Problem B1 (40 pts). Let $H$ be a symmetric positive definite matrix and let $K$ be any symmetric matrix.

(1) Prove that $HK$ is diagonalizable, with real eigenvalues.

(2) If $K$ is also positive definite, then prove that the eigenvalues of $HK$ are positive.

(3) Prove that the number of positive (resp. negative) eigenvalues of $HK$ is equal to the number of positive (resp. negative) eigenvalues of $K$.

Let $A$ be any real or complex $n \times n$ matrix. It can be shown that the sequence $(E_m)$ of matrices

$$E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}$$

converges to a limit denoted

$$e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}$$

and called the exponential of $A$. You may accept this fact without proof.

Problem B2 (Extra Credit 10 pts).

Let $\| \|$ be any operator norm. Prove that for every $m \geq 1$,

$$\|I\| + \sum_{k=1}^{m} \left\| \frac{A^k}{k!} \right\| \leq e^{\|A\|}.$$ 

If you know some analysis, deduce from the above that the sequence $(E_m)$ of matrices

$$E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}$$

converges to a limit denoted $e^A$, and called the exponential of $A$. 

Problem B3 (100 pts). (a) Let $\mathfrak{so}(3)$ be the space of $3 \times 3$ skew symmetric matrices

$$
\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -c & b \\
c & 0 & -a \\
-b & a & 0 \end{pmatrix} \bigg| a, b, c \in \mathbb{R} \right\}.
$$

For any matrix

$$
A = \begin{pmatrix} 0 & -c & b \\
c & 0 & -a \\
-b & a & 0 \end{pmatrix} \in \mathfrak{so}(3),
$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$
B = \begin{pmatrix} a^2 & ab & ac \\
ab & b^2 & bc \\
ac & bc & c^2 \end{pmatrix},
$$

prove that

$$
A^2 = -\theta^2 I + B,
$$
$$
AB = BA = 0.
$$

From the above, deduce that

$$
A^3 = -\theta^2 A.
$$

(b) Prove that the exponential map $\exp: \mathfrak{so}(3) \to \text{SO}(3)$ is given by

$$
\exp A = e^A = \cos \theta \ I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} B,
$$

or, equivalently, by

$$
e^A = I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2, \quad \text{if } \theta \neq 0,
$$

with $\exp(0_3) = I_3$.

(c) Prove that $e^A$ is an orthogonal matrix of determinant +1, i.e., a rotation matrix.

(d) Prove that the exponential map $\exp: \mathfrak{so}(3) \to \text{SO}(3)$ is surjective. For this, proceed as follows: Pick any rotation matrix $R \in \text{SO}(3);

1. The case $R = I$ is trivial.

2. If $R \neq I$ and $\text{tr}(R) \neq -1$, then

$$
\exp^{-1}(R) = \left\{ \frac{\theta}{2 \sin \theta} (R - R^T) \bigg| 1 + 2 \cos \theta = \text{tr}(R) \right\}.
$$

(Recall that $\text{tr}(R) = r_{11} + r_{22} + r_{33}$, the trace of the matrix $R$).

Show that there is a unique skew-symmetric $B$ with corresponding $\theta$ satisfying $0 < \theta < \pi$ such that $e^B = R$. 

2
(3) If $R \neq I$ and $\text{tr}(R) = -1$, then prove that the eigenvalues of $R$ are $1, -1, -1$, that $R = R^\top$, and that $R^2 = I$. Prove that the matrix

$$
S = \frac{1}{2}(R - I)
$$

is a symmetric matrix whose eigenvalues are $-1, -1, 0$. Thus, $S$ can be diagonalized with respect to an orthogonal matrix $Q$ as

$$
S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^\top.
$$

Prove that there exists a skew symmetric matrix

$$
U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}
$$

so that

$$
U^2 = S = \frac{1}{2}(R - I).
$$

Observe that

$$
U^2 = \begin{pmatrix} -(c^2 + d^2) & bc & bd \\ bc & -(b^2 + d^2) & cd \\ bd & cd & -(b^2 + c^2) \end{pmatrix},
$$

and use this to conclude that if $U^2 = S$, then $b^2 + c^2 + d^2 = 1$. Then, show that

$$
\exp^{-1}(R) = \left\{ (2k + 1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}, k \in \mathbb{Z} \right\},
$$

where $(b, c, d)$ is any unit vector such that for the corresponding skew symmetric matrix $U$, we have $U^2 = S$.

(e) To find a skew symmetric matrix $U$ so that $U^2 = S = \frac{1}{2}(R - I)$ as in (d), we can solve the system

$$
\begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S.
$$

We immediately get $b^2, c^2, d^2$, and then, since one of $b, c, d$ is nonzero, say $b$, if we choose the positive square root of $b^2$, we can determine $c$ and $d$ from $bc$ and $bd$.

Implement a computer program to solve the above system.
**Problem B4 (120 pts).** (a) Consider the set of affine maps \( \rho \) of \( \mathbb{R}^3 \) defined such that

\[
\rho(X) = \alpha R X + W,
\]

where \( R \) is a rotation matrix (an orthogonal matrix of determinant +1), \( W \) is some vector in \( \mathbb{R}^3 \), and \( \alpha \in \mathbb{R} \) with \( \alpha > 0 \). Every such a map can be represented by the \( 4 \times 4 \) matrix

\[
\begin{pmatrix}
\alpha R & W \\
0 & 1
\end{pmatrix}
\]

in the sense that

\[
\begin{pmatrix}
\rho(X) \\
1
\end{pmatrix} = \begin{pmatrix}
\alpha R & W \\
0 & 1
\end{pmatrix} \begin{pmatrix}
X \\
1
\end{pmatrix}
\]

iff

\[
\rho(X) = \alpha R X + W.
\]

Prove that these maps form a group, denoted by \( \text{SIM}(3) \) (the *direct affine similitudes* of \( \mathbb{R}^3 \)).

