ODE's, for every pair of vectors $u, v \in T_p M$, there is a unique Jacobi fields J so that $J(0) = u$ and $\frac{DJ}{dt}(0) = v$.

Since $T_p M$ has dimension n, it follows that the dimension of the space of Jacobi fields along γ is $2n$.

Proposition 14.12. *If $J(0)$ and $\frac{DJ}{dt}(0)$ are orthogonal to $\gamma'(0)$, then $J(t)$ is orthogonal to $\gamma'(t)$ for all $t \in [0, 1]$.*

Proposition 14.13. *If J is orthogonal to γ, which means that $J(t)$ is orthogonal to $\gamma'(t)$ for all $t \in [0, 1]$, then $\frac{DJ}{dt}$ is also orthogonal to γ.*

Proposition 14.14. *If $\gamma \in \Omega(p, q)$ is a geodesic in a Riemannian manifold of dimension n, then the following properties hold:*

(1) For all $u, v \in T_pM$, there is a unique Jacobi fields J so that $J(0) = u$ and $\frac{DJ}{dt}(0) = v$. Consequently, the vector space of Jacobi fields has dimension n.

(2) The subspace of Jacobi fields orthogonal to γ has dimension $2n - 2$. The vector fields γ' and $t \mapsto t\gamma'(t)$ are Jacobi fields that form a basis of the subspace of Jacobi fields parallel to γ (that is, such that $J(t)$ is collinear with $\gamma'(t)$, for all $t \in [0, 1]$.) See Figure 14.2.

(3) If J is a Jacobi field, then J is orthogonal to γ iff there exist $a, b \in [0, 1]$, with $a \neq b$, so that $J(a)$ and $J(b)$ are both orthogonal to γ iff there is some $a \in [0, 1]$ so that $J(a)$ and $\frac{DJ}{dt}(a)$ are both orthogonal to γ.

(4) For any two Jacobi fields X, Y along γ, the expression $\langle \nabla_{\gamma'} X, Y\rangle - \langle \nabla_{\gamma'} Y, X\rangle$ is a constant, and if X and Y vanish at some point on γ, then $\langle \nabla_{\gamma'} X, Y\rangle - \langle \nabla_{\gamma'} Y, X\rangle = 0$.

