CIS 189

Lecture 4: Modern Techniques in SAT Solving

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Recap: Iterative DPLL

```
dpll(φ):
    if unit_propagate() = CONFLICT: return UNSAT
    while not all variables have been set:
        let x = pick_variable()
        create new decision level
        set x = T
        while unit_propagate() = CONFLICT:
            if decision_level = 0: return UNSAT
            backtrack()
            set x = F
    return SAT
```

Chronological Backtracking

- DPLL uses **chronological backtracking:** when we find a conflict, backtrack to the previous decision level
- **Issue:** might reach conflicts (contradictions) caused by the same underlying reason over and over again

Chronological Backtracking

$$(1 \vee \overline{2})$$

$$(\overline{1} \vee 3 \vee 4)$$

$$(\overline{1} \vee \overline{3} \vee 4)$$

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Chronological Backtracking

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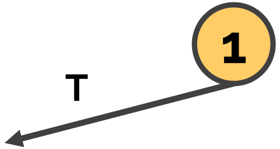

Chronological Backtracking

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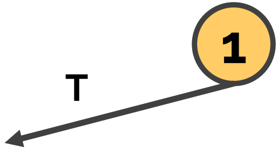
UNSAT subformula



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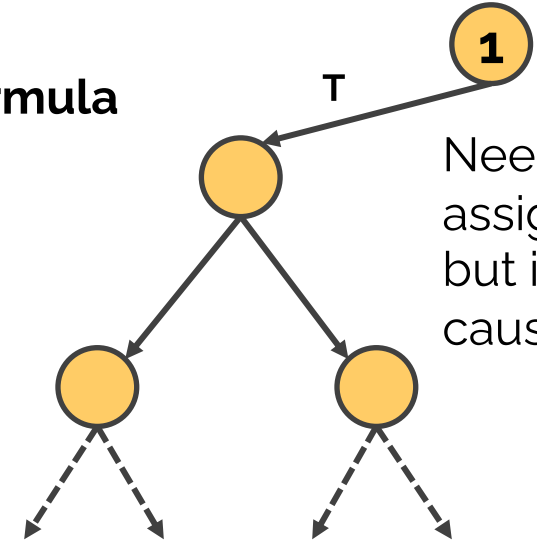

Need to rule out all assignments of 2, 3, 4, but issue was really caused by $1 = T$!

Backjumping

- Not every decision actually contributes to a conflict
- **Idea:** upon conflict, instead of backtracking one level to the last decision, **backjump** to an *important* decision
  - i.e., a decision that contributed to the conflict
- But how do we know what is an important decision?

Implication Graphs

- An **implication graph** G is a DAG whose vertices are literal assignments at a particular decision level
  - Ex: $\overline{x}@3$ represents setting x to False at level 3
  - Assignments can be decisions or due to unit propagation/backtracking
- Can also contain special vertex $\perp$ representing a conflict
- There is an edge $x@i \rightarrow y@j$ if the assignment $x@i$ directly implied the assignment $y@j$
  - i.e., $y@j$ was set by unit propagation from a clause containing $\overline{x}$

Implication Graphs

c_1: $\overline{2} \vee \overline{3} \vee \overline{4}$

c_2: $\overline{3} \vee \overline{5} \vee \overline{6}$

c_3: $4 \vee 6 \vee 7$

c_4: $\overline{7} \vee \overline{8}$

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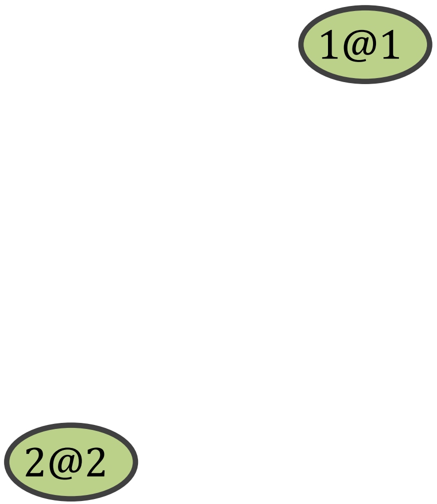

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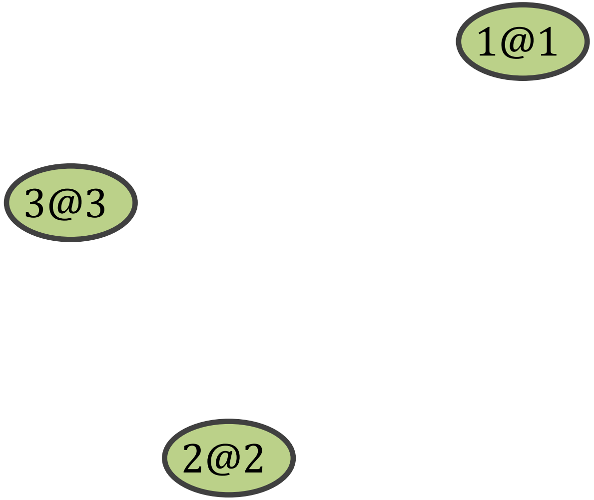

