CIS 1921

Lecture 5: Mixed-Integer & Linear Programming

Reminders

- HW2 is due on Monday, February 24, 11:59PM
  - Start Early!
  - Make sure set-up works!

Today

- Moving away from SAT solving
    - But we will tie it back in later!
- Start looking at “high-level” solvers
- Specify constraints in something closer to mathematical language (as opposed to SAT clauses)

Basic Linear Program

- You're deciding what to bring to a potluck and want a meal with $\geq$ 5000 calories but $\leq$ 200 mg sodium.
- You want to spend as little money as possible.

Item	Price/kg	Calories/kg	Sodium/kg
Rice	1.25	750	15
Pasta	1.65	1200	35
Couscous	1.35	1000	60

Linearity

- $f(x_1, \ldots, x_n)$ is a **linear function** if it is of the form

$$f(x_1, \ldots, x_n) = a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

- A **linear inequality** has the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

Linear Programs

- A **linear program** is a special class of optimization problems with the goal: optimize a **linear function** subject to **linear (in)equalities**
    - Strict inequalities not allowed: <, >, !=
- Widely solved in industry for maximizing value, minimizing cost

Example LP

- LP formalization:

$$
\begin{aligned}
\text{minimize} \quad & 3x_1 + 2x_2 - 4x_3 + 5 \\
\text{subject to} \quad & x_1 \geq 2 \\
& x_2 + 2x_3 \leq 10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

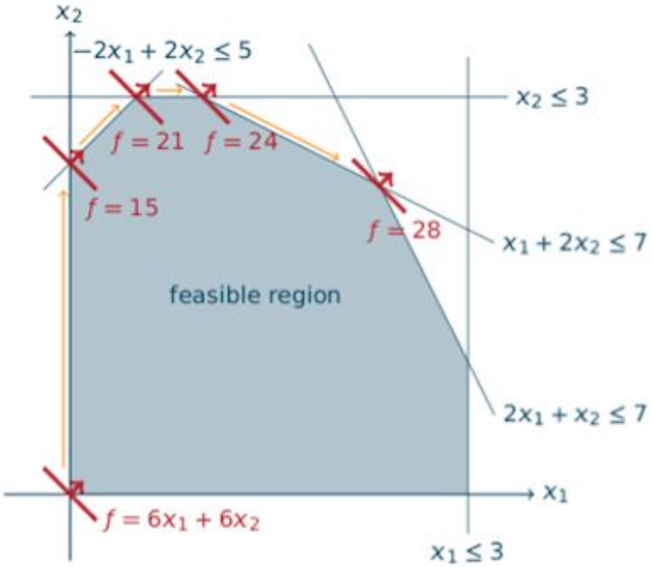
A More Complex Example LP

- LP formalization:

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 5x_2 \\
\text{subject to} \quad & 0 \leq x_1, x_2 \leq 3 \\
& -2x_1 + 2x_2 \leq 5 \\
& x_1 + 2x_2 \leq 7 \\
& 2x_1 + x_2 \leq 7
\end{aligned}
$$

Linear Programming Methods

- Major theory result: any LP is polytime solvable
- In practice: **Simplex algorithm**
  - George Dantzig, 1947
  - Worst-case exponential time
  - Practically fast for most problems
  - Check the corners!



Linear Programming Methods

- More recently: interior-point methods
- Karmarkar's algorithm (1984)
  - Polytime and practically fast

Breakthrough in Problem Solving

By JAMES GLEICK

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

Faster Solutions Seen

These problems are fiendishly com-

"Science has its moments of great progress, and this may well be one of them."

Because problems in linear programming can have billions or more possible answers, even high-speed computers cannot check every one. So computers must use a special procedure, an algorithm, to examine as few answers as possible before finding the best one — typically the one that minimizes cost or maximizes efficiency.

A procedure devised in 1947, the simplex method, is now used for such problems,

Continued on Page A19, Column 1

Karmarkar at Bell Labs: an equation to find a new way through the maze

Folding the Perfect Corner

A young Bell scientist makes a major math breakthrough

Every day 1,200 American Airlines jets crisscross the U.S., Mexico, Canada and the Caribbean, stopping in 110 cities and bearing over 80,000 passengers. More than 4,000 pilots, copilots, flight personnel, maintenance

Indian-born mathematician at Bell Laboratories in Murray Hill, N.J., after only a years' work has cracked the puzzle of linear programming by devising a new algorithm, a step-by-step mathematical formula. He has

NYT (left)
TIME (right)

Basic Linear Program

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Transform into LP

To transform a problem into an instance of LP, identify the following 3 things:

1. Variables
   a. "Variables are the quantities that the solver will give values to. Define your variables in a granular way so when the solver gives values to the variables, you either immediately solve your problem, or can easily derive the solution to your problem."
2. Objective Function
   a. The *linear* function **with respect to your variables** that you intend to minimize or maximize
3. The constraints
   a. The *linear* inequalities **with respect to your variables** that must be followed

Basic Linear Program

- LP formulation:

$$\begin{aligned} \text{minimize} \quad & price \\ \text{subject to} \quad & calories \geq 5000 \\ & sodium \leq 200 \end{aligned}$$

