Problem B1 (40 pts). Give a Turing machine accepting the language
\[ L = \{wcw \mid w \in \{a, b\}^+\} \]
over the alphabet \( \{a, b, c\} \).

Problem B2 (30 pts). (1) Prove that the extended pairing function \( \langle x_1, \ldots, x_n \rangle_n \) defined in the notes (see Definition 3.2 of the notes) satisfies the equation
\[ \langle x_1, \ldots, x_n, x_{n+1} \rangle_{n+1} = \langle x_1, \langle x_2, \ldots, x_{n+1} \rangle_n \rangle. \]

Compute \( \langle 2, 5, 7, 17 \rangle_4 \) (this integer has 10 digits).

(2) Prove that \( \langle x, 0 \rangle = \langle x, 0, \ldots, 0 \rangle_n \) for all \( n \geq 2 \) and all \( x \in \mathbb{N} \).

(3) Prove that
\[ \langle \Pi(1, n, z), \ldots, \Pi(n, n, z) \rangle_n = z \]
for all \( n \geq 1 \) and all \( z \in \mathbb{N} \).

Problem B3 (30 pts). Ackermann’s function \( A \) is defined recursively as follows:
\[
\begin{align*}
A(0, y) &= y + 1, \\
A(x + 1, 0) &= A(x, 1), \\
A(x + 1, y + 1) &= A(x, A(x + 1, y)).
\end{align*}
\]

Prove that
\[
\begin{align*}
A(0, x) &= x + 1, \\
A(1, x) &= x + 2, \\
A(2, x) &= 2x + 3, \\
A(3, x) &= 2^{x+3} - 3,
\end{align*}
\]
and
\[ A(4, x) = 2^{2^{16}}x^2 - 3, \]
with \( A(4, 0) = 16 - 3 = 13 \). Equivalently (and perhaps less confusing)
\[ A(4, x) = 2^{2^{2^2}}x^3 - 3. \]

**Problem B4 (10 pts).** Let \( f : \mathbb{N} \to \mathbb{N} \) be a total computable function. Prove that if \( f \) is a bijection, then its inverse \( f^{-1} \) is also (total) computable.

**Problem B5 (20 pts).** Let \( A, B, C, D \) be the following sets:
\[
A = \{ x \in \mathbb{N} \mid \varphi_x \text{ is constant} \}, \\
B = \{ \langle x, y \rangle \mid \varphi_x = \varphi_y \}, \\
C = \{ x \in \mathbb{N} \mid \varphi_x = \varphi_a \}, \\
D = \{ x \in \mathbb{N} \mid \varphi_x \text{ is undefined for all input} \},
\]
where \( a \) is a given natural number. Prove that the above sets are not computable (not recursive).

**Problem B6 (40 pts).** Given any set, \( X \), for any subset, \( A \subseteq X \), recall that the characteristic function, \( \chi_A \), of \( A \) is the function defined so that
\[
\chi_A(x) = \begin{cases} 
1 & \text{iff } x \in A \\
0 & \text{iff } x \in X - A.
\end{cases}
\]

(i) Prove that, for any two subsets, \( A, B \subseteq X \),
\[
\chi_{A \cap B} = \chi_A \cdot \chi_B, \\
\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B.
\]

(ii) Prove that the union and the intersection of any two Diophantine sets \( A, B \subseteq \mathbb{N} \), is also Diophantine.

(iii) Prove that the union and the intersection of any two listable sets \( A, B \subseteq \mathbb{N} \), is also listable.

(iv) Prove that the union and the intersection of any two computable (recursive) sets, \( A, B \subseteq \mathbb{N} \), is also a computable set (a recursive set).

**TOTAL: 170 points.**