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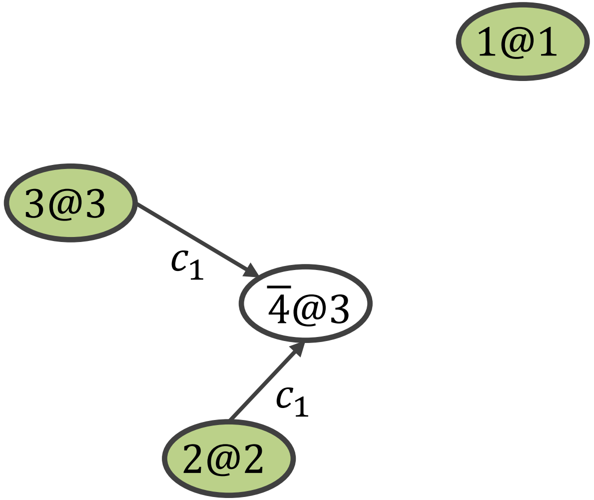

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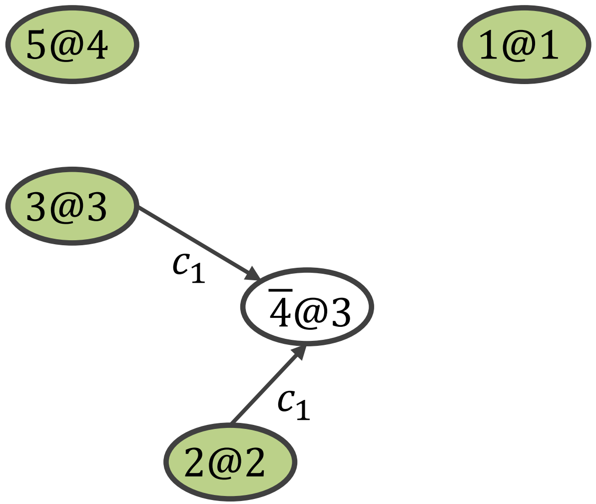

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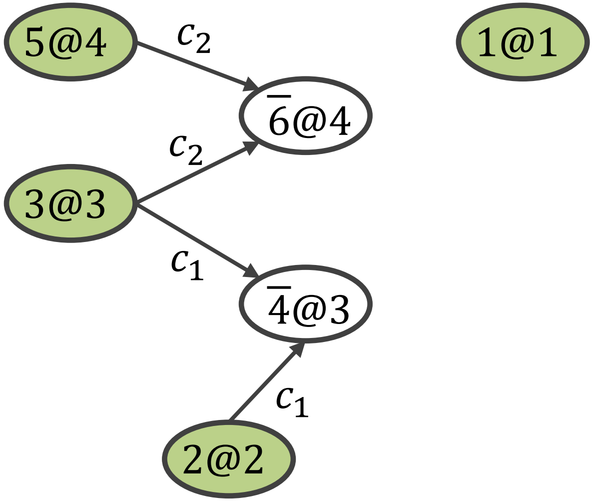

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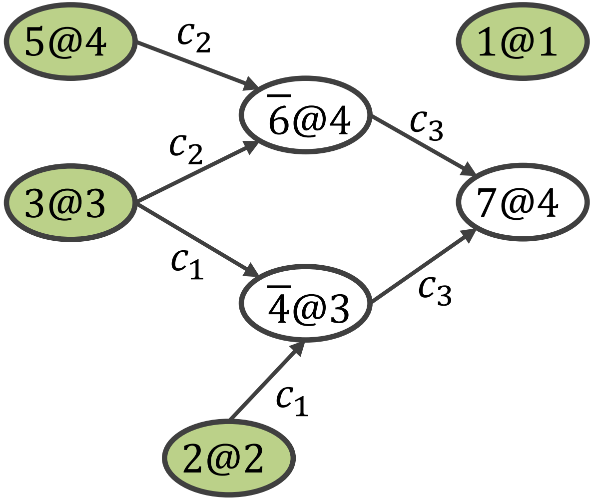

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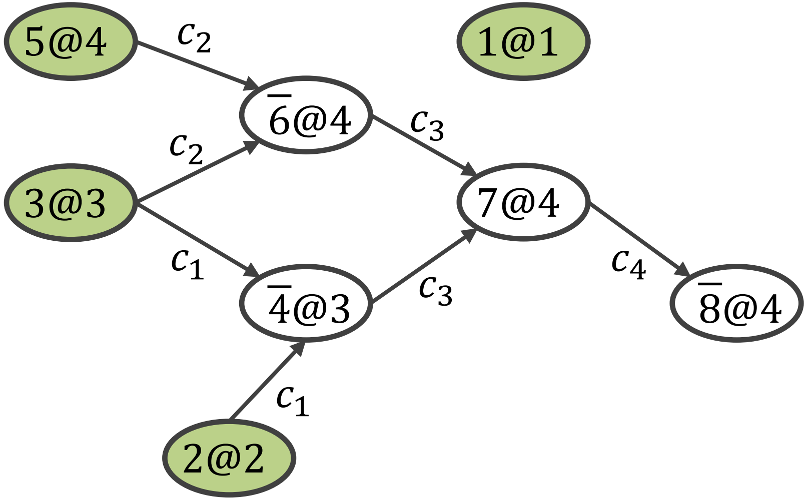

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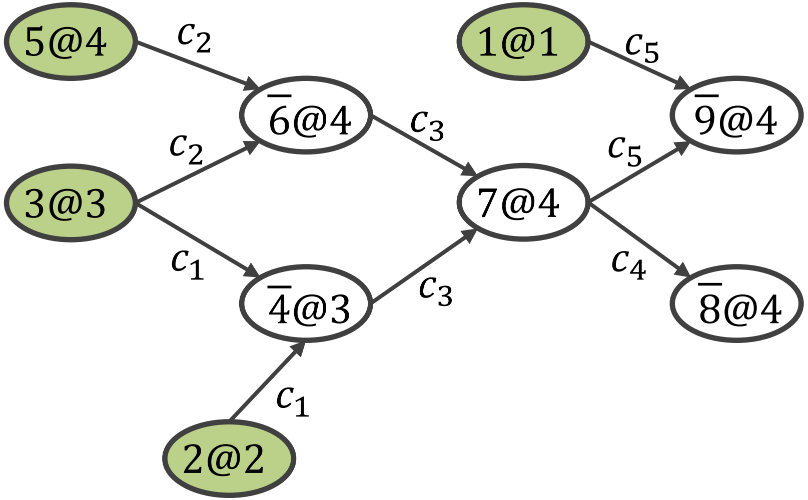

