Strong Induction

Let $n_0$ be a natural number and let $P(n)$ be a predicate for all natural numbers $n \geq n_0$.

Base Case: $P(n_0)$ holds

Induction Hypothesis: $P(j)$ is true for $n_0 \leq j \leq k$

Induction Step: We want to show that $P(k + 1)$ is true. That is, for any $k \geq n_0$,

$P(n_0) \land P(n_0 + 1) \land ... \land P(k) \implies P(k + 1)$ is true.

Strong and ordinary induction are mathematically equivalent.
Graph Terminology

- Undirected graph: \( G = (V, E) \) where
  - \( V \): finite, non-empty set of vertices
  - \( E \): finite (possibly empty) set of edges
  - An edge \( \{u, v\} \) connects vertices \( u \) and \( v \).
Graph Terminology - Continued

- Two vertices \( u, v \) are **adjacent** if \( \{ u, v \} \in E \).
- Nodes adjacent to a vertex \( u \) are called **neighbors** of \( u \).
- **Degree** of a vertex, \( \deg(u) \) is the number of neighbors of \( u \).
- \( \delta(G) = \min_{v \in V} \deg(v) \) is the minimum degree of \( G \).
- \( \Delta(G) = \max_{v \in V} \deg(v) \) is the maximum degree of \( G \).
Graph Lemmas

- The Handshaking Lemma: the sum of the degrees of all vertices in a graph is twice the number of edges

\[ \sum_{v \in V} deg(v) = 2|E| \]

- In any graph, there are an even number of vertices of odd degree
Intro to Probability

- The sample space $\Omega$ is the set of all possible outcomes.
- The probability space is a sample space together with a probability distribution assigned to each outcome $\omega \in \Omega$ s.t.
  
  $$0 \leq Pr[\omega] \leq 1$$

  $$\sum_{\omega \in \Omega} Pr[\omega] = 1$$

- A subset of the sample space is called an event.
- For any event $A \subseteq \Omega$, the probability of $A$ is defined as:
  
  $$Pr[A] = \sum_{\omega \in A} Pr[\omega]$$
A probability space \((\Omega, \Pr)\) is uniform if all outcomes have the same probability

\[
\Pr[\omega] = \frac{1}{|\Omega|}, \text{ for all } \omega \in \Omega
\]

\[
\Pr[E] = \frac{|E|}{|\Omega|}, \text{ for some event } E \subseteq \Omega
\]
Steps to Solve Probability Problems

1. Define a sample space $\Omega$ of the experiment.

2. Define the probability distribution.

3. Find the event of interest $A$ (subset of outcomes $A \subseteq \Omega$ that are of interest).

The Inclusion-Exclusion Formula

- If $A, B, C$ are any events,

$$
Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]
$$
$$
Pr[A \cup B \cup C] = Pr[A] + Pr[B] + Pr[C] - Pr[A \cap B] - Pr[A \cap C] - Pr[B \cap C] + Pr[A \cap B \cap C]
$$

- Union-bound

$$
Pr[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} Pr[A_i]
$$

If the events are pairwise disjoint, the inequality becomes equality.