CIS 189

Lecture 3: Algorithms for SAT

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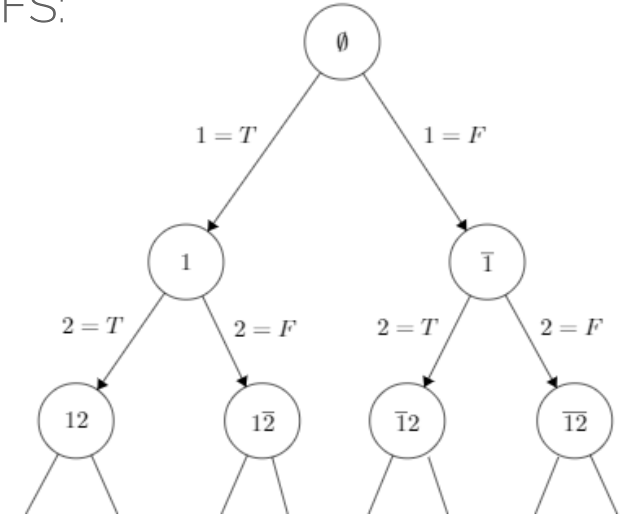
Naive Search for SAT

- Naive algorithm: try every possible assignment until we find a satisfying assignment or exhaust the search space
- Can interpret this as a DFS:

  (search tree)



Trimming the Search Space

- **When we set $x = T$, what happens to the clauses containing x?**
- **Observation 1:** Any clause containing the positive literal x becomes satisfied, so we no longer need to consider those clauses
  - In logic: $(T \vee 1 \vee 2 \vee \cdots) = T$
  - Significance: we should remove all clauses containing x

Trimming the Search Space

- **When we set $x = T$, what happens to the clauses containing $\overline{x}$?**
- **Observation 2:** Any clause containing the negative literal $\overline{x}$ needs to be satisfied by a different literal, so we can ignore $\overline{x}$ in that clause
  - In logic: $(F \vee 1 \vee 2 \vee \cdots) = (1 \vee 2 \vee \cdots)$
  - Significance: we should remove $\overline{x}$ from all clauses containing it

The Splitting Rule

- The previous observations are called the **splitting rule**
- After repeatedly applying the splitting rule to formula φ:
  - If there are **no clauses left**, then all clauses have been satisfied, so φ is satisfied
    - $\varphi = \emptyset$ denotes that there are no clauses left
  - If φ ever contains an **empty clause**, then all literals in that clause are False, so we made a mistake
    - ϵ denotes the empty clause
    - $\epsilon \in \varphi$ denotes that φ contains an empty clause

The Splitting Rule

- The splitting rule allows us to create a smarter recursive **backtracking** algorithm
- Backtracking: repeatedly make a guess to explore partial solutions, and if we hit "dead end" (contradiction) then undo the last guess

Backtracking Notation

- For a CNF φ and a literal x, define $\varphi|x$ (“φ given x”) to be a new CNF produced by:
  - Removing all clauses containing x
  - Removing $\overline{x}$ from all clauses containing it
- Conditioning is “commutative”: $\varphi|x_1|x_2 = \varphi|x_2|x_1$

Backtracking (Pseudocode)

```
# check if φ is satisfiable
backtrack(φ):
    if φ = ∅: return True
    if ϵ ∈ φ: return False
    let x = pick_variable(φ)
    return backtrack(φ|x) OR backtrack(φ|x̄)
```

Example: Backtracking

$(\overline{1} \vee \overline{2})$

$(\overline{1} \vee 2 \vee \overline{3})$

$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee 4 \vee \overline{5})$

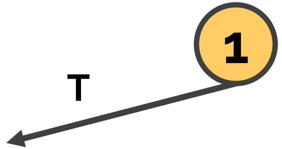
Steps

1	2	3	4	5

Example: Backtracking

$$(\overline{1} \lor \overline{2})$$
$$(\overline{1} \lor 2 \lor \overline{3})$$
$$(3 \lor \overline{4} \lor \overline{5})$$
$$(3 \lor 4 \lor \overline{5})$$

Steps



1	2	3	4	5
T				

Example: Backtracking

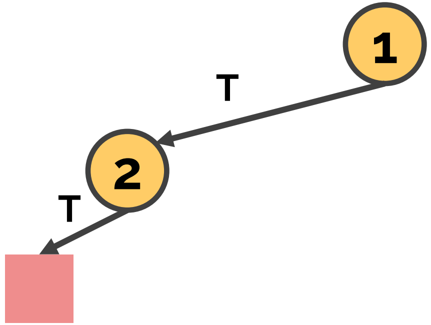
$(\overline{1} \lor \overline{2})$ **Conflict!**