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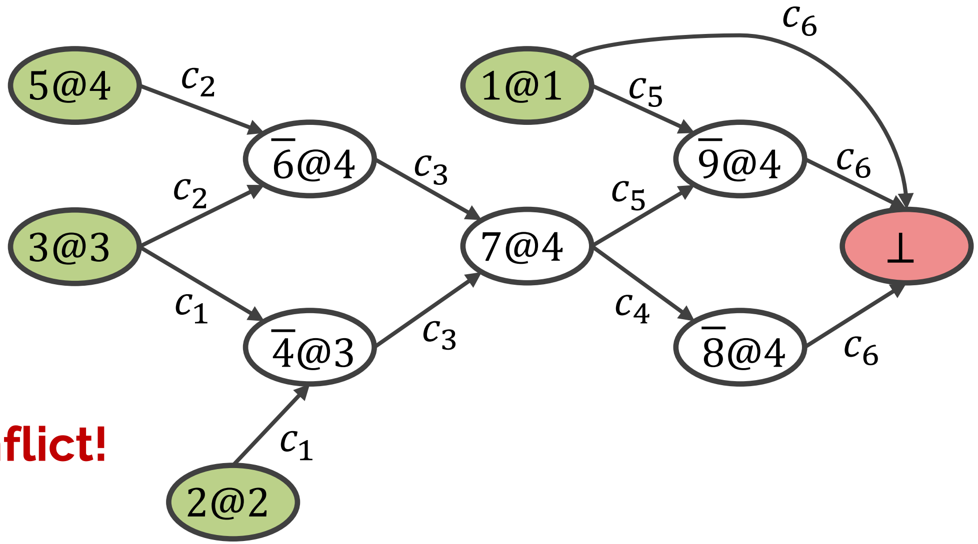
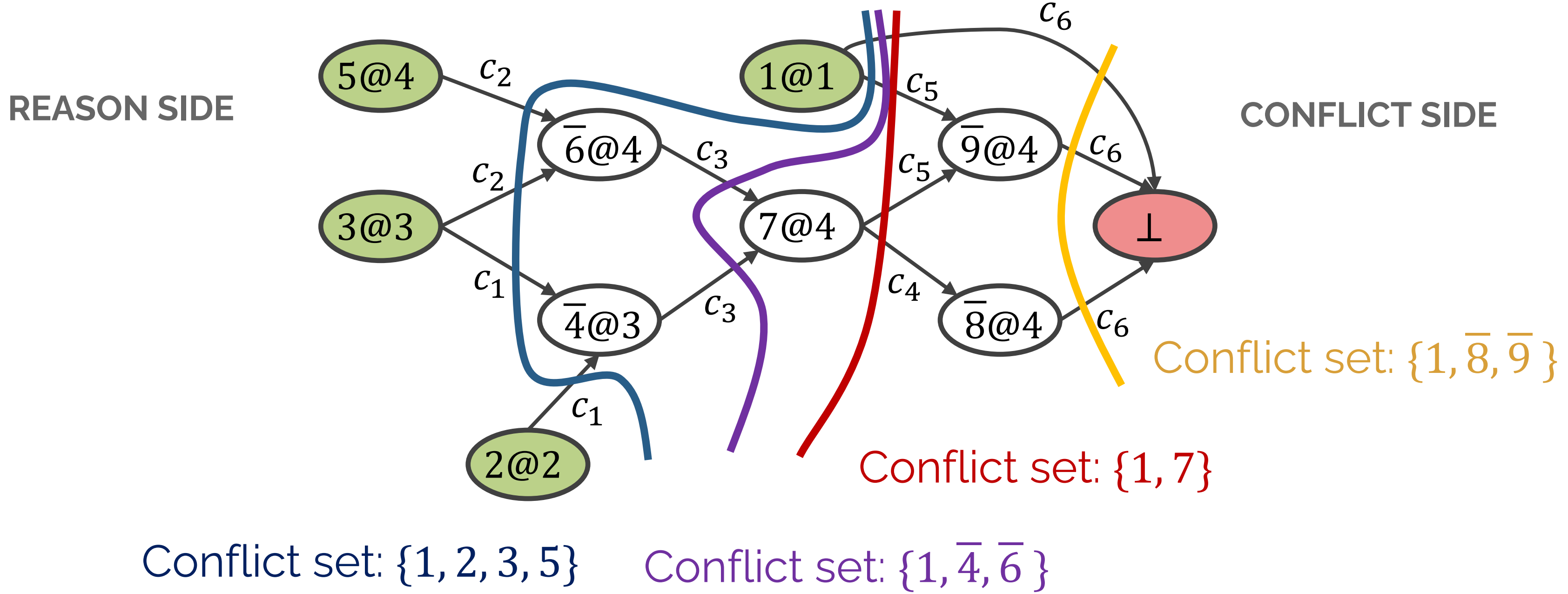


Conflicts

- A **conflict set** of assignments (collectively) imply a conflict
- A **conflict cut** in an implication graph is a bipartition of the vertices $V = R \cup C$ such that:
  - *Reason side* R contains all decisions (source nodes)
  - *Conflict side* C contains the conflict node (a sink)
  - No edges cross $C \to R$, only $R \to C$
- The set of vertices with an outgoing edge crossing a given conflict cut forms a conflict set

Ex: Conflict Cuts



And more...

