Problem B1 (60). (a) Consider the map, $f: \text{GL}(n, \mathbb{R}) \to \mathbb{R}$, given by

$$f(A) = \det(A).$$

Prove that $df(I)(B) = \text{tr}(B)$, the trace of $B$, for any matrix $B$ (here, $I$ is the identity matrix). Then, prove that

$$df(A)(B) = \det(A)\text{tr}(A^{-1}B),$$

where $A \in \text{GL}(n, \mathbb{R})$.

(b) Use the map $A \mapsto \det(A) - 1$ to prove that $\text{SL}(n, \mathbb{R})$ is a manifold of dimension $n^2 - 1$.

(c) Let $J$ be the $(n + 1) \times (n + 1)$ diagonal matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$  

We denote by $\text{SO}(n, 1)$ the group of real $(n + 1) \times (n + 1)$ matrices

$$\text{SO}(n, 1) = \{ A \in \text{GL}(n + 1, \mathbb{R}) \mid A^\top JA = J \quad \text{and} \quad \det(A) = 1 \}.$$  

Check that $\text{SO}(n, 1)$ is indeed a group with the inverse of $A$ given by $A^{-1} = JA^\top J$ (this is the special Lorentz group.) Consider the function $f: \text{GL}^+(n + 1) \to \text{S}(n + 1)$, given by

$$f(A) = A^\top JA - J,$$

where $\text{S}(n + 1)$ denotes the space of $(n + 1) \times (n + 1)$ symmetric matrices. Prove that

$$df(A)(H) = A^\top JH + H^\top JA$$

for any matrix, $H$. Prove that $df(A)$ is surjective for all $A \in \text{SO}(n, 1)$ and that $\text{SO}(n, 1)$ is a manifold of dimension $\frac{n(n+1)}{2}$. 
Problem B2 (30). (a) Given any matrix

\[ B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}), \]

if \( \omega^2 = a^2 + bc \) and \( \omega \) is any of the two complex roots of \( a^2 + bc \), prove that if \( \omega \neq 0 \), then

\[ e^B = \cosh \omega I + \frac{\sinh \omega}{\omega} B, \]

and \( e^B = I + B \), if \( a^2 + bc = 0 \). Observe that \( \text{tr}(e^B) = 2 \cosh \omega \).

Prove that the exponential map, \( \exp: \mathfrak{sl}(2, \mathbb{C}) \to \text{SL}(2, \mathbb{C}) \), is not surjective. For instance, prove that

\[ \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \]

is not the exponential of any matrix in \( \mathfrak{sl}(2, \mathbb{C}) \).

Problem B3 (60 pts). Given a group \( G \), recall that its center is the subset

\[ Z(G) = \{ a \in G, \ ag = ga \text{ for all } g \in G \}. \]

(1) Check that \( Z(G) \) is a commutative normal subgroup of \( G \).

(2) Prove that a matrix \( A \in \text{M}_n(\mathbb{R}) \) commutes with all matrices \( B \in \text{GL}(n, \mathbb{R}) \) iff \( A = \lambda I \) for some \( \lambda \in \mathbb{R} \).

Hint. Remember the elementary matrices.

Prove that

\[ Z(\text{GL}(n, \mathbb{R})) = \{ \lambda I \ | \ \lambda \in \mathbb{R}, \lambda \neq 0 \}. \]

(3) Prove that for any \( m \geq 1 \),

\[ Z(\text{SO}(2m + 1))) = \{ I, -I \} \]

\[ Z(\text{SO}(2m - 1)) = \{ I \} \]

\[ Z(\text{SL}(m, \mathbb{R})) = \{ \lambda I \ | \ \lambda \in \mathbb{R}, \lambda^m = 1 \}. \]

(4) Prove that a matrix \( A \in \text{M}_n(\mathbb{C}) \) commutes with all matrices \( B \in \text{GL}(n, \mathbb{C}) \) iff \( A = \lambda I \) for some \( \lambda \in \mathbb{C} \).