$(\overline{1} \lor 2 \lor \overline{3})$

$(3 \lor \overline{4} \lor \overline{5})$

$(3 \lor 4 \lor \overline{5})$

Steps



1	2	3	4	5
T	T			

Example: Backtracking

$$(\overline{1} \vee \overline{2})$$

$$(\overline{1} \vee 2 \vee \overline{3})$$

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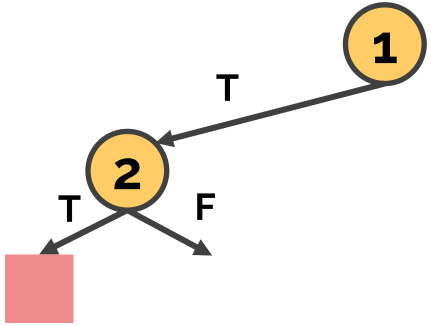
$$(3 \vee \overline{4} \vee 5)$$

Steps



1	2	3	4	5
T	F			

Example: Backtracking

$(\overline{1} \vee \overline{2})$

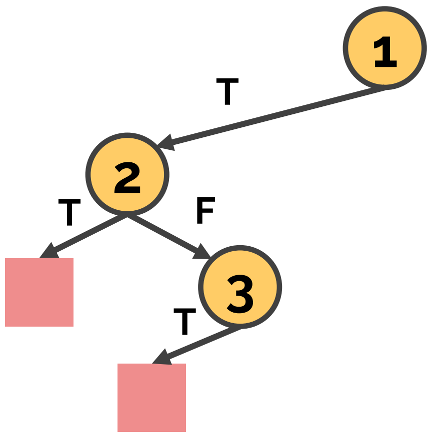
$(\overline{1} \vee 2 \vee \overline{3})$

$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee \overline{4} \vee 5)$

Conflict!

Steps



1	2	3	4	5
T	F	T		

Example: Backtracking

$(\overline{1} \vee \overline{2})$

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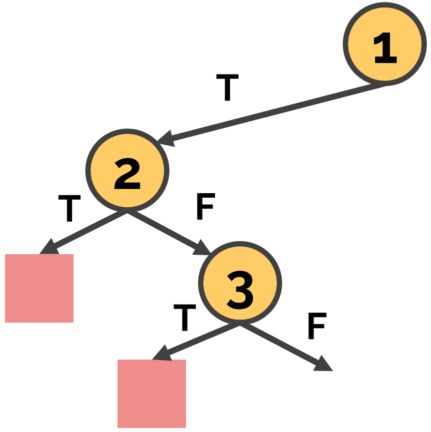
$(3 \vee 4 \vee \overline{5})$

Steps



1	2	3	4	5
T	F	F		

Example: Backtracking

$(\overline{1} \vee \overline{2})$

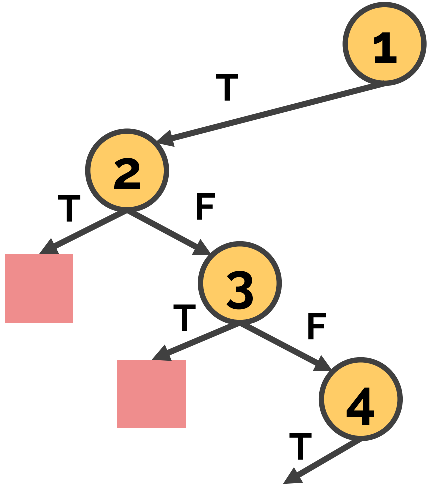
$(\overline{1} \vee 2 \vee \overline{3})$

$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee 4 \vee \overline{5})$

1	2	3	4	5
T	F	F	T	

Steps



Example: Backtracking

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$(\overline{1} \vee 2 \vee \overline{3})$

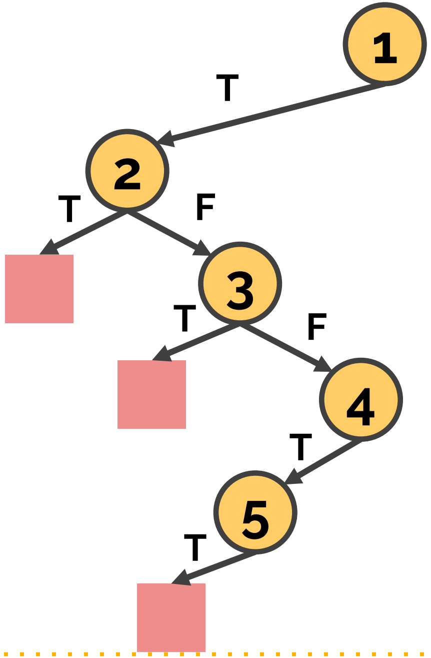
$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee 4 \vee \overline{5})$

Conflict!

