Problem B1 (80). This problem is from Knapp, *Lie Groups Beyond an Introduction*, Introduction, page 21. Recall that the group $SU(2)$ consists of all complex matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad \alpha\beta + \beta\alpha = 1,$$

and the action $\cdot : SU(2) \times (\mathbb{C} \cup \{\infty\}) \to \mathbb{C} \cup \{\infty\}$ is given by

$$A \cdot w = \frac{\alpha w + \beta}{-\beta w + \alpha}, \quad w \in \mathbb{C} \cup \{\infty\}.$$

This is a transitive action. Using the stereographic projection $\sigma_N$ from $S^2$ onto $\mathbb{C} \cup \{\infty\}$ and its inverse $\sigma_N^{-1}$, we can define an action of $SU(2)$ on $S^2$ by

$$A \cdot (x, y, z) = \sigma_N^{-1}(A \cdot \sigma_N(x, y, z)), \quad (x, y, z) \in S^2,$$

and we denote by $\rho(A)$ the corresponding map from $S^2$ to $S^2$.

1. If we write $\alpha = a + ib$ and $\beta = c + id$, prove that $\rho(A)$ is given by the matrix

$$\rho(A) = \begin{pmatrix} a^2 - b^2 - c^2 + d^2 & -2ab - 2cd & -2ac + 2bd \\ 2ab - 2cd & a^2 - b^2 + c^2 - d^2 & -2ad - 2bc \\ 2ac + 2bd & 2ad - 2bc & a^2 + b^2 - c^2 - d^2 \end{pmatrix}.$$

Prove that $\rho(A)$ is indeed a rotation matrix which represents the rotation whose axis is the line determined by the vector $(d, -c, b)$ and whose angle $\theta \in [-\pi, \pi]$ is determined by

$$\cos \frac{\theta}{2} = |a|.$$

*Hint.* Recall that the axis of a rotation matrix $R \in SO(3)$ is specified by any eigenvector of 1 for $R$, and that the angle of rotation $\theta$ satisfies the equation

$$\text{tr}(R) = 2\cos \theta + 1.$$
(2) We can compute the derivative $d\rho_I : \mathfrak{su}(2) \to \mathfrak{so}(3)$ of $\rho$ at $I$ as follows. Recall that $\mathfrak{su}(2)$ consists of all complex matrices of the form

$$\begin{pmatrix} ib & c + id \\ -c + id & -ib \end{pmatrix}, \quad b,c,d \in \mathbb{R},$$

so pick the following basis for $\mathfrak{su}(2)$,

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and define the curves in $\text{SU}(2)$ through $I$ given by

$$c_1(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad c_2(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad c_3(t) = \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix}.$$ 

Prove that $c'_i(0) = X_i$ for $i = 1, 2, 3$, and that

$$d\rho_I(X_1) = 2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad d\rho_I(X_2) = 2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad d\rho_I(X_3) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

Thus, we have

$$d\rho_I(X_1) = 2E_3, \quad d\rho_I(X_2) = -2E_2, \quad d\rho_I(X_3) = 2E_1,$$

where $(E_1, E_2, E_3)$ is the basis of $\mathfrak{so}(3)$ given in Section 2.5. Conclude that $d\rho_I$ is an isomorphism between the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

(3) Recall from Proposition 2.37 that we have the commutative diagram

$$\begin{array}{ccc}
\text{SU}(2) & \xrightarrow{\rho} & \text{SO}(3) \\
\exp & \uparrow & \exp \\
\mathfrak{su}(2) & \xrightarrow{d\rho_I} & \mathfrak{so}(3).
\end{array}$$

Since $d\rho_I$ is surjective and the exponential map $\exp : \mathfrak{so}(3) \to \text{SO}(3)$ is surjective, conclude that $\rho$ is surjective. Prove that $\text{Ker}\rho = \{I, -I\}$.

**Problem B2 (20).** (a) Let $A$ be any invertible (real) $n \times n$ matrix. Prove that for every SVD, $A = VDU^\top$, of $A$, the product $VU^\top$ is the same (i.e., if $V_1DU_1^\top = V_2DU_2^\top$, then $V_1U_1^\top = V_2U_2^\top$). What does $VU^\top$ have to do with the polar form of $A$?