(5) Prove that for any \( n \geq 1 \),

\[ Z(\text{GL}(n, \mathbb{C})) = \{ \lambda I \ | \ \lambda \in \mathbb{C}, \lambda \neq 0 \} \]

\[ Z(\text{SL}(n, \mathbb{C})) = \{ e^{\frac{2k\pi i}{n}} I \ | \ k = 0, 1, \ldots, n - 1 \} \]

\[ Z(\text{U}(n)) = \{ e^{i\theta} I \ | \ 0 \leq \theta < 2\pi \} \]

\[ Z(\text{SU}(n)) = \{ e^{\frac{k2\pi i}{n}} I \ | \ k = 0, 1, \ldots, n - 1 \}. \]
(6) Prove that the groups $\text{SO}(3)$ and $\text{SU}(2)$ are not isomorphic (although their Lie algebras are isomorphic).

**Problem B4 (120 pts).** Recall from Homework 1, Problem B6, the Cayley parametrization of rotation matrices in $\text{SO}(n)$ given by

$$C(B) = (I - B)(I + B)^{-1},$$

where $B$ is any $n \times n$ skew symmetric matrix.

(a) Now, consider $n = 3$, i.e., $\text{SO}(3)$. Let $E_1$, $E_2$ and $E_3$ be the rotations about the $x$-axis, $y$-axis, and $z$-axis, respectively, by the angle $\pi$, i.e.,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that the four maps

$$B \mapsto C(B)$$

$$B \mapsto E_1C(B)$$

$$B \mapsto E_2C(B)$$

$$B \mapsto E_3C(B)$$

where $B$ is skew symmetric, are parametrizations of $\text{SO}(3)$ and that the union of the images of $C$, $E_1C$, $E_2C$ and $E_3C$ covers $\text{SO}(3)$, so that $\text{SO}(3)$ is a manifold.

(b) Let $A$ be any matrix (not necessarily invertible). Prove that there is some diagonal matrix, $E$, with entries $+1$ or $-1$, so that $EA + I$ is invertible.

(c) Prove that every rotation matrix, $A \in \text{SO}(n)$, is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$, and where the number of $-1$ is even. Moreover, prove that every orthogonal matrix $A \in \text{O}(n)$ is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$. The above provide parametrizations for $\text{SO}(n)$ (resp. $\text{O}(n)$) that show that $\text{SO}(n)$ and $\text{O}(n)$ are manifolds. However, observe that the number of these charts grows exponentially with $n$. 

3
Problem B5 (30). Consider the parametric surface given by

\[
\begin{align*}
    x(u, v) &= \frac{8uv}{(u^2 + v^2 + 1)^2}, \\
    y(u, v) &= \frac{4v(u^2 + v^2 - 1)}{(u^2 + v^2 + 1)^2}, \\
    z(u, v) &= \frac{4(u^2 - v^2)}{(u^2 + v^2 + 1)^2}.
\end{align*}
\]

The trace of this surface is called a \textit{crosscap}. In order to plot this surface, make the change of variables

\[
\begin{align*}
    u &= \rho \cos \theta \\
    v &= \rho \sin \theta.
\end{align*}
\]

Prove that we obtain the parametric definition

\[
\begin{align*}
    x &= \frac{4\rho^2}{(\rho^2 + 1)^2} \sin 2\theta, \\
    y &= \frac{4\rho(\rho^2 - 1)}{(\rho^2 + 1)^2} \sin \theta, \\
    z &= \frac{4\rho^2}{(\rho^2 + 1)^2} \cos 2\theta.
\end{align*}
\]

Show that the entire trace of the surface is obtained for \(\rho \in [0, 1]\) and \(\theta \in [-\pi, \pi]\).

\textit{Hint.} What happens if you change \(\rho\) to \(1/\rho\)?

Plot the trace of the surface using the above parametrization. Show that there is a line of self-intersection along the portion of the \(z\)-axis corresponding to \(0 \leq z \leq 1\). What can you say about the point corresponding to \(\rho = 1\) and \(\theta = 0\)?

Plot the portion of the surface for \(\rho \in [0, 1]\) and \(\theta \in [0, \pi]\).

