Fall, 2018   CIS 515

Fundamentals of Linear Algebra and Optimization
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Homework 6

December 3, 2018; Due December 13, 2018

Problem B1 (30 pts). Let E be a real vector space of finite dimension, $n \geq 1$. Say that two bases, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, of E have the *same orientation* iff $\det(P) > 0$, where P the change of basis matrix from $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, namely, the matrix whose jth columns consist of the coordinates of v_j over the basis $(u_1, \ldots, u_n)$.

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An *orientation* of a vector space, E, is the choice of any fixed basis, say $(e_1, \ldots, e_n)$, of E. Any other basis, $(v_1, \ldots, v_n)$, has the *same orientation* as $(e_1, \ldots, e_n)$ (and is said to be *positive* or *direct*) iff $\det(P) > 0$, else it is said to have the *opposite orientation* of $(e_1, \ldots, e_n)$ (or to be *negative* or *indirect*), where P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(v_1, \ldots, v_n)$. An *oriented* vector space is a vector space with some chosen orientation (a positive basis).

(b) Let $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ be two orthonormal bases. For any sequence of vectors, $(w_1, \ldots, w_n)$, in E, let $\det_{B_1}(w_1, \ldots, w_n)$ be the determinant of the matrix whose columns are the coordinates of the w_j's over the basis B_1 and similarly for $\det_{B_2}(w_1, \ldots, w_n)$.

Prove that if B_1 and B_2 have the same orientation, then

$$\det_{B_1}(w_1, \ldots, w_n) = \det_{B_2}(w_1, \ldots, w_n).$$

Given any oriented vector space, E, for any sequence of vectors, $(w_1, \ldots, w_n)$, in E, the common value, $\det_B(w_1, \ldots, w_n)$, for all positive orthonormal bases, B, of E is denoted

$$\lambda_E(w_1, \ldots, w_n)$$

and called a *volume form* of $(w_1, \ldots, w_n)$.

(c) Given any Euclidean oriented vector space, E, of dimension n for any $n-1$ vectors, $w_1, \ldots, w_{n-1}$, in E, check that the map

$$x \mapsto \lambda_E(w_1, \ldots, w_{n-1}, x)$$

is a linear form. Then, prove that there is a unique vector, denoted $w_1 \times \cdots \times w_{n-1}$, such that

$$\lambda_E(w_1, \ldots, w_{n-1}, x) = (w_1 \times \cdots \times w_{n-1}) \cdot x,$$

for all $x \in E$. The vector $w_1 \times \cdots \times w_{n-1}$ is called the *cross-product* of $(w_1, \ldots, w_{n-1})$. It is a generalization of the cross-product in $\mathbb{R}^3$ (when $n = 3$).

Problem B2 (50 pts). Given p vectors $(u_1, \ldots, u_p)$ in a Euclidean space E of dimension $n \geq p$, the *Gram determinant (or Gramian)* of the vectors $(u_1, \ldots, u_p)$ is the determinant

$$\mathrm{Gram}(u_1, \ldots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2\rangle & \ldots & \langle u_1, u_p\rangle \\ \langle u_2, u_1\rangle & \|u_2\|^2 & \ldots & \langle u_2, u_p\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1\rangle & \langle u_p, u_2\rangle & \ldots & \|u_p\|^2 \end{vmatrix}.$$

(1) Prove that

$$\mathrm{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.$$

Hint. If $(e_1, \ldots, e_n)$ is an orthonormal basis and A is the matrix of the vectors $(u_1, \ldots, u_n)$ over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),$$

where A^i denotes the ith column of the matrix A, and $(A^i \cdot A^j)$ denotes the $n \times n$ matrix with entries $A^i \cdot A^j$.

(2) Prove that

$$\|u_1 \times \cdots \times u_{n-1}\|^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}).$$

Hint. Letting $w = u_1 \times \cdots \times u_{n-1}$, observe that

$$\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w\rangle = \|w\|^2,$$

and show that

$$\begin{aligned} \|w\|^4 &= \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}, w) \\ &= \mathrm{Gram}(u_1, \ldots, u_{n-1})\|w\|^2. \end{aligned}$$

Problem B3 (20 pts). Let $\varphi\colon E \times E \to \mathbb{R}$ be a bilinear form on a real vector space E of finite dimension n. Given any basis $(e_1, \ldots, e_n)$ of E, let $A = (a_{i\,j})$ be the matrix defined such that

$$a_{i\,j} = \varphi(e_i, e_j),$$

$1 \leq i, j \leq n$. We call A *the matrix of* φ *w.r.t. the basis* $(e_1, \ldots, e_n)$.