1	2	3	4	5
T	F	F	T	T

Steps



Example: Backtracking

$(\overline{1} \vee \overline{2})$

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$(3 \vee \overline{4} \vee \overline{5})$

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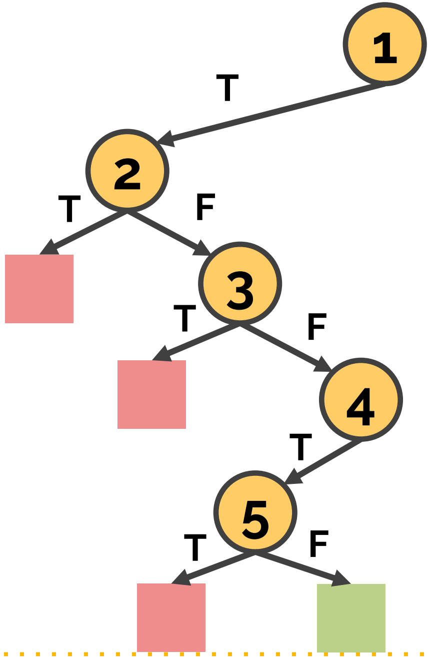
1	2	3	4	5
T	F	F	T	F

Steps



Efficient Splitting

- How do we compute $\varphi|x$?
- Goals:
  - Support fast searching for empty clauses
  - Support fast backtracking
  - Fast to actually compute $\varphi|x$

Naïve Idea 1

- Transform φ into $\varphi|x$ by deleting satisfied clauses and False literals from φ
    - Deletion not too expensive if we use linked lists
    - Can quickly recognize an empty clause (linked list will be empty), but need to check all clauses
    - Big issue: how do we backtrack?

Naïve Idea 2

- Simple fix: instead of modifying φ directly, create a copy first and modify that
    - Easy backtracking – just restore the old formula
    - Big issue: too expensive (time and memory) to copy formula every time we split
        - What if we have hundreds of thousands, even millions of clauses?

Towards a smarter scheme

- Don't modify or copy the formula
- **Observation:** if we set $x = T$, the only clauses that become empty must contain $\overline{x}$
  - Store a dictionary mapping each literal to a list of all clauses that contain it
  - But we can do even better!`

1 Watched Literal Scheme

- **Observation:** a clause can only become empty if it has just one unassigned literal remaining
  - Ideally, only need to check these clauses
- Each clause "watches" one literal and maintains watching invariant: the watched literal is True or unassigned
  - If the watched literal becomes False, watch another
  - If there are no more True/unassigned literals to watch, then the clause must be empty

1 Watched Literal Scheme

- **Watchlists** data structure: maps each literal to a list of clauses currently watching it
- When setting $x = T$, only need to check watchlist of $\overline{x}$
  - Suppose we successfully maintain the watching invariant. What can we say about the watchlist of $\overline{x}$?

Example: 1 Watched Literal

$(\overline{1} \vee \overline{2})$

$(\overline{1} \vee 2 \vee \overline{3})$

$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee 4 \vee \overline{5})$

1	2	3	4	5

Steps

Example: 1 Watched Literal

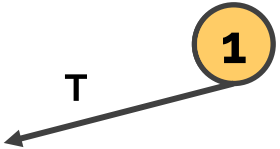
$(\overline{1} \vee \overline{2})$

$(\overline{1} \vee 2 \vee \overline{3})$

$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee 4 \vee \overline{5})$

Steps



1	2	3	4	5
T				

Example: 1 Watched Literal

$(\overline{1} \lor \overline{2})$

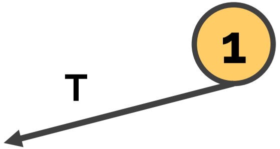
$(\overline{1} \lor 2 \lor \overline{3})$