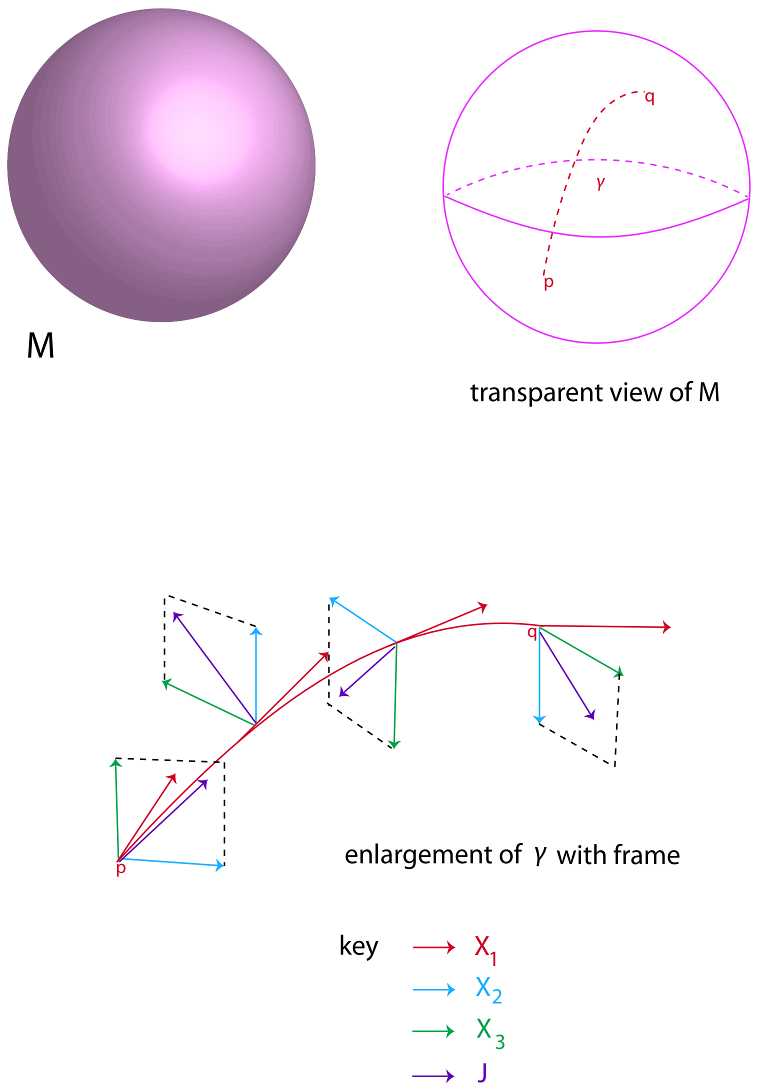

Figure 14.2: An orthogonal Jacobi field J for a three dimensional manifold M. Note that J is in the plane spanned by X_2 and X_3, while X_1 is in the direction of the velocity field.

Following Milnor, we will show that the Jacobi fields in $T_\gamma\Omega(p,q)$ are exactly the vector fields in the nullspace of the index form I.

First, we define the important notion of conjugate points.

Definition 14.9. Let $\gamma \in \Omega(p,q)$ be a geodesic. Two distinct parameter values $a, b \in [0,1]$ with $a < b$ are *conjugate along γ* iff there is some Jacobi field J, not identically zero, such that $J(a) = J(b) = 0$.

The dimension k of the space $\mathfrak{J}_{a,b}$ consisting of all such Jacobi fields is called the *multiplicity* (or *order of conjugacy)* of a and b as conjugate parameters. We also say that the points $p_1 = \gamma(a)$ and $p_2 = \gamma(b)$ are *conjugate along γ*.

Remark: As remarked by Milnor and others, as γ may have self-intersections, the above definition is ambiguous if we replace a and b by $p_1 = \gamma(a)$ and $p_2 = \gamma(b)$, even though many authors make this slight abuse.

Although it makes sense to say that the points p_1 and p_2 are conjugate, the space of Jacobi fields vanishing at p_1 and p_2 is not well defined.

Indeed, if $p_1 = \gamma(a)$ for distinct values of a (or $p_2 = \gamma(b)$ for distinct values of b), then we don't know which of the spaces, $\mathfrak{J}_{a,b}$, to pick.

We will say that some points p_1 and p_2 on γ are *conjugate* iff there are parameter values, $a < b$, such that $p_1 = \gamma(a)$, $p_2 = \gamma(b)$, and a and b are conjugate along γ.

However, for the endpoints p and q of the geodesic segment γ, we may assume that $p = \gamma(0)$ and $q = \gamma(1)$, so that when we say that p and q are conjugate we consider the space of Jacobi fields vanishing for $t = 0$ and $t = 1$.

In view of Proposition 14.14 (3), the Jacobi fields involved in the definition of conjugate points are orthogonal to γ.

The dimension of the space of Jacobi fields such that $J(a) = 0$ is obviously n, since the only remaining parameter determining J is $\frac{dJ}{dt}(a)$.

Furthermore, the Jacobi field $t \mapsto (t-a)\gamma'(t)$ vanishes at a but not at b, so the multiplicity of conjugate parameters (points) is at most $n - 1$.

For example, if M is a flat manifold, that is if its curvature tensor is identically zero, then the Jacobi equation becomes

$$\frac{D^2 J}{dt^2} = 0.$$

It follows that $J \equiv 0$, and thus, there are no conjugate points. More generally, the Jacobi equation can be solved explicitly for spaces of constant curvature.