(a) For any two vectors x and y, if X and Y denote the column vectors of coordinates of x and y w.r.t. the basis $(e_1, \ldots, e_n)$, prove that

$$\varphi(x, y) = X^\top AY.$$

(b) Recall that A is a *symmetric* matrix if $A = A^\top$. Prove that φ is symmetric if A is a symmetric matrix.

(c) If $(f_1, \ldots, f_n)$ is another basis of E and P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, prove that the matrix of φ w.r.t. the basis $(f_1, \ldots, f_n)$ is

$$P^\top AP.$$

The common rank of all matrices representing φ is called the *rank* of φ.

Problem B4 (50 pts). Let $\varphi\colon E \times E \to \mathbb{R}$ be a symmetric bilinear form on a real vector space E of finite dimension n. Two vectors x and y are said to be *conjugate or orthogonal w.r.t.* φ if $\varphi(x, y) = 0$. The main purpose of this problem is to prove that there is a basis of vectors that are pairwise conjugate w.r.t. φ.

(a) Prove that if $\varphi(x, x) = 0$ for all $x \in E$, then φ is identically null on E.

Otherwise, we can assume that there is some vector $x \in E$ such that $\varphi(x, x) \neq 0$.

Use induction to prove that there is a basis of vectors $(u_1, \ldots, u_n)$ that are pairwise conjugate w.r.t. φ.

Hint. For the induction step, proceed as follows. Let $(u_1, e_2, \ldots, e_n)$ be a basis of E, with $\varphi(u_1, u_1) \neq 0$. Prove that there are scalars $\lambda_2, \ldots, \lambda_n$ such that each of the vectors

$$v_i = e_i + \lambda_i u_1$$

is conjugate to u_1 w.r.t. φ, where $2 \leq i \leq n$, and that $(u_1, v_2, \ldots, v_n)$ is a basis.

(b) Let $(e_1, \ldots, e_n)$ be a basis of vectors that are pairwise conjugate w.r.t. φ, and assume that they are ordered such that

$$\varphi(e_i, e_i) = \begin{cases} \theta_i \neq 0 & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } r + 1 \leq i \leq n, \end{cases}$$

where r is the rank of φ. Show that the matrix of φ w.r.t. $(e_1, \ldots, e_n)$ is a diagonal matrix, and that

$$\varphi(x, y) = \sum_{i=1}^{r} \theta_i x_i y_i,$$

where $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{i=1}^{n} y_i e_i$.

Prove that for every symmetric matrix A, there is an invertible matrix P such that

$$P^\top AP = D,$$

where D is a diagonal matrix.

(c) Prove that there is an integer p, $0 \leq p \leq r$ (where r is the rank of φ), such that $\varphi(u_i, u_i) > 0$ for exactly p vectors of every basis $(u_1, \ldots, u_n)$ of vectors that are pairwise conjugate w.r.t. φ (*Sylvester's inertia theorem*).

Proceed as follows. Assume that in the basis $(u_1, \ldots, u_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \alpha_1 x_1^2 + \cdots + \alpha_p x_p^2 - \alpha_{p+1} x_{p+1}^2 - \cdots - \alpha_r x_r^2,$$

where $x = \sum_{i=1}^n x_i u_i$, and that in the basis $(v_1, \ldots, v_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \beta_1 y_1^2 + \cdots + \beta_q y_q^2 - \beta_{q+1} y_{q+1}^2 - \cdots - \beta_r y_r^2,$$

where $x = \sum_{i=1}^n y_i v_i$, with $\alpha_i > 0$, $\beta_i > 0$, $1 \leq i \leq r$.

Assume that $p > q$ and derive a contradiction. First, consider x in the subspace F spanned by

$$(u_1, \ldots, u_p, u_{r+1}, \ldots, u_n),$$

and observe that $\varphi(x, x) \geq 0$ if $x \neq 0$. Next, consider x in the subspace G spanned by

$$(v_{q+1}, \ldots, v_r),$$

and observe that $\varphi(x, x) < 0$ if $x \neq 0$. Prove that $F \cap G$ is nontrivial (i.e., contains some nonnull vector), and derive a contradiction. This implies that $p \leq q$. Finish the proof.

The pair $(p, r - p)$ is called the *signature* of φ.

(d) A symmetric bilinear form φ is *definite* if for every $x \in E$, if $\varphi(x, x) = 0$, then $x = 0$.