$(3 \lor \overline{4} \lor \overline{5})$

$(3 \lor 4 \lor \overline{5})$

1	2	3	4	5
T				

Steps



Example: 1 Watched Literal

$$(\bar{1} \vee \bar{2})$$

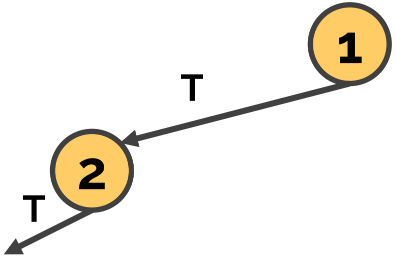
$$(\bar{1} \vee 2 \vee \bar{3})$$

$$(3 \vee \bar{4} \vee \bar{5})$$

$$(3 \vee 4 \vee \bar{5})$$

1	2	3	4	5
T	T			

Steps



Example: 1 Watched Literal

$(\overline{1} \vee \boxed{\overline{2}})$ **Conflict!**

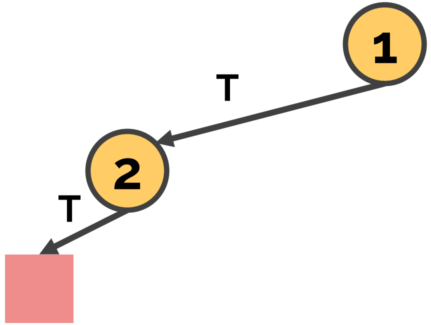
$(\overline{1} \vee 2 \vee \overline{3})$

$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee 4 \vee \overline{5})$

1	2	3	4	5
T	T			

Steps



Example: 1 Watched Literal

$(\bar{1} \vee \bar{2})$

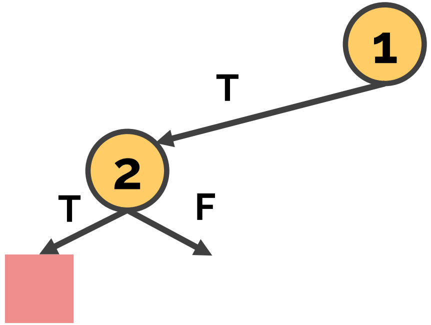
$(\bar{1} \vee 2 \vee \bar{3})$

$(3 \vee \bar{4} \vee \bar{5})$

$(3 \vee 4 \vee \bar{5})$

1	2	3	4	5
T	F			

Steps



Example: 1 Watched Literal

$(\overline{1} \vee \overline{2})$

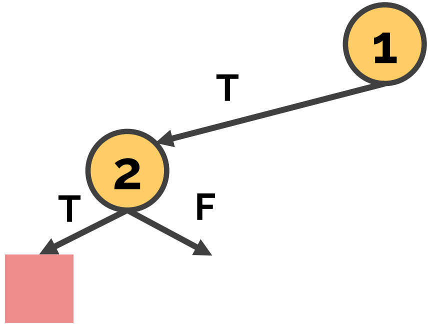
$(\overline{1} \vee 2 \vee \overline{3})$

$(3 \vee \overline{4} \vee \overline{5})$

$(3 \vee 4 \vee \overline{5})$

1	2	3	4	5
T	F			

Steps



Example: 1 Watched Literal

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$(3 \vee \bar{4} \vee 5)$

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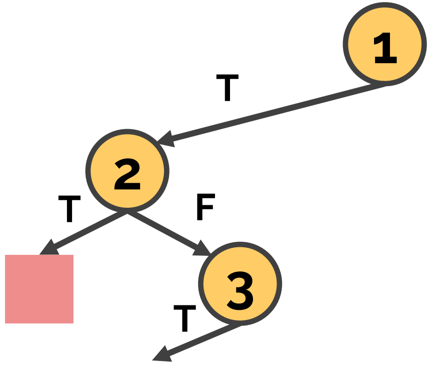
1	2	3	4	5
T	F	T		

Steps



Example: 1 Watched Literal

$(\bar{1} \lor \bar{2})$

$(\bar{1} \lor 2 \lor \bar{3})$

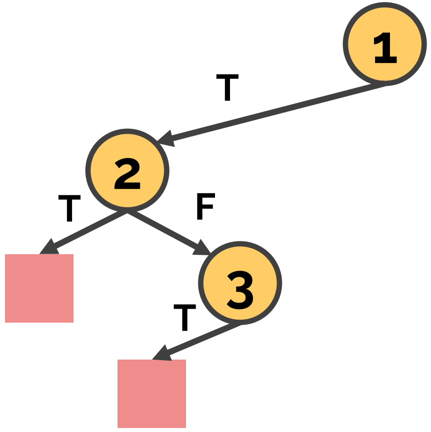
$(3 \lor \bar{4} \lor \bar{5})$

$(3 \lor 4 \lor \bar{5})$

Conflict!