(b) Express the trigonometric functions in terms of \(u = \tan(\theta/2)\), and letting \(v = \rho\), show that we get

\[
\begin{align*}
    x &= \frac{16uv^2(1 - u^2)}{(u^2 + 1)^2(v^2 + 1)^2}, \\
    y &= \frac{8uv(u^2 + 1)(v^2 - 1)}{(u^2 + 1)^2(v^2 + 1)^2}, \\
    z &= \frac{4v^2(u^4 - 6u^2 + 1)}{(u^2 + 1)^2(v^2 + 1)^2}.
\end{align*}
\]
Problem B6 (30). Consider the parametric surface given by

\[
\begin{aligned}
x(u, v) &= \frac{4v(u^2 + v^2 - 1)}{(u^2 + v^2 + 1)^2}, \\
y(u, v) &= \frac{4u(u^2 + v^2 - 1)}{(u^2 + v^2 + 1)^2}, \\
z(u, v) &= \frac{4(u^2 - v^2)}{(u^2 + v^2 + 1)^2}.
\end{aligned}
\]

The trace of this surface is called the *Steiner Roman surface*. In order to plot this surface, make the change of variables

\[
\begin{aligned}
u &= \rho \cos \theta \\
v &= \rho \sin \theta.
\end{aligned}
\]

Prove that we obtain the parametric definition

\[
\begin{aligned}
x &= \frac{4\rho(\rho^2 - 1)}{(\rho^2 + 1)^2} \sin \theta, \\
y &= \frac{4\rho(\rho^2 - 1)}{(\rho^2 + 1)^2} \cos \theta, \\
z &= \frac{4\rho^2}{(\rho^2 + 1)^2} \cos 2\theta.
\end{aligned}
\]

Show that the entire trace of the surface is obtained for \(\rho \in [0, 1]\) and \(\theta \in [-\pi, \pi]\). Plot the trace of the surface using the above parametrization.

Plot the portion of the surface for \(\rho \in [0, 1]\) and \(\theta \in [0, \pi]\).

Prove that this surface has five singular points.

(b) Express the trigonometric functions in terms of \(u = \tan(\theta/2)\), and letting \(v = \rho\), show that we get

\[
\begin{aligned}
x &= \frac{8uv(u^2 + 1)(v^2 - 1)}{(u^2 + 1)^2(v^2 + 1)^2}, \\
y &= \frac{4v(1 - u^4)(v^2 - 1)}{(u^2 + 1)^2(v^2 + 1)^2}, \\
z &= \frac{4v^2(u^4 - 6u^2 + 1)}{(u^2 + 1)^2(v^2 + 1)^2}.
\end{aligned}
\]

Problem B7 (160). Consider the map \(\mathcal{H}: \mathbb{R}^3 \to \mathbb{R}^4\) defined such that

\[(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2).\]
Prove that when it is restricted to the sphere $S^2$ (in $\mathbb{R}^3$), we have $H(x, y, z) = H(x', y', z')$ iff $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$. In other words, the inverse image of every point in $H(S^2)$ consists of two antipodal points.

(a) Prove that the map $H$ induces an injective map from the projective plane onto $H(S^2)$, and that it is a homeomorphism.

(b) The map $H$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$
\psi_1(u, v) = \left(\frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{u}{u^2 + v^2 + 1}, \frac{u^2 - v^2}{u^2 + v^2 + 1}\right),
$$

$$
\psi_2(u, v) = \left(\frac{u}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{u^2 - 1}{u^2 + v^2 + 1}\right),
$$

$$
\psi_3(u, v) = \left(\frac{u}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{1 - u^2}{u^2 + v^2 + 1}\right).
$$

Observe that $\psi_1$ is the composition $H \circ \alpha_1$, where $\alpha_1: \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}\right),$$

that $\psi_2$ is the composition $H \circ \alpha_2$, where $\alpha_2: \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}\right),$$

and $\psi_3$ is the composition $H \circ \alpha_3$, where $\alpha_3: \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}\right).$$

Prove that each $\psi_i$ is injective, continuous and nonsingular (i.e., the Jacobian has rank 2).