Clause Learning

- **Observation:** Given a conflict set $\{x_1, x_2, \dots, x_k\}$, we know at least one literal in the set must be False
- Can derive the **conflict clause** $(\overline{x_1} \vee \overline{x_2} \vee \cdots \vee \overline{x_k})$
- **Conflict-driven clause learning (CDCL):** add conflict clauses to the original CNF we're solving
  - Introduced by GRASP (1996); revolutionized SAT solving
  - Many solvers have **aggressive deletion policy** for long, "inactive," "unhelpful" learned clauses – avoid explosion in CNF size

Asserting Clauses

- Many conflict cuts – how do we decide which to choose to build a conflict clause?
- **Goal:** after backjumping, be able to apply new knowledge from learned clause right away
  - Want learned clause to become a unit clause right after backjumping

Asserting Clauses

- A learned clause is **asserting** if it contains only one variable set on the same decision level as conflict
  - Is it possible for any conflict clause to contain zero?
- **Observation:** iff a clause is asserting, it will become a unit clause after backtracking
- How far can we backjump and still have asserting clauses become unit clauses?
  - Backjump to *second-largest* (i.e., deepest) decision level in asserting clause (or zeroth level if asserting clause has size 1)
    - i.e., return to that decision level (don't undo the decision)
  - Called the **asserting level**

CDCL (Pseudocode)

```
cdcl(φ):
    if unit_propagate() = CONFLICT: return UNSAT
    while not all variables have been set:
        let x = pick_variable()
        create new decision level; set x = T
        while unit_propagate() = CONFLICT:
            if level = 0: return UNSAT
            let (conflict_cls, assrt_lvl) = analyze_conflict()
            let φ = φ ∪ { conflict_cls }
            # discard all assignments after asserting level
            backjump(assrt_lvl)
    return SAT
```

Asserting Level Backjump

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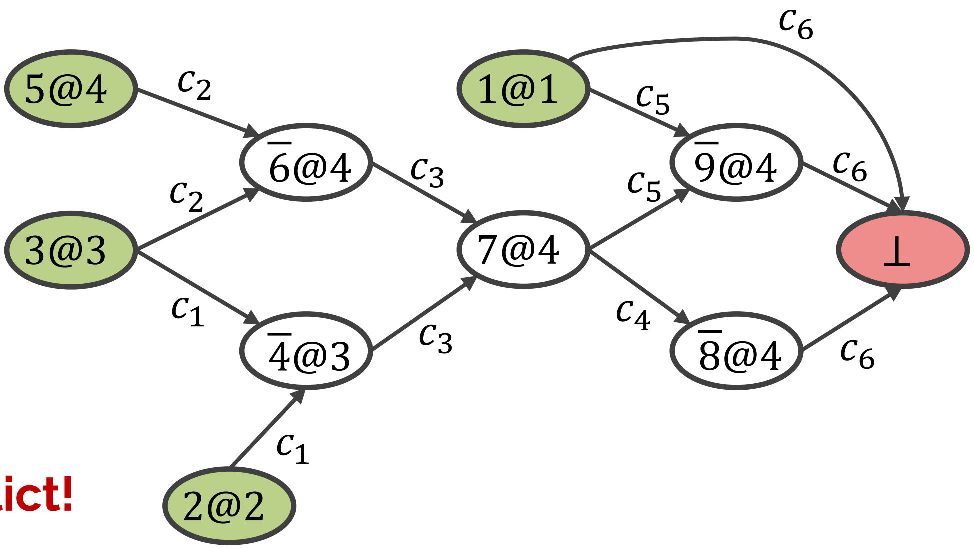

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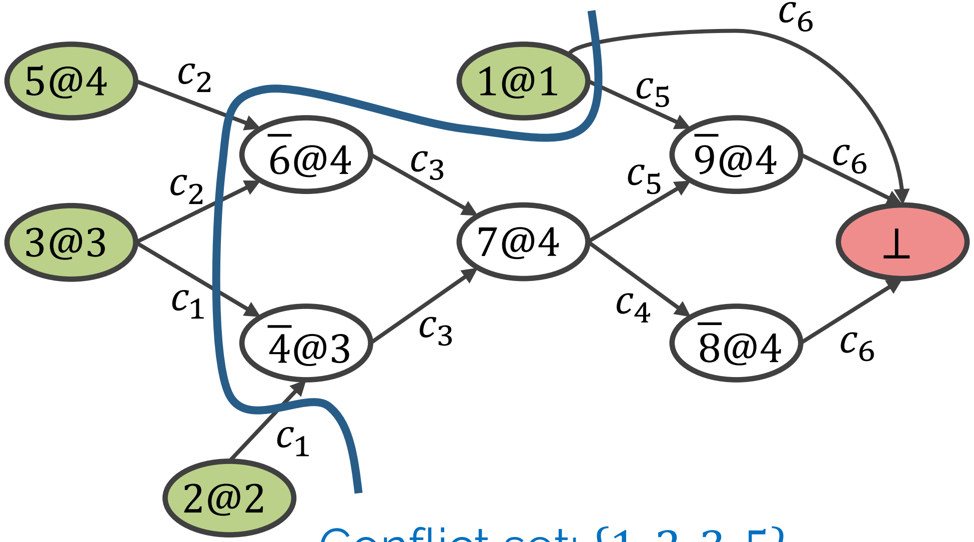
c: $\overline{1} \vee \overline{2} \vee \overline{3} \vee \overline{5}$