1	2	3	4	5
T	F	T		

Steps



Example: 1 Watched Literal

$(\bar{1} \vee \boxed{\bar{2}})$

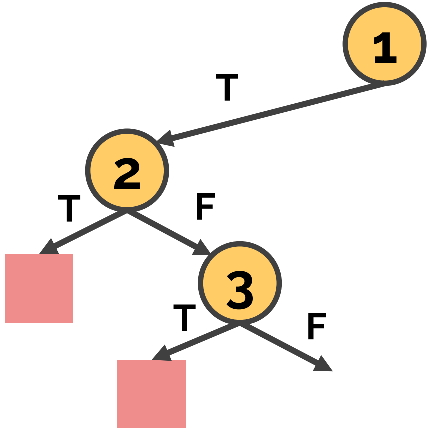
$(\bar{1} \vee 2 \vee \boxed{\bar{3}})$

$(\boxed{3} \vee \bar{4} \vee \bar{5})$

$(\boxed{3} \vee 4 \vee \bar{5})$

1	2	3	4	5
T	F	F		

Steps



Example: 1 Watched Literal

$(\bar{1} \vee \bar{2})$

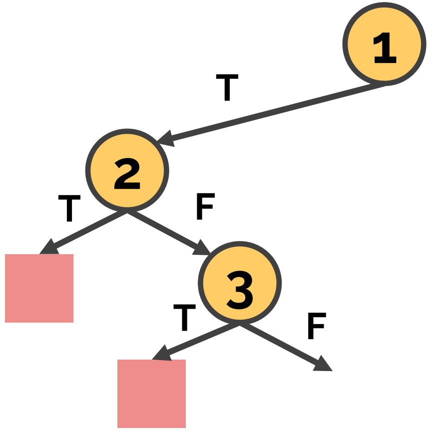
$(\bar{1} \vee 2 \vee \bar{3})$

$(3 \vee \bar{4} \vee 5)$

$(3 \vee 4 \vee \bar{5})$

1	2	3	4	5
T	F	F		

Steps



Example: 1 Watched Literal

$(\overline{1} \vee \boxed{\overline{2}})$

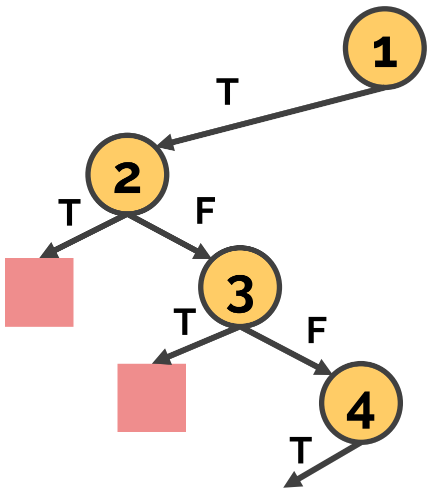
$(\overline{1} \vee 2 \vee \boxed{\overline{3}})$

$(3 \vee \boxed{\overline{4}} \vee \overline{5})$

$(3 \vee \boxed{4} \vee \overline{5})$

1	2	3	4	5
T	F	F	T	

Steps



Example: 1 Watched Literal

$(\overline{1} \lor \overline{2})$

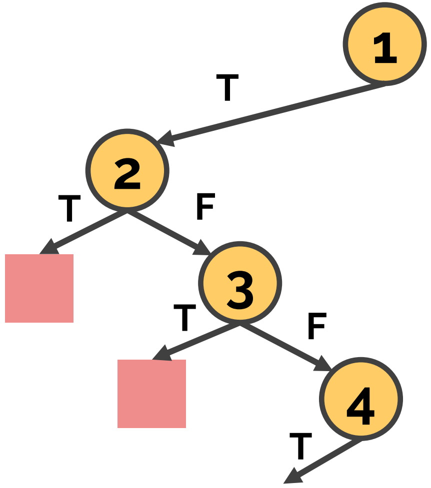
$(\overline{1} \lor 2 \lor \overline{3})$