Theorem 14.15. *Let $\gamma \in \Omega(p, q)$ be a geodesic. A vector field $W \in T_\gamma \Omega(p, q)$ belongs to the nullspace of the index form I iff W is a Jacobi field. Hence, I is degenerate if p and q are conjugate. The nullity of I is equal to the multiplicity of p and q.*

Theorem 14.15 implies that the nullity of I is finite, since the vector space of Jacobi fields vanishing at 0 and 1 has dimension at most n.

Corollary 14.16. *The nullity ν of I satisfies $0 \leq \nu \leq n - 1$, where $n = \dim(M)$.*

As our (connected) Riemannian manifold M is a metric space, (see Proposition 13.13), the path space $\Omega(p, q)$ is also a metric space if we use the metric d^* given by

$$d^*(\omega_1, \omega_2) = \max_t(d(\omega_1(t), \omega_2(t))),$$

where d is the metric on M induced by the Riemannian metric.

Remark: The topology induced by d^* turns out to be the compact open topology on $\Omega(p, q)$.

Theorem 14.17. *Let $\gamma \in \Omega(p, q)$ be a geodesic. Then the following properties hold:*

(1) If there are no conjugate points to p along γ, then there is some open subset $\mathcal{V}$ of $\Omega(p, q)$, with $\gamma \in \mathcal{V}$, such that

$$L(\omega) \geq L(\gamma) \quad \text{and} \quad E(\omega) \geq E(\gamma), \quad \text{for all } \omega \in \mathcal{V},$$

with strict inequality when $\omega([0, 1]) \neq \gamma([0, 1])$. We say that γ is a local minimum.

(2) If there is some $t \in (0, 1)$ such that p and $\gamma(t)$ are conjugate along γ, then there is a fixed endpoints variation α, such that

$$L(\widetilde{\alpha}(u)) < L(\gamma) \quad \text{and} \quad E(\widetilde{\alpha}(u)) < E(\gamma),$$

for u small enough.

14.6 Jacobi Fields and Geodesic Variations

Jacobi fields turn out to be induced by certain kinds of variations called *geodesic variations*.

Definition 14.10. Given a geodesic $\gamma \in \Omega(p, q)$, a *geodesic variation of* γ is a smooth map

$$\alpha\colon (-\epsilon, \epsilon) \times [0, 1] \to M,$$

such that

(1) $\alpha(0, t) = \gamma(t)$, for all $t \in [0, 1]$.

(2) For every $u \in (-\epsilon, \epsilon)$, the curve $\widetilde{\alpha}(u)$ is a geodesic, where

$$\widetilde{\alpha}(u)(t) = \alpha(u, t), \qquad t \in [0, 1].$$

Note that the geodesics $\widetilde{\alpha}(u)$ do not necessarily begin at p and end at q, and so a geodesic variation is not a "fixed endpoints" variation. See Figure 14.3.

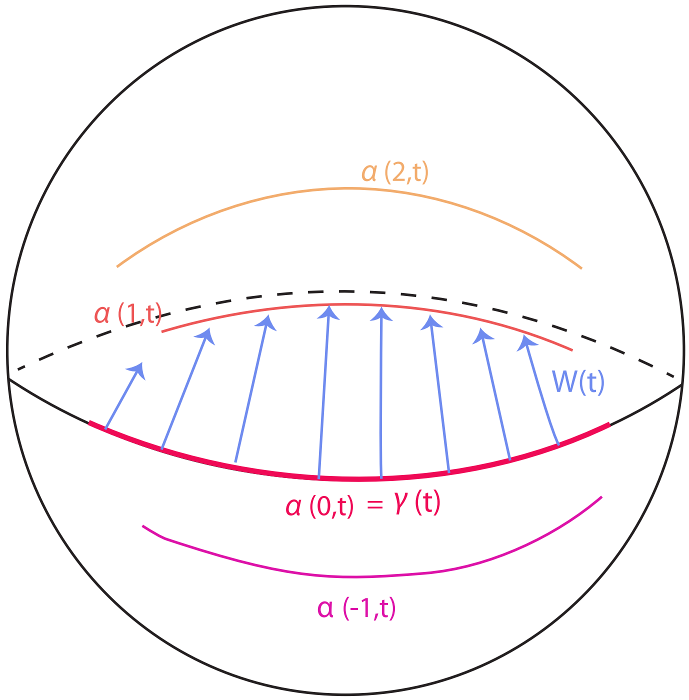

Figure 14.3: A geodesic variation for S^2 with its associated Jacobi field $W(t)$.