Conflict set: $\{1, 2, 3, 5\}$
Asserting level = 3 (from 3@3)

Asserting Level Backjump

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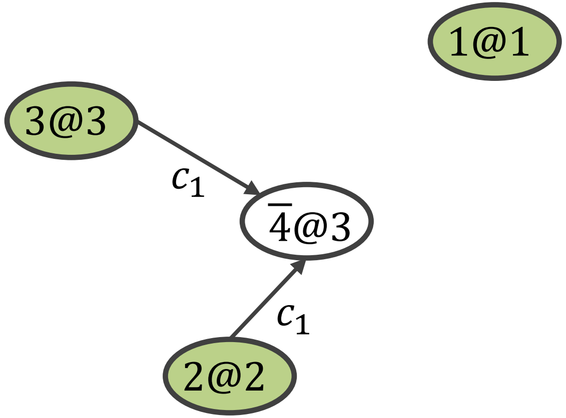
c: $\overline{1} \vee \overline{2} \vee \overline{3} \vee \overline{5}$

Backjump!



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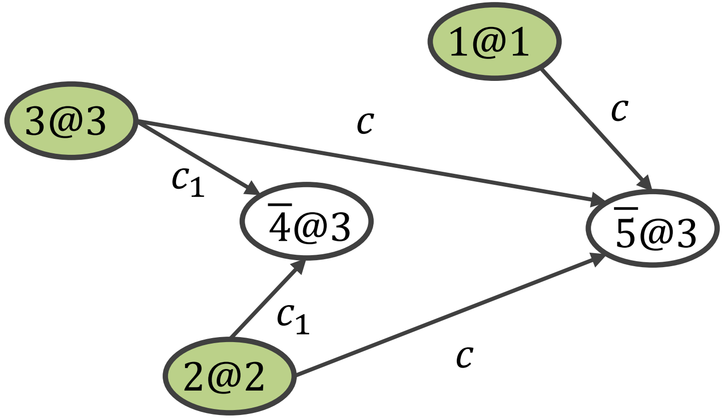
c: $\overline{1} \vee \overline{2} \vee \overline{3} \vee \overline{5}$ **Unit!**



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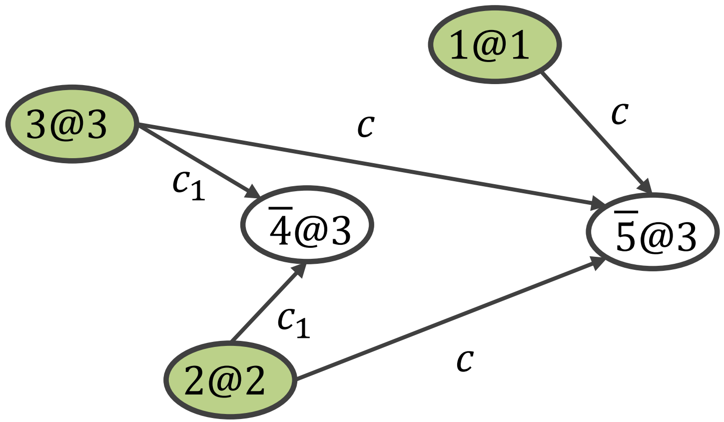
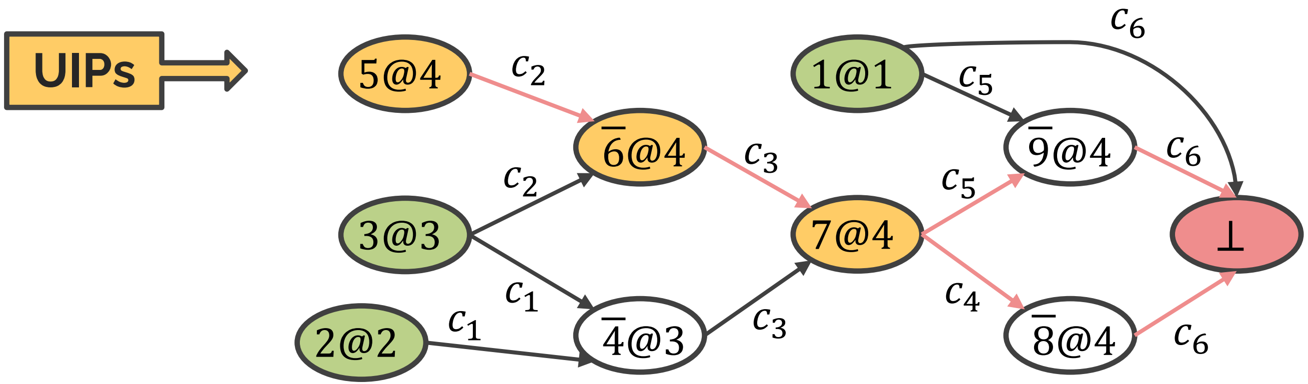
c: $\overline{1} \vee \overline{2} \vee \overline{3} \vee \overline{5}$ **Unit!**

Wait a second... same outcome as backtracking with DPLL.