(b) Given any invertible (real) $n \times n$ matrix, $A$, prove that there is a unique orthogonal matrix, $Q \in \text{O}(n)$, such that $\|A - Q\|_F$ is minimal (under the Frobenius norm). In fact, prove that $Q = VU^\top$, where $A = VDU^\top$ is an SVD of $A$. Moreover, if $\det(A) > 0$, show that $Q \in \text{SO}(n)$.
What can you say if $A$ is singular (i.e., non-invertible)?

**Problem B3 (40 pts).** Consider the action of the group $\text{SL}(2, \mathbb{R})$ on the upper half-plane, $H = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, given by

$$(a \ b) \cdot z = \frac{az + b}{cz + d},$$

(a) Check that for any $g \in \text{SL}(2, \mathbb{R})$,

$$\Im(g \cdot z) = \frac{\Im(z)}{|cz + d|^2},$$

and conclude that if $z \in H$, then $g \cdot z \in H$, so that the action of $\text{SL}(2, \mathbb{R})$ on $H$ is indeed well-defined (Recall, $\Re(z) = x$ and $\Im(z) = y$, where $z = x + iy$.)

(b) Check that if $c \neq 0$, then

$$\frac{az + b}{cz + d} = \frac{-1}{c^2z + cd} + \frac{a}{c}.$$ 

Prove that the group of M"obius transformations induced by $\text{SL}(2, \mathbb{R})$ is generated by M"obius transformations of the form

1. $z \mapsto z + b$,
2. $z \mapsto kz$,
3. $z \mapsto -1/z$,

where $b \in \mathbb{R}$ and $k \in \mathbb{R}$, with $k > 0$. Deduce from the above that the action of $\text{SL}(2, \mathbb{R})$ on $H$ is transitive and that transformations of type (1) and (2) suffice for transitivity.

(c) Now, consider the action of the discrete group $\text{SL}(2, \mathbb{Z})$ on $H$, where $\text{SL}(2, \mathbb{Z})$ consists of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}.$$ 

Why is this action not transitive? Consider the two transformations

$S: z \mapsto -1/z$

associated with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$T: z \mapsto z + 1$

associated with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. 

3
Define the subset, \( D \), of \( H \), as the set of points, \( z = x + iy \), such that \(-1/2 \leq x \leq -1/2\) and \( x^2 + y^2 \geq 1 \). Observe that \( D \) contains the three special points, \( i \), \( \rho = e^{2\pi i/3} \) and \(-\rho = e^{\pi i/3} \).

Draw a picture of this set, known as a fundamental domain of the action of \( G = \text{SL}(2, \mathbb{Z}) \) on \( H \).

Remark: Gauss proved that the group \( G = \text{SL}(2, \mathbb{Z}) \) is generated by \( S \) and \( T \).

**Problem B4 (30 pts).** Let \( J \) be the \( 2 \times 2 \) matrix

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and let \( \text{SU}(1, 1) \) be the set of \( 2 \times 2 \) complex matrices

\[
\text{SU}(1, 1) = \{ A \mid A^* JA = J, \quad \det(A) = 1 \},
\]

where \( A^* \) is the conjugate transpose of \( A \).

(a) Prove that \( \text{SU}(1, 1) \) is the group of matrices of the form

\[
A = \begin{pmatrix} a & b \\ b & \overline{a} \end{pmatrix}, \quad \text{with} \quad a\overline{a} - b\overline{b} = 1.
\]

If

\[
g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}
\]

prove that the map from \( \text{SL}(2, \mathbb{R}) \) to \( \text{SU}(1, 1) \) given by

\[
A \mapsto gAg^{-1}
\]

is a group isomorphism.

(b) Prove that the Möbius transformation associated with \( g \),

\[
z \mapsto \frac{z - i}{z + i}
\]

is a bijection between the upper half-plane, \( H \), and the unit open disk, \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \). Prove that the map from \( \text{SU}(1, 1) \) to \( S^1 \times D \) given by

\[
\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto (a/|a|, b/a)
\]

is a continuous bijection (in fact, a homeomorphism). Conclude that \( \text{SU}(1, 1) \) is topologically an open solid torus.
(c) Check that $\text{SU}(1, 1)$ acts transitively on $D$ by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \cdot z = \frac{az + b}{\bar{b}z + \bar{a}}.$$ 

Find the stabilizer of $z = 0$ and conclude that

$$\text{SU}(1, 1)/\text{SO}(2) \cong D.$$ 

**Problem B5 (80 pts).** Given a finite dimensional Lie algebra $\mathfrak{g}$ (as a vector space over $\mathbb{R}$), we define the function $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$$B(X, Y) = \text{tr}(\text{ad}(X) \circ \text{ad}(Y)), \quad X, Y \in \mathfrak{g}.$$ 

(1) Check that $B$ is $\mathbb{R}$-bilinear and symmetric.