(c) Prove that if $\psi_1(u, v) = (x, y, z, t)$, then

$$y^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad y^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that if $\psi_1(u, v) = (x, y, z, t)$, then $u$ and $v$ satisfy the equations

$$(y^2 + z^2)u^2 - zu + z^2 = 0$$

$$(y^2 + z^2)v^2 - yv + y^2 = 0.$$

Prove that if $y^2 + z^2 \neq 0$, then

$$u = \frac{z(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,$$
\[
\begin{align*}
    u &= \frac{z(1 + \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if } u^2 + v^2 \geq 1
\end{align*}
\]

and there are similar formulae for \(v\). Prove that the expression giving \(u\) in terms of \(y\) and \(z\) is continuous everywhere in \(\{(y, z) \mid y^2 + z^2 \leq \frac{1}{4}\}\) and similarly for the expression giving \(v\) in terms of \(y\) and \(z\). Conclude that \(\psi_1 : \mathbb{R}^2 \to \psi_1(\mathbb{R}^2)\) is a homeomorphism onto its image. Therefore, \(U_1 = \psi_1(\mathbb{R}^2)\) is an open subset of \(\mathcal{H}(S^2)\).

Prove that if \(\psi_2(u, v) = (x, y, z, t)\), then
\[
    x^2 + y^2 \leq \frac{1}{4} \quad \text{and} \quad x^2 + y^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.
\]

Prove that if \(\psi_2(u, v) = (x, y, z, t)\), then \(u\) and \(v\) satisfy the equations
\[
\begin{align*}
    (x^2 + y^2)u^2 - xu + x^2 &= 0 \\
    (x^2 + y^2)v^2 - yv + y^2 &= 0.
\end{align*}
\]

Conclude that \(\psi_2 : \mathbb{R}^2 \to \psi_2(\mathbb{R}^2)\) is a homeomorphism onto its image and that the set \(U_2 = \psi_2(\mathbb{R}^2)\) is an open subset of \(\mathcal{H}(S^2)\).

Prove that if \(\psi_3(u, v) = (x, y, z, t)\), then
\[
    x^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad x^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.
\]

Prove that if \(\psi_3(u, v) = (x, y, z, t)\), then \(u\) and \(v\) satisfy the equations
\[
\begin{align*}
    (x^2 + z^2)u^2 - xu + x^2 &= 0 \\
    (x^2 + z^2)v^2 - zv + z^2 &= 0.
\end{align*}
\]

Conclude that \(\psi_3 : \mathbb{R}^2 \to \psi_3(\mathbb{R}^2)\) is a homeomorphism onto its image and that the set \(U_3 = \psi_3(\mathbb{R}^2)\) is an open subset of \(\mathcal{H}(S^2)\).

Prove that the union of the \(U_i\)'s covers \(\mathcal{H}(S^2)\). Conclude that \(\psi_1, \psi_2, \psi_3\) are parametrizations of \(\mathbb{RP}^2\) as a smooth manifold in \(\mathbb{R}^4\).

(d) Plot the surfaces obtained by dropping the fourth coordinate and the third coordinates, respectively (with \(u, v \in [-1, 1]\)).

(e) Prove that if \((x, y, z, t) \in \mathcal{H}(S^2)\), then
\[
\begin{align*}
    x^2y^2 + x^2z^2 + y^2z^2 &= xyz \\
    x(z^2 - y^2) &= yzt.
\end{align*}
\]

Prove that the zero locus of these equations strictly contains \(\mathcal{H}(S^2)\). This is a “famous mistake” of Hilbert and Cohn-Vossen in *Geometry and the Imagination*!
Finding a set of equations defining exactly $\mathcal{H}(S^2)$ appears to be an open problem.

**Problem B8 (80).** Recall that $\text{ad}_A = L_A - R_A$, and that $L_A$ and $R_A$ commute. Prove that

$$d(\exp)_A = e^{L_A} \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} (L_A - R_A)^j.$$ 

*Hint.* Recall from Homework 1 Problem B3 that

$$d(\exp)_A = \sum_{h,k \geq 0} \frac{L_A^h R_A^k}{(h+k+1)!}.$$ 

To simplify notation, write $a$ for $L_A$ and $b$ for $L_B$. Then, the problem is to prove that

$$e^a \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} (a - b)^j = \sum_{h,k \geq 0} \frac{a^h b^k}{(h+k+1)!},$$

assuming that $ab = ba$.

Expand the expression on the left and equate the coefficients of the monomial $a^h b^k$. To conclude, you will need to prove the following identity:

$$\sum_{i=0}^{h} (-1)^{h-i} \binom{h+k+1}{i} \binom{h+k-i}{k} = 1.$$ 

**TOTAL:** 570 points.