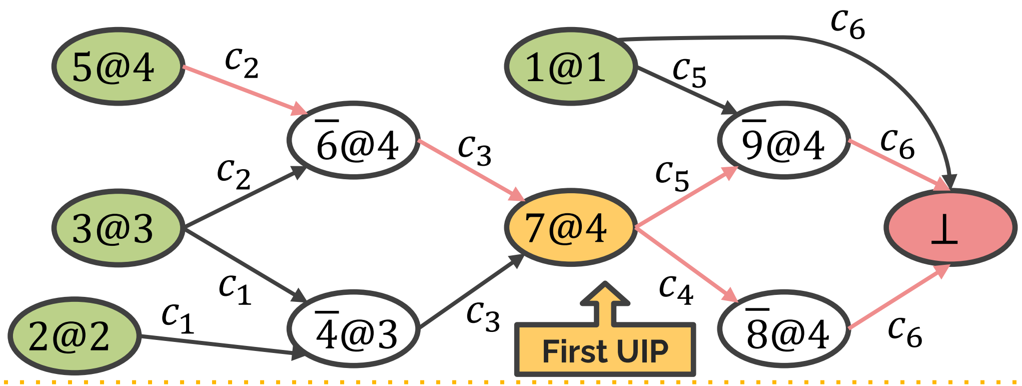
Unique Implication Points

- **Unique implication point (UIP):** a node in the implication graph that all paths from the most recent decision variable to the conflict must pass through
- **Intuition:** at the decision level of the conflict, the UIP is a literal that, by itself, implies a contradiction



The 1-UIP Scheme

- The **“first” UIP** is the closest UIP to the conflict node
  - i.e., the “rightmost”
- When we reach a conflict, cut after the first UIP
  - Generally produces shortest learned clauses



1-UIP Backjump

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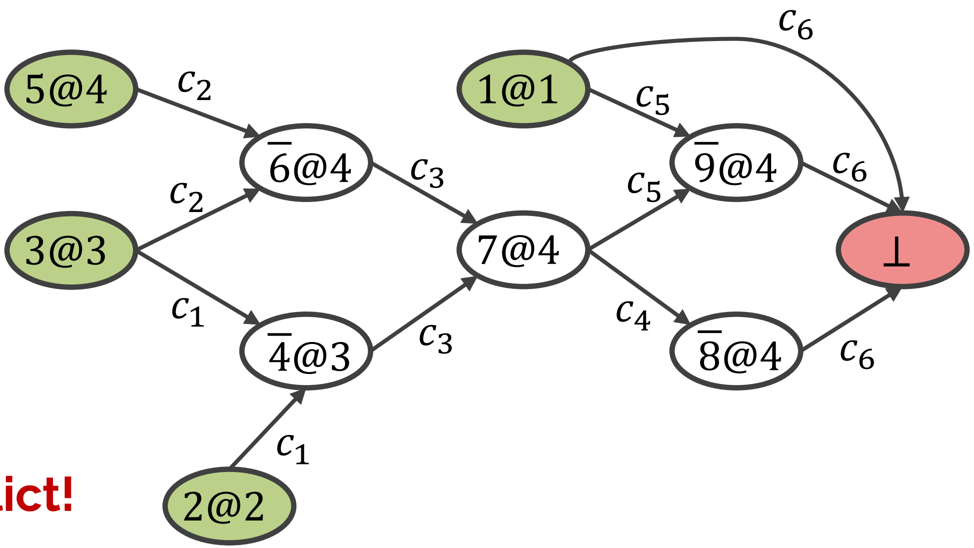

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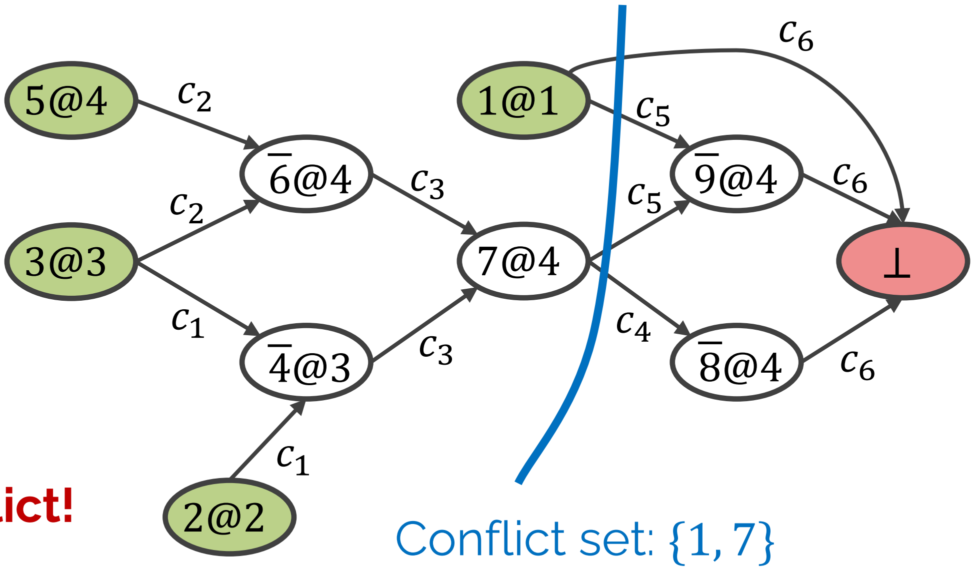
c: $\overline{1} \vee \overline{7}$



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Backjump!

