Fundamentals of Linear Algebra and Optimization
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Homework 2
September, 21 2020; Due October 5, 2020
Beginning of class

Problem B1 (10 pts). Let \( f : E \to F \) be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function \( f^{-1} : F \to E \) is linear.

Problem B2 (10 pts). Given two vector spaces \( E \) and \( F \), let \((u_i)_{i \in I}\) be any basis of \( E \) and let \((v_i)_{i \in I}\) be any family of vectors in \( F \). Prove that the unique linear map \( f : E \to F \) such that \( f(u_i) = v_i \) for all \( i \in I \) is surjective iff \((v_i)_{i \in I}\) spans \( F \).

Problem B3 (10 pts). Let \( f : E \to F \) be a linear map with \( \dim(E) = n \) and \( \dim(F) = m \). Prove that \( f \) has rank 1 iff \( f \) is represented by an \( m \times n \) matrix of the form

\[
A = uv^\top
\]

with \( u \) a nonzero column vector of dimension \( m \) and \( v \) a nonzero column vector of dimension \( n \).

Problem B4 (120 pts). (Haar extravaganza) Consider the matrix

\[
W_{3,3} = \begin{pmatrix}
  1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

(1) Show that given any vector \( c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) \), the result \( W_{3,3}c \) of applying \( W_{3,3} \) to \( c \) is

\[
W_{3,3}c = (c_1 + c_5, c_1 - c_5, c_2 + c_6, c_2 - c_6, c_3 + c_7, c_3 - c_7, c_4 + c_8, c_4 - c_8),
\]

the last step in reconstructing a vector from its Haar coefficients.
(2) Prove that the inverse of $W_{3,3}$ is $(1/2)W_{3,3}^T$. Prove that the columns and the rows of $W_{3,3}$ are orthogonal.

(3) Let $W_{3,2}$ and $W_{3,1}$ be the following matrices:

\[
W_{3,2} = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
W_{3,1} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,2}c$ of applying $W_{3,2}$ to $c$ is

\[
W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8),
\]

the second step in reconstructing a vector from its Haar coefficients, and the result $W_{3,1}c$ of applying $W_{3,1}$ to $c$ is

\[
W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8),
\]

the first step in reconstructing a vector from its Haar coefficients.

Conclude that

\[
W_{3,3}W_{3,2}W_{3,1} = W_3,
\]

the Haar matrix

\[
W_3 = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1
\end{pmatrix}.
\]

Hint. First, check that

\[
W_{3,2}W_{3,1} = \begin{pmatrix} W_2 & \mathbf{0}_{4,4} \\ \mathbf{0}_{4,4} & I_4 \end{pmatrix},
\]

where

\[
W_2 = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & -1 & 0 \\
1 & -1 & 0 & 1 \\
1 & -1 & 0 & -1
\end{pmatrix}.
\]
(4) Prove that the columns and the rows of $W_{3,2}$ and $W_{3,1}$ are orthogonal. Deduce from this that the columns of $W_3$ are orthogonal, and the rows of $W_3^{-1}$ are orthogonal. Are the rows of $W_3$ orthogonal? Are the columns of $W_3^{-1}$ orthogonal? Find the inverse of $W_{3,2}$ and the inverse of $W_{3,1}$.

(5) For any $n \geq 2$, the $2^n \times 2^n$ matrix $W_{n,n}$ is obtained form the two rows

\[
\begin{array}{c}
1,0,\ldots,0,1,0,\ldots,0 \\
1,0,\ldots,0,-1,0,\ldots,0 \\
\end{array}
\]

by shifting them $2^{n-1} - 1$ times over to the right by inserting a zero on the left each time.

Given any vector $c = (c_1, c_2, \ldots, c_{2^n})$, show that $W_{n,n}c$ is the result of the last step in the process of reconstructing a vector from its Haar coefficients $c$. Prove that $W_{n,n}^{-1} = (1/2)W_{n,n}^\top$, and that the columns and the rows of $W_{n,n}$ are orthogonal.

Extra credit (30 pts.)

