Problem 1: Let $T$ be a tree where the maximum degree is $\Delta$. Prove that $T$ has at least $\Delta$ leaves by contradiction.

Solution:
Assume that $\Delta \geq 2$, since the cases of $\Delta = 0$ and $\Delta = 1$ are clearly true. Suppose for the sake of contradiction that there are at most $\psi < \Delta$ leaves. Let $v \in V$ have degree $\Delta$. Consider $S = \{u \in V \mid \{u, v\} \in E\}$. Note that $S$ is the set of $v$’s neighbors, and $|S| = \Delta$.

For all $u_i \in S$, there exists at least one path that starts with $\{v, u_i\}$ that ends with a leaf. We pick any such leaf for each edge $\{v, u_i\}$ and call the leaf $l_i$. Note there is a unique $l_i$ corresponding to each $u_i$, as trees are acyclic, so we have $\Delta$ $l_i$’s in total. Hence, by the Pigeonhole Principle, where the pigeons are the terminating leaves $l_i$ of each path and the holes are the $\psi$ leaves available, we know that $\left\lceil \frac{\Delta}{\psi} \right\rceil \geq \left\lceil \frac{\Delta}{\Delta - 1} \right\rceil = \left\lceil 1 + \frac{1}{\Delta - 1} \right\rceil$ (since $\Delta \geq 2$) = 2 paths share the same terminating leaf, say $\ell_\omega$.

This is a contradiction, since the path between $\ell_\omega$ and $v$ are unique in a tree.

For each $u_i \in S$, let $p_i$ be a maximal path starting from $v - u_i$. Note that there must be $\Delta$ such paths. We know from the lemma proven above that all such $p_i$ must terminate in a leaf $\ell_i$. 
Problem 2:
Prove that \( G \) or the complement of \( G \) is connected. Note that the complement of a graph \( G = (V, E) \) is \( G^c = (V, E') \) and \( \forall u, v \in V, \{u, v\} \in E' \iff \{u, v\} \notin E. \)

Solution:
If \( G \) is connected we are done.

If \( G \) is not connected then \( G \) is composed of multiple connected components. We want to prove that given two arbitrary vertices in \( G \) there must be a path between them in \( G^c \). Let these two arbitrary vertices be \( u \) and \( v \).

Case 1: \( u \) and \( v \) do not share an edge in \( G \)
This means they must share an edge in \( G^c \) and thus there is a path from \( u \) to \( v \) in \( G^c \).

Case 2: \( u \) and \( v \) share an edge in \( G \)
This means they were part of the same connected component in \( G \). Take an arbitrary vertex \( x \) in a different connected component in \( G \). Edges \( u - x \) and \( v - x \) must both exist in \( G^c \). Thus, there is a path \( u - x - v \) between vertices \( u \) and \( v \).

Thus, we have shown that there exists a path between any two arbitrary vertices in \( G^c \). By definition \( G^c \) must be connected. The claim is proved.