$(3 \lor \overline{4} \lor \overline{5})$

$(3 \lor 4 \lor \overline{5})$

1	2	3	4	5
T	F	F	T	

Steps



Example: 1 Watched Literal

$(\overline{1} \vee \overline{2})$

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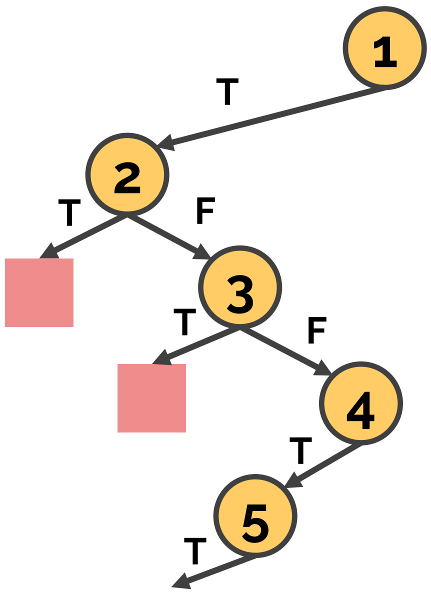
$(3 \vee 4 \vee \overline{5})$

1	2	3	4	5
T	F	F	T	T

Steps



Example: 1 Watched Literal

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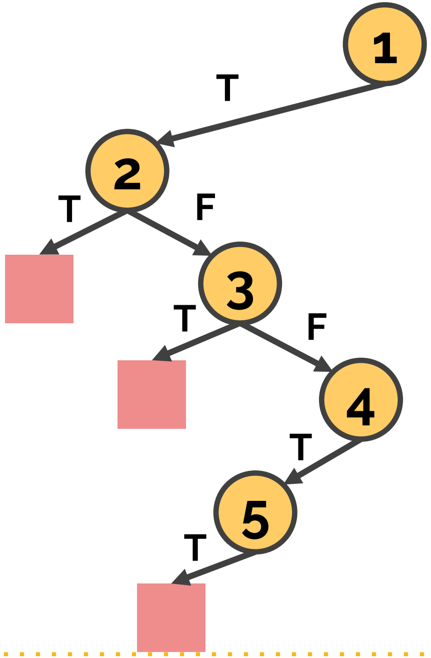
Conflict!

1	2	3	4	5
T	F	F	T	T

Steps



Example: 1 Watched Literal

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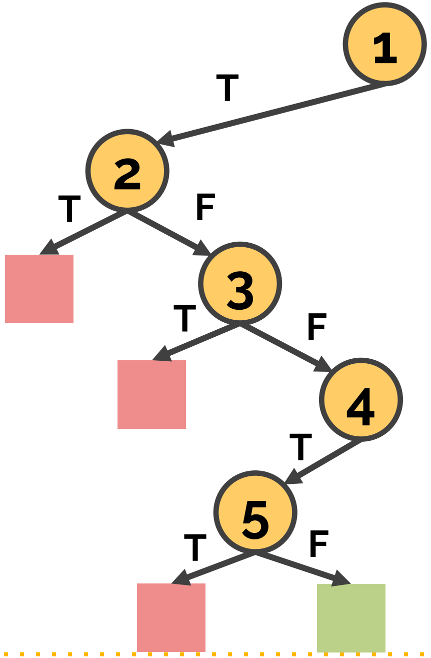
1	2	3	4	5
T	F	F	T	F

Steps

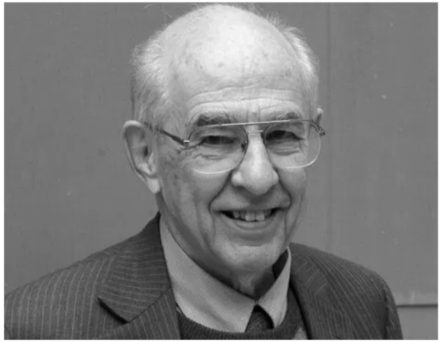


Unit Propagation (UP)

- A **unit clause** is a clause containing only one literal
- **Unit propagation rule:** for any unit clause $\{\ell\}$, we must set $\ell = T$
- Applying unit propagation can massively speed up the backtracking algorithm in practice
  - Combining with the splitting rule can lead to a “domino effect” of cascading unit propagation

The DPLL Algorithm

- Davis-Putnam-Logemann-Loveland (1962)
- Improved upon naive backtracking **(search)** with unit propagation **(inference)**
- Still the basic algorithm behind most state-of-the-art SAT solvers today!