(2) Let $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R}) = M_2(\mathbb{R})$. Given any matrix $A \in M_2(\mathbb{R})$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

show that in the basis $(E_{12}, E_{11}, E_{22}, E_{21})$, the matrix of $\text{ad}(A)$ is given by

$$\begin{pmatrix} a - d & -b & b & 0 \\ -c & 0 & 0 & b \\ c & 0 & 0 & -b \\ 0 & c & -c & d - a \end{pmatrix}.$$ 

Show that

$$\det(xI - \text{ad}(A)) = x^2(x^2 - ((a - d)^2 + 4bc)).$$

(3) Given $A, A' \in M_2(\mathbb{R})$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

prove that

$$B(A, A') = 2(d - a)(d' - a') + 4bc' + 4c'b' = 4\text{tr}(AA') - 2\text{tr}(A)\text{tr}(A').$$

(4) Next, let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Check that the following three matrices form a basis of $\mathfrak{sl}(2, \mathbb{R})$:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
Prove that in the basis \((H, X, Y)\), for any

\[
A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}),
\]

the matrix of \(\text{ad}(A)\) is

\[
\begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}.
\]

Prove that

\[
\det(xI - \text{ad}(A)) = x(x^2 - 4(a^2 + bc)).
\]

(5) Given \(A, A' \in \mathfrak{sl}(2, \mathbb{R})\) with

\[
A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad A' = \begin{pmatrix} a' & b' \\ c' & -a'' \end{pmatrix},
\]

prove that

\[
B(A, A') = 8aa' + 4bc' + 4cb' = 4\text{tr}(AA').
\]

(6) Let \(g = \mathfrak{so}(3)\). For any \(A \in \mathfrak{so}(3)\), with

\[
A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},
\]

we know from Proposition 2.39 that in the basis \((E_1, E_2, E_3)\), the matrix of \(\text{ad}(A)\) is \(A\) itself. Prove that

\[
B(A, A') = -2(aa' + bb' + cc') = \text{tr}(AA').
\]

(7) Recall that a symmetric bilinear form \(B\) is nondegenerate if for every \(X\), if \(B(X, Y) = 0\) for all \(Y\), then \(X = 0\).

Prove that \(B\) on \(\mathfrak{gl}(2, \mathbb{R}) = M_2(\mathbb{R})\) is degenerate; \(B\) on \(\mathfrak{sl}(2, \mathbb{R})\) is nondegenerate but neither positive definite nor negative definite; \(B\) on \(\mathfrak{so}(3)\) is nondegenerate negative definite.

(8) Extra Credit (45) points. Recall that a subspace \(\mathfrak{h}\) of a Lie algebra \(g\) is a subalgebra of \(g\) if \([x, y] \in \mathfrak{h}\) for all \(x, y \in \mathfrak{h}\), and an ideal if \([h, x] \in \mathfrak{h}\) for all \(h \in \mathfrak{h}\) and all \(x \in g\). Check that \(\mathfrak{sl}(n, \mathbb{R})\) is an ideal in \(\mathfrak{gl}(n, \mathbb{R})\), and that \(\mathfrak{so}(n)\) is a subalgebra of \(\mathfrak{sl}(n, \mathbb{R})\), but not an ideal. Prove that if \(\mathfrak{h}\) is an ideal in \(g\), then the bilinear form \(B\) on \(\mathfrak{h}\) is equal to the restriction of the bilinear form \(B\) on \(g\) to \(\mathfrak{h}\).