Proposition 14.18. *If* $\alpha\colon (-\epsilon, \epsilon) \times [0, 1] \to M$ *is a geodesic variation of* $\gamma \in \Omega(p, q)$*, then the vector field* $W(t) = \frac{\partial \alpha}{\partial u}(0, t)$ *is a Jacobi field along* γ*.*

For example, on the sphere S^n, for any two antipodal points p and q, rotating the sphere keeping p and q fixed, the variation field along a geodesic γ through p and q (a great circle) is a Jacobi field vanishing at p and q.

Rotating in $n-1$ different directions one obtains $n-1$ linearly independent Jacobi fields and thus, p and q are conjugate along γ with multiplicity $n-1$.

Interestingly, the converse of Proposition 14.18 holds.

Proposition 14.19. *For every Jacobi field $W(t)$ along a geodesic $\gamma \in \Omega(p, q)$, there is some geodesic variation $\alpha\colon (-\epsilon, \epsilon) \times [0, 1] \to M$ of γ such that $W(t) = \frac{\partial \alpha}{\partial u}(0, t)$. Furthermore, for every point $\gamma(a)$, there is an open subset U containing $\gamma(a)$ such that the Jacobi fields along a geodesic segment in U are uniquely determined by their values at the endpoints of the geodesic.*

Remark: The proof of Proposition 14.19 also shows that there is some open interval $(-\delta, \delta)$ such that if $t \in (-\delta, \delta)$, then $\gamma(t)$ is not conjugate to $\gamma(0)$ along γ.

Using Proposition 14.19 it is easy to characterize conjugate points in terms of geodesic variations.

Proposition 14.20. *If $\gamma \in \Omega(p, q)$ is a geodesic, then q is conjugate to p iff there is a geodesic variation α of γ such that every geodesic $\widetilde{\alpha}(u)$ starts from p, the Jacobi field $J(t) = \frac{\partial \alpha}{\partial u}(0, t)$ does not vanish identically, and $J(1) = 0$.*

Jacobi fields, as characterized by Proposition 14.18, can be used to compute the sectional curvature of the sphere S^n and the sectional curvature of hyperbolic space $H^n = \mathcal{H}_n^+(1)$, both equipped with their respective canonical metrics.

This requires knowing the geodesics in S^n and H^n.

This is done in Section 20.7 for the sphere. The hyperbolic space $H^n = \mathcal{H}_n^+(1)$ is shown to be a symmetric space in Section 20.9, and it would be easy to derive its geodesics by analogy with what we did for the sphere.

For the sake of brevity, we will assume without proof that we know these geodesics. The reader may consult Gallot, Hulin and Lafontaine [19] or O'Neill [38] for details.

First we consider the sphere S^n. For any $p \in S^n$, the geodesic from p with initial velocity a unit vector v is

$$\gamma(t) = (\cos t)p + (\sin t)v.$$

We find that the sectional curvature of the sphere is constant and equal to $+1$.

Let us now consider the hyperbolic space H^n. This time the geodesic from p with initial velocity a unit vector v is

$$\gamma(t) = (\cosh t)p + (\sinh t)v.$$

We find that the sectional curvature of the hyperbolic is constant and equal to -1.

it can be shown that $\mathbb{RP}^n$ with the canonical metric also has constant sectional curvature equal to $+1$; see Gallot, Hulin and Lafontaine [19] (Chapter III, section 3.49).