DPLL (Pseudocode)

```
dpll(φ):
    if φ = ∅: return TRUE
    if ϵ ∈ φ: return FALSE
    if φ contains unit clause {ℓ}:
        return dpll(φ|ℓ)
    let x = pick_variable(φ)
    return dpll(φ|x) OR dpll(φ|x̄)
```

Example: DPLL

$(\overline{1} \vee \overline{2})$

$(\overline{1} \vee 2)$

$(1 \vee \overline{2} \vee 3)$

$(1 \vee 2 \vee \overline{4})$

1	2	3	4

Steps

Example: DPLL

$(\overline{1} \vee \overline{2})$ **Unit!**

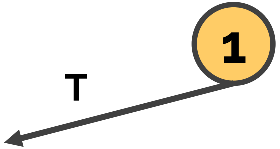
$(\overline{1} \vee 2)$

$(1 \vee \overline{2} \vee 3)$

$(1 \vee 2 \vee \overline{4})$

1	2	3	4
T			

Steps



Example: DPLL

$(\overline{1} \lor \overline{2})$

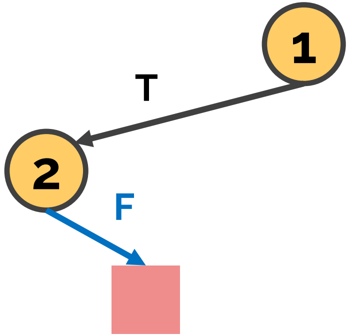
$(\overline{1} \lor 2)$

$(1 \lor \overline{2} \lor 3)$

$(1 \lor 2 \lor \overline{4})$

Conflict!

Steps



1	2	3	4
T	F		

Example: DPLL

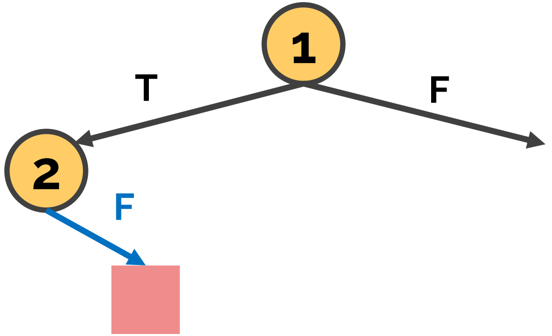
$(\overline{1} \vee \overline{2})$

$(\overline{1} \vee 2)$

$(1 \vee \overline{2} \vee 3)$

$(1 \vee 2 \vee \overline{4})$

Steps



1	2	3	4
F			

Example: DPLL

$(\overline{1} \vee \overline{2})$

$(\overline{1} \vee 2)$

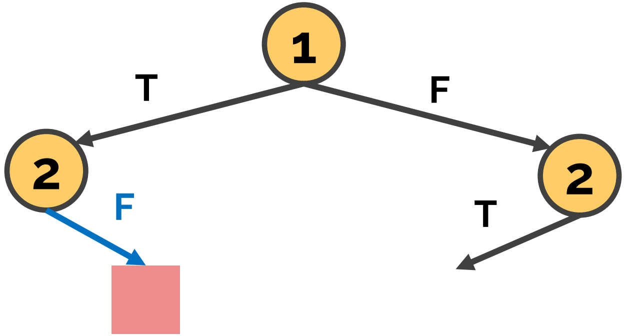
$(1 \vee \overline{2} \vee 3)$

$(1 \vee 2 \vee \overline{4})$

Unit!