1@1

1-UIP Backjump

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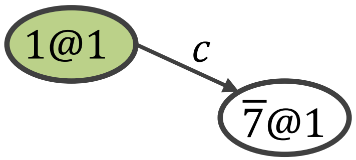
c: $\overline{1} \vee \overline{7}$ **Unit!**



Restarts

- Problem: if we make bad early guesses, can get stuck in fruitless areas of search tree
- Solution: periodically **restart** the search – throw away the current partial assignment
  - Modern solvers favor aggressive restart policy
    - MiniSAT, PicoSAT: every ~100 conflicts
- **Key idea:** CDCL is deterministic, so why won't we end up back where we were?
  - Learned clauses remain in formula after restart

Incremental SAT Solving

- CDCL solvers give us a new method in our toolkit!

add_clause(C): add clause C to the formula

- New clauses can only rule out previously satisfying assignments
- Can re-solve CNF with new clauses added
- **Key:** keep learned clauses generated during last call to solve()
- Simple use case: generating all satisfying assignments

Introducing: PennSAT

- **HW2: PennSAT (due on Tue 2/21 by 4pm)**
- Features:
  - DPLL-based
  - Iterative
  - Maintains propagation queue
  - No Two-Watched Literals
  - Static most-frequent decision heuristic
- This assignment is tricky – **start early!**
  - Requires solid understanding

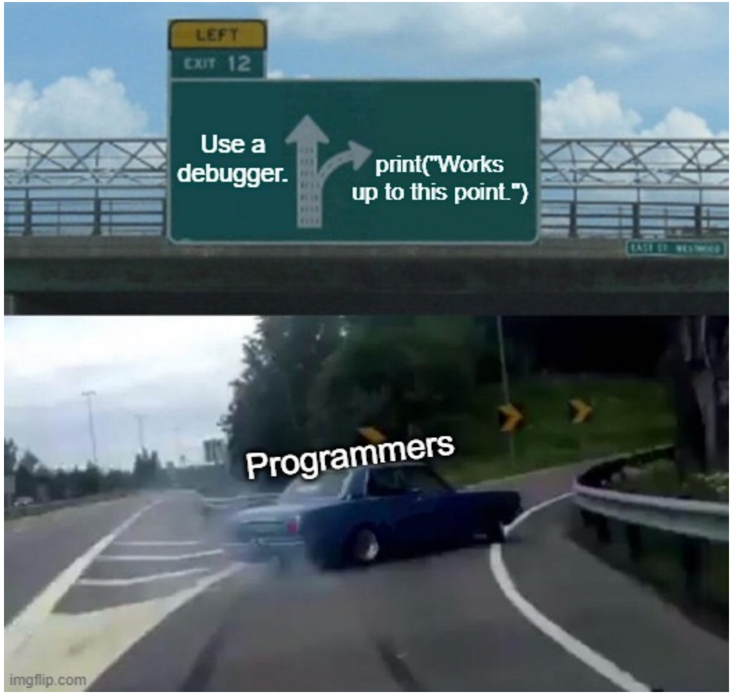
Testing a SAT Solver

- SAT solvers have tons of complicated logic… how to check for soundness bugs?
    - Hard and tedious to figure out all cases to unit test
- **Random testing:** generate random CNF formulas to test against reference solver
- If reference solver is not available, can at least check that satisfying assignments are valid

Debugging a SAT Solver

- Once we've found a bug, how do we find the mistake in the code?
- **Print debugging:** stick a bunch of print statements in relevant places and look at the console
- Easy, but not as effective for complex systems
    - Easy to forget to print something, or print in wrong place

Debugging a SAT Solver



Debugging in VS Code

- **Debugger:** allows us to stop program mid-execution, run code line-by-line, inspect values of local variables

```
def __init__(self, n: int, cnf: CNF, activity_heu
    # The number of variables
    self.n = n
    # The CNF as a list of clauses
    self.cnf = preprocess(cnf)
    # A stack of partial truth assignments: lists
    self.assignment_stack = [[None] * (n+1)]
```

Breakpoint

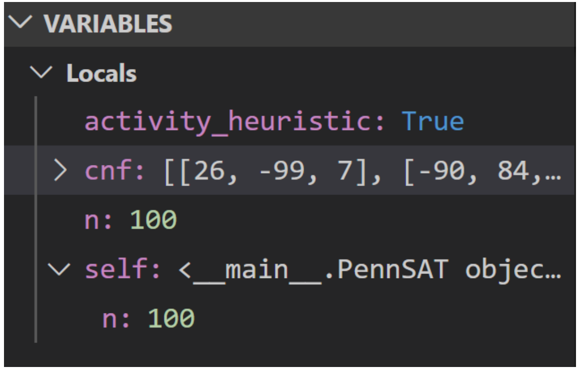
- **Breakpoint:** STOP at this line of code

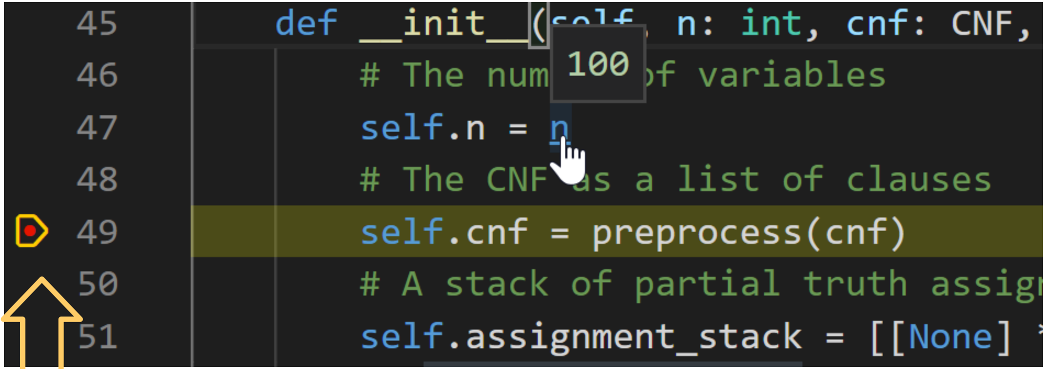
Debugging in VS Code

- After breakpoints set: Run > Start Debugging (F5)

View or *modify* current variables & values

Hover over variables to inspect values





Stopped right before line 49!