Prove the following facts: for all \(n \geq 2:\)

\[
\begin{align*}
\mathfrak{gl}(n, \mathbb{R}): \quad & B(X, Y) = 2n\text{tr}(XY) - 2\text{tr}(X)\text{tr}(Y) \\
\mathfrak{sl}(n, \mathbb{R}): \quad & B(X, Y) = 2n\text{tr}(XY) \\
\mathfrak{so}(n): \quad & B(X, Y) = (n - 2)\text{tr}(XY).
\end{align*}
\]
Problem B6 (100 pts). As in Problem B5, consider a finite dimensional Lie algebra $\mathfrak{g}$, but this time a vector space over $\mathbb{C}$, and define the function $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$$B(x, y) = \text{tr}(\text{ad}(x) \circ \text{ad}(y)), \quad x, y \in \mathfrak{g}.$$  

The bilinear form $B$ is called the Killing form of $\mathfrak{g}$. Recall that a homomorphism $\varphi: \mathfrak{g} \to \mathfrak{g}$ is a linear map such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$, or equivalently such that $\varphi \circ \text{ad}(x) = \text{ad}(\varphi(x)) \circ \varphi$, for all $x \in \mathfrak{g}$,

and that an automorphism of $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ that has an inverse which is also a homomorphism of $\mathfrak{g}$.

(1) Prove that for every automorphism $\varphi: \mathfrak{g} \to \mathfrak{g}$, we have

$$B(\varphi(x), \varphi(y)) = B(x, y), \quad \text{for all } x, y \in \mathfrak{g}.$$ 

Prove that for all $x, y, z \in \mathfrak{g}$, we have

$$B(\text{ad}(x)(y), z) = -B(y, \text{ad}(x)(z)),$$

or equivalently

$$B([y, x], z) = B(y, [x, z]).$$

(2) Review the primary decomposition theorem, Section 16.3 of my notes Fundamentals of Linear Algebra and Optimization (linalg.pdf), especially Theorem 16.16. For any $x \in \mathfrak{g}$, we can apply the primary decomposition theorem to the linear map $\text{ad}(x)$. Write

$$m(X) = (X - \lambda_1)^{r_1} \cdots (X - \lambda_k)^{r_k}$$

for the minimal polynomial of $\text{ad}(x)$, where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of $\text{ad}(x)$, and let

$$\mathfrak{g}_x^{\lambda_i} = \text{Ker} (\lambda_i I - \text{ad}(x))^{r_i}, \quad i = 1, \ldots, k.$$  

We know that 0 is an eigenvalue of $\text{ad}(x)$, and we agree that $\lambda_0 = 0$. Then, we have a direct sum

$$\mathfrak{g} = \bigoplus_{\lambda_i} \mathfrak{g}_x^{\lambda_i}.$$  

It is convenient to define $\mathfrak{g}_x^{\lambda}$ when $\lambda$ is not an eigenvalue of $\text{ad}(x)$ as

$$\mathfrak{g}_x^{\lambda} = (0).$$

Prove that

$$[\mathfrak{g}_x^{\lambda}, \mathfrak{g}_x^{\mu}] \subseteq \mathfrak{g}_x^{\lambda + \mu}, \quad \text{for all } \lambda, \mu \in \mathbb{C}.$$
Hint. First, show that
\[
((\lambda + \mu)I - \text{ad}(x))[y, z] = [(\lambda I - \text{ad}(x))(y), z] + [y, (\mu I - \text{ad}(x))(z)],
\]
for all \(x, y, z \in \mathfrak{g}\), and then that
\[
((\lambda + \mu)I - \text{ad}(x))^n[y, z] = \sum_{p=0}^{n} \binom{n}{p} [(\lambda I - \text{ad}(x))^p(y), (\mu I - \text{ad}(x))^{n-p}(z)],
\]
by induction on \(n\).

Prove that \(\mathfrak{g}^0_x\) is a Lie subalgebra of \(\mathfrak{g}\).

(3) Prove that if \(\lambda + \mu \neq 0\), then \(\mathfrak{g}^\lambda_x\) and \(\mathfrak{g}^\mu_x\) are orthogonal with respect to \(B\) (which means that \(B(X, Y) = 0\) for all \(X \in \mathfrak{g}^\lambda_x\) and all \(Y \in \mathfrak{g}^\mu_x\)).