- Plus implicit constraint: can't buy negative amounts

Basic Linear Program

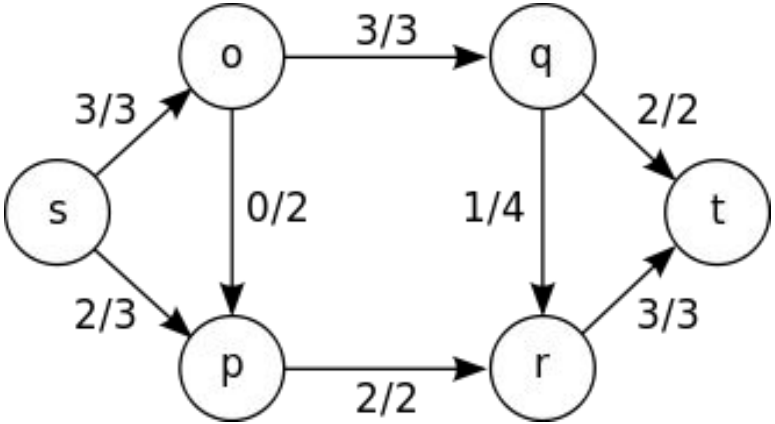
- LP formulation:

$$
\begin{array}{ll}
\text{minimize} & 1.25r + 1.65p + 1.35c \\
\text{subject to} & 750r + 1200p + 1000c \geq 5000 \\
 & 15r + 35p + 60c \leq 200 \\
 & r \geq 0,\ \ p \geq 0,\ \ c \geq 0
\end{array}
$$

Demo

Max Flows with LP

- Max flow problem has a natural LP formulation
- Recall the problem: how much flow can we send along the edges from s to t?
    - Flow conservation
    - Capacity constraints



Max Flows with LP

- Variables:

  f_{uv} = total flow along edge (u, v)

- Objective:

  $$\text{maximize} \sum_{v \in N_{\text{Out}}(s)} f_{sv}$$

  - Alternatively: add ∞-capacity feedback edge (t, s) and maximize f_{ts}

Max Flows with LP

- Capacity constraints:

$$0 \le f_{uv} \le c(u,v) \quad \forall (u,v) \in E$$

- Conservation constraints:

$$\sum_{v \in N_{\text{In}}(u)} f_{vu} - \sum_{v \in N_{\text{Out}}(u)} f_{uv} = 0 \quad \forall u \in V - \{s,t\}$$

  - If we added feedback edge, don't exclude s, t

Demo

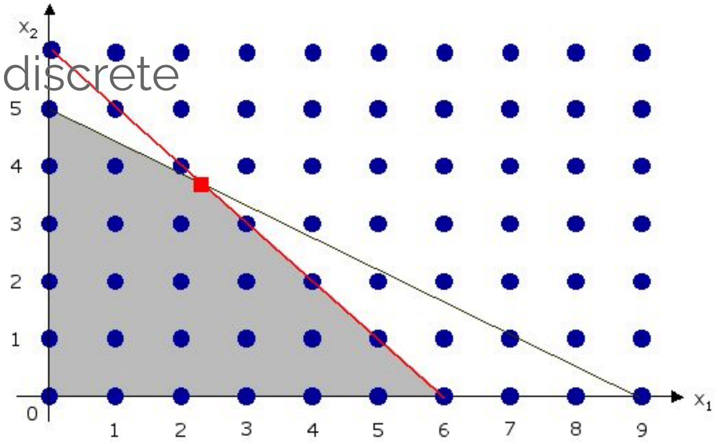
Applications of Max Flow

- Baseball Elimination
- Airline Scheduling
- Image Segmentation

https://en.wikipedia.org/wiki/Maximum_flow_problem

Integer (Linear) Programming

- **Integer linear program (ILP):** a linear program with the additional constraint that variables must take integer values
    - aka **integer program (IP)**
    - In real life, items often come in discrete units



Integer (Linear) Programming

- **Ex:** What should we buy to maximize profit?
  - We have \$45 to buy pies (\$5) and cakes (\$9).
  - We can resell pies for \$5 profit and cakes for \$8 profit.
  - But we can only carry 6 items total.
- IP formulation:

maximize	$5p + 8c$
subject to	$5p + 9c \leq 45$
	$p + c \leq 6$
	$p, c \geq 0$ are integers

Rounding LPs from ILPs?

- What if we just solve the **LP relaxation** and round?

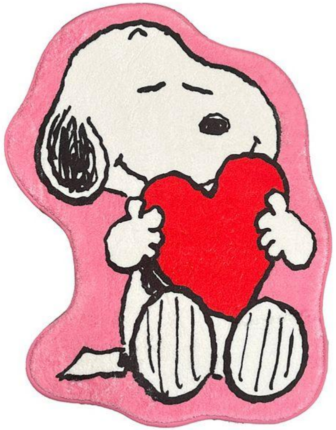
Bad Theory News

- **Bad news 1:** we can construct ILPs whose rounded LP solution is arbitrarily far away
  - Sometimes, we can “round” in a clever way so that the rounded solution is not too far
- **Bad news 2:** integer programming is NP-complete!
- Good practical news: lots of work on robust solvers for real-world IPs

Mixed-Integer Programming

- **Mixed-integer program (MIP):** some variables may be constrained to be integers, and some may not
- Objectives & constraints are still linear!
- We'll just talk about MIP, since it generalizes IP

Share Love!



“Love's such an old-fashioned word, and love dares you to care for the people on the edge of the night, and love dares you to change our way of caring about ourselves.” ~Lyrics from “Under Pressure” by David Bowie & Queen