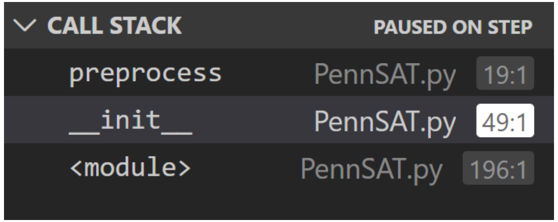
Debugging in VS Code

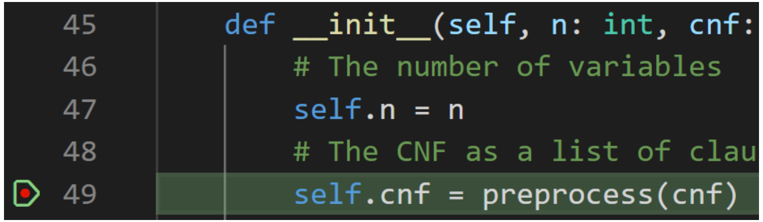
- Control flow: 

-  Continue (F5): run until next breakpoint hit
-  Step Over (F10): run just one more line of code
-  Restart (Ctrl+Shift+F5): start over from beginning
-  Stop (Shift+F5): quit the debugger

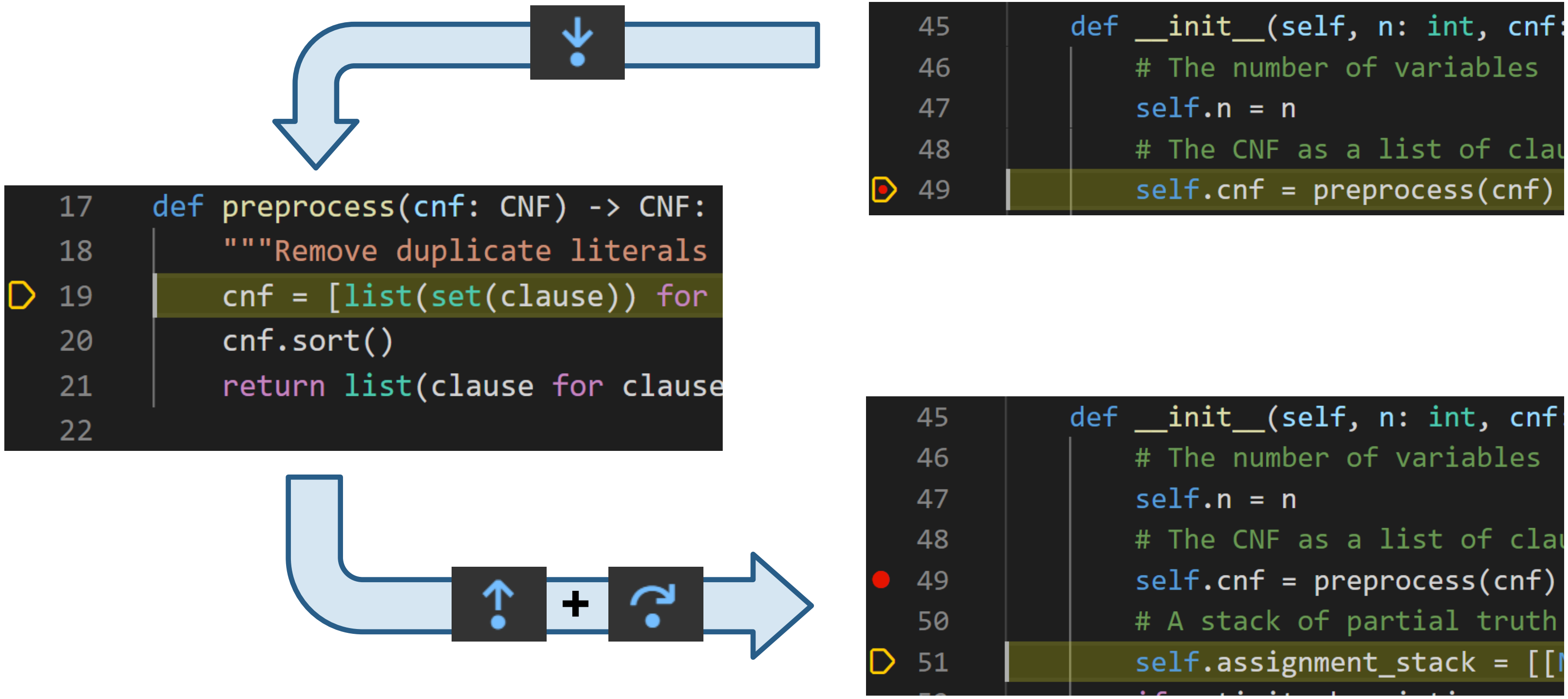
Debugging in VS Code

- **Step Into (F11):** enter code of first function called on the current line and resume debugging there
- **Step Out (Shift+F11):** run until the current function returns; resume debugging from parent function
- Can click to view different levels of the call stack
  - Useful for inspecting values of local vars in different scopes





Debugging in VS Code



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