Hint. For any \(X \in \mathfrak{g}^\lambda_x\) and any \(Y \in \mathfrak{g}^\mu_x\), prove that \(\text{ad}(X) \circ \text{ad}(Y)\) is nilpotent. Note that for any \(\nu\) and any \(Z \in \mathfrak{g}^\nu_x\),
\[
(\text{ad}(X) \circ \text{ad}(Y))(Z) = [X, [Y, Z]],
\]
so by (2),
\[
[\mathfrak{g}^\lambda_x, [\mathfrak{g}^\mu_x, \mathfrak{g}^\nu_x]] \subseteq \mathfrak{g}^{\lambda+\mu+\nu}_x.
\]

Conclude that we have an orthogonal direct sum decomposition
\[
\mathfrak{g} = \mathfrak{g}^0_x \oplus \bigoplus_{\lambda \neq 0} (\mathfrak{g}^\lambda_x \oplus \mathfrak{g}^{-\lambda}_x).
\]

Prove that if \(B\) is nondegenerate, then \(B\) is nondegenerate on each of the summands.

**Problem B7 (60 pts).** We can let the group \(\text{SO}(3)\) act on itself by conjugation, so that
\[
R \cdot S = RSR^{-1} = RSR^\top.
\]
The orbits of this action are the *conjugacy classes* of \(\text{SO}(3)\).

(1) Prove that the conjugacy classes of \(\text{SO}(3)\) are in bijection with the following sets:

1. \(C_0 = \{(0,0,0)\}\), the sphere of radius 0.
2. \(C_\theta\), with \(0 < \theta < \pi\) and
\[
C_\theta = \{u \in \mathbb{R}^3 \mid \|u\| = \theta\},
\]
the sphere of radius \(\theta\).
3. \(C_\pi = \mathbb{R}\mathbb{P}^2\), viewed as the quotient of the sphere of radius \(\pi\) by the equivalence relation of being antipodal.
(2) Give $M_3(\mathbb{R})$ the Euclidean structure where
\[
\langle A, B \rangle = \frac{1}{2} \text{tr}(A^\top B).
\]
Consider the following three curves in $\text{SO}(3)$:
\[
c(t) = \begin{pmatrix}
\cos t & -\sin t & 0 \\
\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
for $0 \leq t \leq 2\pi$,
\[
\alpha(\theta) = \begin{pmatrix}
-\cos 2\theta & 0 & \sin 2\theta \\
0 & -1 & 0 \\
\sin 2\theta & 0 & \cos 2\theta
\end{pmatrix},
\]
for $-\pi/2 \leq \theta \leq \pi/2$, and
\[
\beta(\theta) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -\cos 2\theta & \sin 2\theta \\
0 & \sin 2\theta & \cos 2\theta
\end{pmatrix},
\]
for $-\pi/2 \leq \theta \leq \pi/2$.

Check that $c(t)$ is a rotation of angle $t$ and axis $(0,0,1)$, that $\alpha(\theta)$ is a rotation of angle $\pi$ whose axis is in the $(x,z)$-plane, and that $\beta(\theta)$ is a rotation of angle $\pi$ whose axis is in the $(y,z)$-plane. Show that a log of $\alpha(\theta)$ is
\[
B_\alpha = \pi \begin{pmatrix}
0 & -\cos \theta & 0 \\
\cos \theta & 0 & -\sin \theta \\
0 & \sin \theta & 0
\end{pmatrix},
\]
and that a log of $\beta(\theta)$ is
\[
B_\beta = \pi \begin{pmatrix}
0 & -\cos \theta & \sin \theta \\
\cos \theta & 0 & 0 \\
-\sin \theta & 0 & 0
\end{pmatrix}.
\]

(3) The curve $c(t)$ is a closed curve starting and ending at $I$ that intersects $C_\pi$ for $t = \pi$, and $\alpha, \beta$ are contained in $C_\pi$ and coincide with $c(\pi)$ for $\theta = 0$. Compute the derivative $c'(\pi)$ of $c(t)$ at $t = \pi$, and the derivatives $\alpha'(0)$ and $\beta'(0)$, and prove that they are pairwise orthogonal (under the inner product $\langle -,- \rangle$).

Conclude that $c(t)$ intersects $C_\pi$ transversally in $\text{SO}(3)$, which means that
\[
T_{c(\pi)} c + T_{c(\pi)} C_\pi = T_{c(\pi)} \text{SO}(3).
\]
This fact can be used to prove that all closed curves smoothly homotopic to \( c(t) \) must intersect \( C_{\pi} \) transversally, and consequently \( c(t) \) is not (smoothly) homotopic to a point. This implies that \( \text{SO}(3) \) is not simply connected, but this will have to wait for another homework!

**TOTAL:** 410 + 45 points.