1	2	3	4
F	T		

Steps



Example: DPLL

$(\overline{1} \vee \overline{2})$

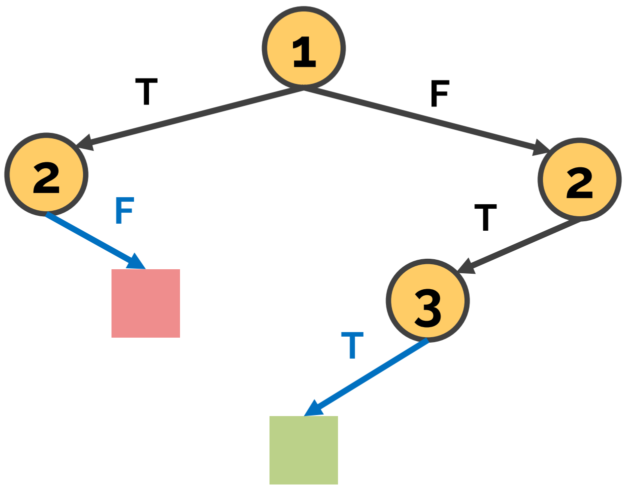
$(\overline{1} \vee 2)$

$(1 \vee \overline{2} \vee 3)$

$(1 \vee 2 \vee \overline{4})$

1	2	3	4
F	T	T	

Steps



Engineering Matters

- Since the main DPLL subroutine might run exponentially many times, every speedup counts
- DPLL spends by far the most time on UP
  - How can we speed this up?
- Although DPLL has a natural recursive formulation, recursion is slow — lots of overhead
  - We can make DPLL **iterative** using a stack

2 Watched Literals (2WL)

- **Key observation:** a clause can only be unsatisfied or unit if it has at most one non-False literal
  - Optimize unit propagation: only visit those clauses
- Each clause “watches” two literals and maintains watching **invariant:** the watched literals are not False, unless the clause is satisfied
  - If a watched literal becomes False, watch another
- If can’t maintain invariant, clause is unit (can propagate)

2 Watched Literals (2WL)

- Still use watchlists (list of all clauses watching each lit)
- Best part: since backtracking only unassigns variables, it can never break the 2WL invariant
  - Don't need to update watchlists

Unit!

$$(\bar{1} \vee 2 \vee \bar{3}) \xrightarrow{\text{Set } 1 = T} (\bar{1} \vee 2 \vee \bar{3}) \xrightarrow{\text{Set } 2 = F} (\bar{1} \vee 2 \vee \bar{3})$$

Iterative DPLL

- A **decision** refers to any time our algorithm *arbitrarily* assigns a variable (without being forced to do so)
  - Selecting a literal and assigning it True is a decision
  - Unit propagation & reassigning selected literal after backtracking are not decisions
- All assignments implied by the i^{th} decision are said to be on the i^{th} **decision level**
  - Can assignments ever be on the zeroth decision level?

Iterative DPLL

- Maintain an **assignment stack** with the assignments from each decision level
    - Whenever we make a new decision, copy the current assignment onto the top of the stack
- To backtrack: pop the current assignment off the stack, restoring the previous one
- Keep a **propagation queue** of literals that are set to False
    - Take literals from the queue and check if their watching clauses are empty/unit

Assignment Stack

T	T	F	T	T
T	T	F		
T				
1	2	3	4	5

Set $2 = T$. Propagate $3 = F$.

Set $1 = T$

Assignment Stack

Pop!

T	T	F	T	T

Backtrack!

T	T	F		
T				
1	2	3	4	5

Set $2 = T$. Propagate $3 = F$.

Set $1 = T$

Iterative DPLL (Pseudocode)

```
dpll(φ):
    if unit_propagate() = CONFLICT: return UNSAT
    while not all variables have been set:
        let x = pick_variable()
        create new decision level
        set x = T
        while unit_propagate() = CONFLICT:
            if decision_level = 0: return UNSAT
            backtrack()
            set x = F
   return SAT
```

Iterative DPLL (Pseudocode)

```
dpll(φ):
    if unit_propagate() = CONFLICT: return UNSAT
    while not all variables have been set:
        let x = pick_variable()
        create new decision level
        set x = T
        while unit_propagate() = CONFLICT:
            if decision_level = 0: return UNSAT
            backtrack()
            set x = F
    return SAT
```

How to implement this?

How should we branch?

- Order of assigning variables greatly affects runtime
- Want to find a satisfying assignment quicker and find conflicts (rule out bad assignments) quicker

- Ex: $\{1\bar{2}34, \overline{12}3, 12\bar{3}5, 23\bar{5}, 3\overline{45}, \ldots, 67, \bar{6}7, 6\bar{7}, \overline{67}\}$
  - If we assign 6 first, then we can find conflicts right away

Decision Heuristics

- **Static heuristics:** variable ordering fixed at the start
- **Dynamic heuristics:** variable ordering is updated as the solver runs
    - More effective, but also more expensive
- Basic examples of decision heuristics:
    - Random ordering
    - Most-frequent static ordering
    - Most-frequent dynamic ordering

References

A. Biere, *Handbook of satisfiability*. Amsterdam: IOS Press, 2009.

N. Eén and N. Sörensson, “An Extensible SAT-solver,” *Theory and Applications of Satisfiability Testing Lecture Notes in Computer Science*, pp. 502–518, 2004.