Prove that a symmetric bilinear form is definite iff its signature is either $(n, 0)$ or $(0, n)$. In other words, a symmetric definite bilinear form has rank n and is either positive or negative.

Problem B5 (40 pts). Let H be a symmetric positive definite matrix and let K be any symmetric matrix.

(1) Prove that HK is diagonalizable, with real eigenvalues.

(2) If K is also positive definite, then prove that the eigenvalues of HK are positive.

(3) Prove that the number of positive (resp. negative) eigenvalues of HK is equal to the number of positive (resp. negative) eigenvalues of K.

Let A be any real or complex $n \times n$ matrix. It can be shown that the sequence (E_m) of matrices

$$E_m = I + \sum_{k=1}^m \frac{A^k}{k!}$$

converges to a limit denoted

$$e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}$$

and called the *exponential* of A. You may accept this fact without proof.

Problem B6 (Extra Credit 10 pts).

Let $\| \ \|$ be any operator norm. Prove that for every $m \geq 1$,

$$\|I\| + \sum_{k=1}^{m} \left\| \frac{A^k}{k!} \right\| \leq e^{\|A\|}.$$

If you know some analysis, deduce from the above that the sequence (E_m) of matrices

$$E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}$$

converges to a limit denoted e^A, and called the *exponential* of A.

Problem B7 (100 pts). (a) Let $\mathfrak{so}(3)$ be the space of 3×3 skew symmetric matrices

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

prove that

$$\begin{aligned} A^2 &= -\theta^2 I + B, \\ AB &= BA = 0. \end{aligned}$$

From the above, deduce that

$$A^3 = -\theta^2 A.$$

(b) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is given by

$$\exp A = e^A = \cos\theta\, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2, \quad \text{if } \theta \neq 0,$$

with $\exp(0_3) = I_3$.

(c) Prove that e^A is an orthogonal matrix of determinant $+1$, i.e., a rotation matrix.

(d) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective. For this, proceed as follows: Pick any rotation matrix $R \in \mathbf{SO}(3)$;

(1) The case $R = I$ is trivial.

(2) If $R \neq I$ and $\mathrm{tr}(R) \neq -1$, then

$$\exp^{-1}(R) = \left\{ \frac{\theta}{2\sin\theta}(R - R^T) \;\middle|\; 1 + 2\cos\theta = \mathrm{tr}(R) \right\}.$$

(Recall that $\mathrm{tr}(R) = r_{11} + r_{22} + r_{33}$, the *trace* of the matrix R).

Show that there is a unique skew-symmetric B with corresponding θ satisfying $0 < \theta < \pi$ such that $e^B = R$.

(3) If $R \neq I$ and $\mathrm{tr}(R) = -1$, then prove that the eigenvalues of R are $1, -1, -1$, that $R = R^\top$, and that $R^2 = I$. Prove that the matrix

$$S = \frac{1}{2}(R - I)$$

is a symmetric matrix whose eigenvalues are $-1, -1, 0$. Thus, S can be diagonalized with respect to an orthogonal matrix Q as

$$S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^\top.$$

Prove that there exists a skew symmetric matrix

$$U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}$$

so that

$$U^2 = S = \frac{1}{2}(R - I).$$

Observe that

$$U^2 = \begin{pmatrix} -(c^2 + d^2) & bc & bd \\ bc & -(b^2 + d^2) & cd \\ bd & cd & -(b^2 + c^2) \end{pmatrix},$$

and use this to conclude that if $U^2 = S$, then $b^2 + c^2 + d^2 = 1$. Then, show that

$$\exp^{-1}(R) = \left\{ (2k+1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}, \ k \in \mathbb{Z} \right\},$$

where (b, c, d) is any unit vector such that for the corresponding skew symmetric matrix U, we have $U^2 = S$.

(e) To find a skew symmetric matrix U so that $U^2 = S = \frac{1}{2}(R - I)$ as in (d), we can solve the system

$$\begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S.$$

We immediately get b^2, c^2, d^2, and then, since one of b, c, d is nonzero, say b, if we choose the positive square root of b^2, we can determine c and d from bc and bd.

Implement a computer program to solve the above system.