(b) Let us now consider the set of \( 4 \times 4 \) real matrices of the form

\[
B = \begin{pmatrix}
\Gamma & W \\
0 & 0
\end{pmatrix},
\]

where \( \Gamma \) is a matrix of the form

\[
\Gamma = \lambda I_3 + \Omega,
\]

with

\[
\Omega = \begin{pmatrix}
0 & -c & b \\
c & 0 & -a \\
-b & a & 0
\end{pmatrix},
\]

so that

\[
\Gamma = \begin{pmatrix}
\lambda & -c & b \\
c & \lambda & -a \\
-b & a & \lambda
\end{pmatrix},
\]

and \( W \) is a vector in \( \mathbb{R}^3 \).

Verify that this set of matrices is a vector space isomorphic to \( (\mathbb{R}^7, +) \). This vector space is denoted by \( \text{sim}(3) \).

(c) Given a matrix

\[
B = \begin{pmatrix}
\Gamma & W \\
0 & 0
\end{pmatrix}
\]

as in (b), prove that

\[
B^n = \begin{pmatrix}
\Gamma^n & \Gamma^{n-1} W \\
0 & 0
\end{pmatrix}
\]
where \( \Gamma^0 = I_3 \). Prove that
\[
e^B = \begin{pmatrix} e^\Gamma & VW \\ 0 & 1 \end{pmatrix},
\]
where
\[
V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!}.
\]

(d) Prove that if \( \Gamma = \lambda I_3 + \Omega \) as in (b), then
\[
V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!} = \int_0^1 e^{\Gamma t} dt.
\]

(e) For any matrix \( \Gamma = \lambda I_3 + \Omega \), with
\[
\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},
\]
if we let \( \theta = \sqrt{a^2 + b^2 + c^2} \), then prove that
\[
e^\Gamma = e^\lambda e^\Omega = e^\lambda \left( I_3 + \frac{\sin \theta}{\theta} \Omega + \frac{(1 - \cos \theta)}{\theta^2} \Omega^2 \right), \quad \text{if } \theta \neq 0,
\]
and \( e^\Gamma = e^\lambda I_3 \) if \( \theta = 0 \).

Hint. You may use the fact that if \( AB = BA \), then \( e^{A+B} = e^A e^B \). In general, \( e^{A+B} \neq e^A e^B \)!

(f) Prove that
1. If \( \theta = 0 \) and \( \lambda = 0 \), then
\[
V = I_3.
\]
2. If \( \theta = 0 \) and \( \lambda \neq 0 \), then
\[
V = \frac{(e^\lambda - 1)}{\lambda} I_3;
\]
3. If \( \theta \neq 0 \) and \( \lambda = 0 \), then
\[
V = I_3 + \frac{(1 - \cos \theta)}{\theta^2} \Omega + \frac{(\theta - \sin \theta)}{\theta^3} \Omega^2.
\]
4. If \( \theta \neq 0 \) and \( \lambda \neq 0 \), then
\[
V = \frac{(e^\lambda - 1)}{\lambda} I_3 + \frac{(\theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta)}{\theta(\lambda^2 + \theta^2)} \Omega
\]
\[
+ \left( \frac{(e^\lambda - 1)}{\lambda \theta^2} - \frac{e^\lambda \sin \theta}{\theta(\lambda^2 + \theta^2)} - \frac{\lambda(e^\lambda \cos \theta - 1)}{\theta^2(\lambda^2 + \theta^2)} \right) \Omega^2.
\]
Hint. You will need to compute \( \int_0^1 e^{\lambda t} \sin \theta t \, dt \) and \( \int_0^1 e^{\lambda t} \cos \theta t \, dt \).

(g) Prove that \( V \) is invertible iff \( \lambda \neq 0 \) or \( \theta \neq k2\pi \), with \( k \in \mathbb{Z} - \{0\} \).

Hint. Express the eigenvalues of \( V \) in terms of the eigenvalues of \( \Gamma \).

In the special case where \( \lambda = 0 \), show that

\[
V^{-1} = I - \frac{1}{2} \Omega + \frac{1}{\theta^2} \left( 1 - \frac{\theta \sin \theta}{2(1 - \cos \theta)} \right) \Omega^2, \quad \text{if } \theta \neq 0.
\]

Hint. Assume that the inverse of \( V \) is of the form

\[
Z = I_3 + a\Omega + b\Omega^2,
\]

and show that \( a, b \), are given by a system of linear equations that always has a unique solution.

(h) Prove that the exponential map \( \exp : \mathfrak{so}(3) \to \text{SO}(3) \), given by \( \exp(B) = e^B \), is surjective. You may use the fact that \( \exp : \mathfrak{so}(3) \to \text{SO}(3) \) is surjective, proved in another Problem.

Remark: Curves in \( \text{SIM}(3) \) can be used to describe certain deformations of bodies in \( \mathbb{R}^3 \).

TOTAL: 260 points + 10 points Extra credit