There is also an intimate connection between Jacobi fields and the differential of the exponential map, and between conjugate points and critical points of the exponential map.

Recall that if $f\colon M \to N$ is a smooth map between manifolds, a point $p \in M$ is a *critical* point of f iff the tangent map at p

$$df_p\colon T_pM \to T_{f(p)}N$$

is not surjective.

If M and N have the same dimension, which will be the case in the sequel, df_p is not surjective iff it is not injective, so p is a critical point of f iff there is some nonzero vector $u \in T_pM$ such that $df_p(u) = 0$.

If $\exp_p\colon T_pM \to M$ is the exponential map, for any $v \in T_pM$ where $\exp_p(v)$ is defined, we have the derivative of $\exp_p$ at v:

$$(d\exp_p)_v\colon T_v(T_pM) \to T_pM.$$

Since T_pM is a finite-dimensional vector space, $T_v(T_pM)$ is isomorphic to T_pM, so we identify $T_v(T_pM)$ with T_pM.

Jacobi fields can be used to compute the derivative of the exponential (see Gallot, Hulin and Lafontaine [19], Chapter 3, Corollary 3.46).

Proposition 14.21. *Given any point $p \in M$, for any vectors $u, v \in T_pM$, if $\exp_p v$ is defined, then*

$$J(t) = (d\exp_p)_{tv}(tu), \qquad 0 \leq t \leq 1,$$

is a Jacobi field such that $\frac{DJ}{dt}(0) = u$.

Remark: If $u, v \in T_pM$ are orthogonal unit vectors, then $R(u, v, u, v) = K(u, v)$, the sectional curvature of the plane spanned by u and v in T_pM, and for t small enough, we have

$$\|J(t)\| = t - \frac{1}{6}K(u, v)t^3 + o(t^3).$$

(Here, $o(t^3)$ stands for an expression of the form $t^4R(t)$, such that $\lim_{t\mapsto 0} R(t) = 0$.)

Intuitively, this formula tells us how fast the geodesics that start from p and are tangent to the plane spanned by u and v spread apart.

Locally, for $K(u, v) > 0$ the radial geodesics spread apart less than the rays in T_pM, and for $K(u, v) < 0$ they spread apart more than the rays in T_pM.

Jacobi fields can also be used to obtain a Taylor expansion for the matrix coefficients g_{ij} representing the metric g in a normal coordinate system near a point $p \in M$.

Proposition 14.22. *With respect to a normal coordinate system $x = (x_1, \ldots, x_n)$ around a point $p \in M$, the matrix coefficients g_{ij} representing the metric g near 0 are given by*

$$g_{ij}(x_1, \ldots, x_n) = \delta_{ij} + \frac{1}{3} \sum_{k,l} R_{ikjl}(p)\, x_k x_l + o(\|x\|^3).$$

The above formula shows that the deviation of the Riemannian metric on M near p from the canonical Euclidean metric is measured by the curvature coefficients R_{ikjl}.

For any $x \neq 0$, write $x = tu$ with $u = x/\|x\|$ and $t = \|x\|$, where $x = (x_1, \ldots, x_n)$ are local coordinates at p. Proposition 14.22 can used used to give an expression for $\det(g_{ij}(tu))$ in terms of the Ricci curvature $\mathrm{Ric}_p(u, u)$.

Proposition 14.23. *With respect to a normal coordinate system* $(x_1, \ldots, x_n) = tu$ *with* $\|u\| = 1$ *around a point* $p \in M$*, we have*

$$\det(g_{ij}(tu)) = 1 - \frac{1}{3}\mathrm{Ric}_p(u, u)\, t^2 + o(t^3).$$

The above formula shows that the Ricci curvature at p in the direction u is a measure of the deviation of the determinant $\det(g_{ij}(tu))$ to be equal to 1 (as in the case of the canonical Euclidean metric).

We now establish a relationship between conjugate points and critical points of the exponential map. These are points where the exponential is no longer a diffeomorphism.