Problem B8 (120 pts). (a) Consider the set of affine maps ρ of $\mathbb{R}^3$ defined such that

$$\rho(X) = \alpha RX + W,$$

where R is a rotation matrix (an orthogonal matrix of determinant $+1$), W is some vector in $\mathbb{R}^3$, and $\alpha \in \mathbb{R}$ with $\alpha > 0$. Every such a map can be represented by the 4×4 matrix

$$\begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = \alpha RX + W.$$

Prove that these maps form a group, denoted by $\mathbf{SIM}(3)$ (the *direct affine similitudes* of $\mathbb{R}^3$).

(b) Let us now consider the set of 4×4 real matrices of the form

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix},$$

where Γ is a matrix of the form

$$\Gamma = \lambda I_3 + \Omega,$$

with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

so that

$$\Gamma = \begin{pmatrix} \lambda & -c & b \\ c & \lambda & -a \\ -b & a & \lambda \end{pmatrix},$$

and W is a vector in $\mathbb{R}^3$.

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^7, +)$. This vector space is denoted by $\mathfrak{sim}(3)$.

(c) Given a matrix

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix}$$

as in (b), prove that

$$B^n = \begin{pmatrix} \Gamma^n & \Gamma^{n-1}W \\ 0 & 0 \end{pmatrix}$$

where $\Gamma^0 = I_3$. Prove that

$$e^B = \begin{pmatrix} e^\Gamma & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!}.$$

(d) Prove that if $\Gamma = \lambda I_3 + \Omega$ as in (b), then

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!} = \int_0^1 e^{\Gamma t} dt.$$

(e) For any matrix $\Gamma = \lambda I_3 + \Omega$, with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$, then prove that

$$e^{\Gamma} = e^{\lambda} e^{\Omega} = e^{\lambda}\left(I_3 + \frac{\sin\theta}{\theta}\Omega + \frac{(1-\cos\theta)}{\theta^2}\Omega^2\right), \quad \text{if } \theta \neq 0,$$

and $e^{\Gamma} = e^{\lambda} I_3$ if $\theta = 0$.

Hint. You may use the fact that if $AB = BA$, then $e^{A+B} = e^A e^B$. In general, $e^{A+B} \neq e^A e^B$!

(f) Prove that

1. If $\theta = 0$ and $\lambda = 0$, then
$$V = I_3.$$
2. If $\theta = 0$ and $\lambda \neq 0$, then
$$V = \frac{(e^{\lambda} - 1)}{\lambda} I_3;$$
3. If $\theta \neq 0$ and $\lambda = 0$, then
$$V = I_3 + \frac{(1-\cos\theta)}{\theta^2}\Omega + \frac{(\theta - \sin\theta)}{\theta^3}\Omega^2.$$
4. If $\theta \neq 0$ and $\lambda \neq 0$, then
$$V = \frac{(e^{\lambda} - 1)}{\lambda} I_3 + \frac{(\theta(1 - e^{\lambda}\cos\theta) + e^{\lambda}\lambda\sin\theta)}{\theta(\lambda^2 + \theta^2)}\Omega + \left(\frac{(e^{\lambda}-1)}{\lambda\theta^2} - \frac{e^{\lambda}\sin\theta}{\theta(\lambda^2+\theta^2)} - \frac{\lambda(e^{\lambda}\cos\theta - 1)}{\theta^2(\lambda^2+\theta^2)}\right)\Omega^2.$$

Hint. You will need to compute $\int_0^1 e^{\lambda t}\sin\theta t\, dt$ and $\int_0^1 e^{\lambda t}\cos\theta t\, dt$.

(g) Prove that V is invertible iff $\lambda \neq 0$ or $\theta \neq k2\pi$, with $k \in \mathbb{Z} - \{0\}$.

Hint. Express the eigenvalues of V in terms of the eigenvalues of Γ.

In the special case where $\lambda = 0$, show that

$$V^{-1} = I - \frac{1}{2}\Omega + \frac{1}{\theta^2}\left(1 - \frac{\theta\sin\theta}{2(1-\cos\theta)}\right)\Omega^2, \quad \text{if } \theta \neq 0.$$

Hint. Assume that the inverse of V is of the form

$$Z = I_3 + a\Omega + b\Omega^2,$$

and show that a, b, are given by a system of linear equations that always has a unique solution.

(h) Prove that the exponential map $\exp\colon \mathfrak{sim}(3) \to \mathbf{SIM}(3)$, given by $\exp(B) = e^B$, is surjective. You may use the fact that $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective, proved in another Problem.

Remark: Curves in $\mathbf{SIM}(3)$ can be used to describe certain deformations of bodies in $\mathbb{R}^3$.

TOTAL: 410 points+ 10 points Extra credit