Given a $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix $B = (b_{ij})$, the Kronecker product (or tensor product) $A \otimes B$ of $A$ and $B$ is the $mp \times nq$ matrix

\[
A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{pmatrix}.
\]

It can be shown (and you may use these facts without proof) that $\otimes$ is associative and that

\[
(A \otimes B)(C \otimes D) = AC \otimes BD
\]

\[
(A \otimes B)^\top = A^\top \otimes B^\top,
\]

whenever $AC$ and $BD$ are well defined.

Check that

\[
W_{n,n} = \left( I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \left( I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),
\]

and that

\[
W_n = \left( W_{n-1} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \left( I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right).
\]

Use the above to reprove that

\[
W_{n,n}W_{n,n}^\top = 2I_{2^n}.
\]
Let
\[ B_1 = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \]
and for \( n \geq 1, \)
\[ B_{n+1} = 2 \begin{pmatrix} B_n & 0 \\ 0 & I_{2^n} \end{pmatrix}. \]
Prove that
\[ W_n^T W_n = B_n, \quad \text{for all } n \geq 1. \]

(6) The matrix \( W_{n,i} \) is obtained from the matrix \( W_{i,i} \) \((1 \leq i \leq n - 1)\) as follows:
\[ W_{n,i} = \begin{pmatrix} W_{i,i} & 0_{2^i,2^{n-2^i}} \\ 0_{2^{n-2^i},2^i} & I_{2^{n-2^i}} \end{pmatrix}. \]
It consists of four blocks, where \( 0_{2^i,2^{n-2^i}} \) and \( 0_{2^{n-2^i},2^i} \) are matrices of zeros and \( I_{2^{n-2^i}} \) is the identity matrix of dimension \( 2^n - 2^i \).

Explain what \( W_{n,i} \) does to \( c \) and prove that
\[ W_n W_{n,n-1} \cdots W_{n,1} = W_n, \]
where \( W_n \) is the Haar matrix of dimension \( 2^n \).

Hint. Use induction on \( k \), with the induction hypothesis
\[ W_{n,k} W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k,2^{n-2^k}} \\ 0_{2^{n-2^k},2^k} & I_{2^{n-2^k}} \end{pmatrix}. \]

Prove that the columns and rows of \( W_{n,k} \) are orthogonal, and use this to prove that the columns of \( W_n \) and the rows of \( W_n^{-1} \) are orthogonal. Are the rows of \( W_n \) orthogonal? Are the columns of \( W_n^{-1} \) orthogonal? Prove that
\[ W_n^{-1} = \begin{pmatrix} \frac{1}{2} W_k^T & 0_{2^k,2^{n-2^k}} \\ 0_{2^{n-2^k},2^k} & I_{2^{n-2^k}} \end{pmatrix}. \]

**Problem B5 (20 pts).** Prove that for every vector space \( E \), if \( f: E \to E \) is an idempotent linear map, i.e., \( f \circ f = f \), then we have a direct sum
\[ E = \text{Ker } f \oplus \text{Im } f, \]
so that \( f \) is the projection onto its image \( \text{Im } f \).

**Problem B6 (40 pts).** Given any vector space \( E \), a linear map \( f: E \to E \) is an **involution** if \( f \circ f = \text{id} \).

(1) Prove that an involution \( f \) is invertible. What is its inverse?
(2) Let $E_1$ and $E_{-1}$ be the subspaces of $E$ defined as follows:
\[
E_1 = \{ u \in E \mid f(u) = u \} \\
E_{-1} = \{ u \in E \mid f(u) = -u \}.
\]
Prove that we have a direct sum
\[
E = E_1 \oplus E_{-1}.
\]
*Hint.* For every $u \in E$, write
\[
u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.
\]

(3) If $E$ is finite-dimensional and $f$ is an involution, prove that there is some basis of $E$ with respect to which the matrix of $f$ is of the form
\[
I_{k,n-k} = \begin{pmatrix}
I_k & 0 \\
0 & -I_{n-k}
\end{pmatrix},
\]
where $I_k$ is the $k \times k$ identity matrix (similarly for $I_{n-k}$) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of $f$ (especially when $k = n - 1$)?

**TOTAL: 210 + 30 points.**