Proposition 14.24. *Let* $\gamma \in \Omega(p, q)$ *be a geodesic. The point* $r = \gamma(t)$*, with* $t \in (0, 1]$*, is conjugate to* p *along* γ *iff* $v = t\gamma'(0)$ *is a critical point of* $\exp_p$*. Furthermore, the multiplicity of* p *and* r *as conjugate points is equal to the dimension of the kernel of* $(d \exp_p)_v$*.*

14.7 Applications of Jacobi Fields and Conjugate Points

Jacobi fields and conjugate points are basic tools that can be used to prove many global results of Riemannian geometry.

The flavor of these results is that certain constraints on curvature (sectional, Ricci, scalar) have a significant impact on the topology.

One may want consider the effect of non-positive curvature, constant curvature, curvature bounded from below by a positive constant, *etc*.

This is a vast subject and we highly recommend Berger's Panorama of Riemannian Geometry [5] for a masterly survey.

We will content ourselves with three results:

(1) Hadamard and Cartan's Theorem about complete manifolds of non-positive sectional curvature.

(2) Myers' Theorem about complete manifolds of Ricci curvature bounded from below by a positive number.

(3) The Morse Index Theorem.

First, on the way to Hadamard and Cartan, we begin with a proposition.

Proposition 14.25. *Let M be a complete Riemannian manifold with non-positive curvature $K \leq 0$. Then, for every geodesic $\gamma \in \Omega(p, q)$, there are no conjugate points to p along γ. Consequently, the exponential map $\exp_p \colon T_pM \to M$ is a local diffeomorphism for all $p \in M$.*

Theorem 14.26. *(Hadamard–Cartan) Let M be a complete Riemannian manifold. If M has non-positive sectional curvature $K \leq 0$, then the following hold:*

(1) For every $p \in M$, the map $\exp_p\colon T_pM \to M$ is a Riemannian covering.

(2) If M is simply connected then M is diffeomorphic to $\mathbb{R}^n$, where $n = \dim(M)$; more precisely, $\exp_p\colon T_pM \to M$ is a diffeomorphism for all $p \in M$. Furthermore, any two points on M are joined by a unique minimal geodesic.

Remark: A version of Theorem 14.26 was first proved by Hadamard and then extended by Cartan.

Theorem 14.26 was generalized by Kobayashi, see Kobayashi and Nomizu [26] (Chapter VIII, Remark 2 after Corollary 8.2).

Also, it is shown in Milnor [33] that if M is complete, assuming non-positive sectional curvature, then all homotopy groups $\pi_i(M)$ vanish for $i > 1$, and that $\pi_1(M)$ has no element of finite order except the identity.

Finally, non-positive sectional curvature implies that the exponential map does not decrease distance (Kobayashi and Nomizu [26], Chapter VIII, Section 8, Lemma 3).

We now turn to manifolds with strictly positive curvature bounded away from zero and to Myers' Theorem.

The first version of such a theorem was first proved by Bonnet for surfaces with positive sectional curvature bounded away from zero.

It was then generalized by Myers in 1941. For these reasons, this theorem is sometimes called the *Bonnet-Myers' Theorem*. The proof of Myers Theorem involves a beautiful "trick."

Given any metric space X, recall that the *diameter* of X is defined by

$$\mathrm{diam}(X) = \sup\{d(p, q) \mid p, q \in X\}.$$

The diameter of X may be infinite.

Theorem 14.27. *(Myers) Let M be a complete Riemannian manifold of dimension n and assume that*

$$\mathrm{Ric}(u, u) \geq (n-1)/r^2, \quad \textit{for all unit vectors, } u \in T_pM, \textit{ and for all } p \in M,$$

with $r > 0$. Then,

(1) The diameter of M is bounded by πr and M is compact.

(2) The fundamental group of M is finite.

Remarks:

(1) The condition on the Ricci curvature cannot be weakened to $\mathrm{Ric}(u, u) > 0$ for all unit vectors.

Indeed, the paraboloid of revolution $z = x^2 + y^2$ satisfies the above condition, yet it is not compact.

(2) Theorem 14.27 also holds under the stronger condition that the sectional curvature $K(u, v)$ satisfies

$$K(u, v) \geq (n - 1)/r^2,$$

for all orthonormal vectors, u, v. In this form, it is due to Bonnet (for surfaces).

It would be a pity not to include in this section a beautiful theorem due to Morse. This theorem has to do with the index of $I\colon T_\gamma\Omega(p,q)\times T_\gamma\Omega(p,q)\to\mathbb{R}$, which is defined as follows.

Definition 14.11. For any geodesic $\gamma\in\Omega(p,q)$, we define the *index* λ of

$$I\colon T_\gamma\Omega(p,q)\times T_\gamma\Omega(p,q)\to\mathbb{R}$$

as the maximum dimension of a subspace of $T_\gamma\Omega(p,q)$ on which I is negative definite.

Proposition 14.11 says that the index of I is zero for a minimal geodesic γ. It turns out that the index of I is finite for any geodesic γ.

Theorem 14.28. *(Morse Index Theorem) Given a geodesic $\gamma \in \Omega(p,q)$, the index λ of the index form $I\colon T_\gamma\Omega(p,q) \times T_\gamma\Omega(p,q) \to \mathbb{R}$ is equal to the number of points $\gamma(t)$, with $0 \leq t \leq 1$, such that $\gamma(t)$ is conjugate to $p = \gamma(0)$ along γ, each such conjugate point counted with its multiplicity. The index λ is always finite.*

As a corollary of Theorem 14.28, we see that there are only finitely many points which are conjugate to $p = \gamma(0)$ along γ.

A proof of Theorem 14.28 can be found in Milnor [33] (Part III, Section 15) and also in Do Carmo [13] (Chapter 11) or Kobayashi and Nomizu [26] (Chapter VIII, Section 6).

In the next section, we will use conjugate points to give a more precise characterization of the cut locus.

14.8 Cut Locus and Injectivity Radius: Some Properties

We begin by reviewing the definition of the cut locus from a slightly different point of view.

Let M be a complete Riemannian manifold of dimension n. There is a bundle UM, called the *unit tangent bundle*, such that the fibre at any $p \in M$ is the unit sphere $S^{n-1} \subseteq T_pM$ (check the details).

As usual, we let $\pi\colon UM \to M$ denote the projection map which sends every point in the fibre over p to p.

Then, we have the function

$$\rho\colon UM \to \mathbb{R},$$

defined so that for all $p \in M$, for all $v \in S^{n-1} \subseteq T_pM$,

$$\begin{aligned}\rho(v) &= \sup_{t\in\mathbb{R}\cup\{\infty\}} d(\pi(v), \exp_p(tv)) = t \\ &= \sup\{t \in \mathbb{R} \cup \{\infty\} \mid \\ &\quad \text{the geodesic} \quad t \mapsto \exp_p(tv) \quad \text{is minimal on } [0, t]\}.\end{aligned}$$

The number $\rho(v)$ is called the *cut value* of v.

It can be shown that ρ is continuous, and for every $p \in M$, we let

$$\widetilde{\mathrm{Cut}}(p) = \{\rho(v)v \in T_pM \mid v \in UM \cap T_pM, \rho(v) \text{ is finite}\}$$

be the *tangential cut locus of* p, and

$$\mathrm{Cut}(p) = \exp_p(\widetilde{\mathrm{Cut}}(p))$$

be the *cut locus of* p.

The point $\exp_p(\rho(v)v)$ in M is called the *cut point* of the geodesic $t \mapsto \exp_p(vt)$, and so the cut locus of p is the set of cut points of all the geodesics emanating from p.

Also recall from Definition 13.12 that

$$\mathcal{U}_p = \{v \in T_pM \mid \rho(v) > 1\},$$

and that $\mathcal{U}_p$ is open and star-shaped. It can be shown that

$$\widetilde{\mathrm{Cut}}(p) = \partial\mathcal{U}_p,$$

and the following property holds:

Theorem 14.29. *If M is a complete Riemannian manifold, then for every $p \in M$, the exponential map* $\exp_p$ *is a diffeomorphism between $\mathcal{U}_p$ and its image* $\exp_p(\mathcal{U}_p) = M - \mathrm{Cut}(p)$ *in M.*

Theorem 14.29 implies that the cut locus is closed.

Remark: In fact, $M - \mathrm{Cut}(p)$ can be retracted homeomorphically onto a ball around p, and $\mathrm{Cut}(p)$ is a deformation retract of $M - \{p\}$.

The following Proposition gives a rather nice characterization of the cut locus in terms of minimizing geodesics and conjugate points:

Proposition 14.30. *Let M be a complete Riemannian manifold. For every pair of points $p, q \in M$, the point q belongs to the cut locus of p iff one of the two (not mutually exclusive from each other) properties hold:*

(a) There exist two distinct minimizing geodesics from p to q.

(b) There is a minimizing geodesic γ from p to q, and q is the first conjugate point to p along γ.

Observe that Proposition 14.30 implies the following symmetry property of the cut locus: $q \in \mathrm{Cut}(p)$ iff $p \in \mathrm{Cut}(q)$. Furthermore, if M is compact, we have

$$p = \bigcap_{q \in \mathrm{Cut}(p)} \mathrm{Cut}(q).$$

Recall from Definition 13.13 the definition of the injectivity radius,

$$i(M) = \inf_{p \in M} d(p, \mathrm{Cut}(p)).$$

Proposition 14.30 admits the following sharpening:

Proposition 14.31. *Let M be a complete Riemannian manifold. For all $p, q \in M$, if $q \in \mathrm{Cut}(p)$, then:*

(a) If among the minimizing geodesics from p to q, there is one, say γ, such that q is not conjugate to p along γ, then there is another minimizing geodesic $\omega \neq \gamma$ from p to q.

(b) Suppose $q \in \mathrm{Cut}(p)$ realizes the distance from p to $\mathrm{Cut}(p)$ (i.e. $d(p, q) = d(p, \mathrm{Cut}(p))$). If there are no minimal geodesics from p to q such that q is conjugate to p along this geodesic, then there are exactly two minimizing geodesics γ_1 and γ_2 from p to q, with $\gamma_2'(1) = -\gamma_1'(1)$. Moreover, if $d(p, q) = i(M)$ (the injectivity radius), then γ_1 and γ_2 together form a closed geodesic.

We also have the following characterization of $\widetilde{\mathrm{Cut}}(p)$:

Proposition 14.32. *Let M be a complete Riemannian manifold. For any $p \in M$, the set of vectors $u \in \widetilde{\mathrm{Cut}}(p)$ such that is some $v \in \widetilde{\mathrm{Cut}}(p)$ with $v \neq u$ and $\exp_p(u) = \exp_p(v)$ is dense in $\widetilde{\mathrm{Cut}}(p)$.*

We conclude this section by stating a classical theorem of Klingenberg about the injectivity radius of a manifold of bounded positive sectional curvature.

Theorem 14.33. *(Klingenberg) Let M be a complete Riemannian manifold and assume that there are some positive constants $K_{\min}$, $K_{\max}$, such that the sectional curvature of K satisfies*

$$0 < K_{\min} \leq K \leq K_{\max}.$$

Then, M is compact, and either

(a) $i(M) \geq \pi/\sqrt{K_{\max}}$, or

(b) There is a closed geodesic γ of minimal length among all closed geodesics in M and such that

$$i(M) = \frac{1}{2} L(\gamma).$$

The proof of Theorem 14.33 is quite hard. A proof using Rauch's comparison Theorem can be found in Do Carmo [13] (Chapter 13, Proposition 2.13).