Topics Covered: Proofs, Counting

Problem 1: Let $m$ and $n$ be two integers. Prove that $mn + m$ is odd if and only if $m$ is odd and $n$ is even.

Solution:

Lemma 1: For two integers $x$ and $y$, if $xy$ is odd, then $x$ and $y$ are both odd.

We will prove the Lemma through a proof by contrapositive. In other words, we will prove “If $x$ or $y$ is even, then $xy$ is even.”

WLOG, let $x$ be the even integer.

We can write $x = 2k$, for some $k \in \mathbb{Z}$. Then, we have,

$$xy = (2k)(y) = 2(ky)$$

which is even, as $ky \in \mathbb{Z}$.

($\implies$) If $mn + m$ is odd, then $m$ is odd and $n$ is even.

We can write $mn + m = m(n + 1)$. Then, according to Lemma 1, $m$ and $n + 1$ must both be odd.

Since we know that $n + 1$ is odd and 1 is odd, then $n$ must be an even integer (because only even + odd = odd).

Therefore, $m$ is odd and $n$ is even.

($\iff$) If $m$ is odd and $n$ is even, then $mn + m$ is odd.

We can write $m = 2k + 1$, for some $k \in \mathbb{Z}$ and $n = 2l$, for some $l \in \mathbb{Z}$. Then, we have,

$$mn + m = (2k + 1)(2l) + (2k + 1)$$

$$= 4kl + 2l + 2k + 1$$

$$= 2(2kl + l + k) + 1$$

which is odd, as $2kl + l + k \in \mathbb{Z}$. 

**Problem 2:** Let $A = \{n \mid n = 2k + 5 \text{ for some } k \in \mathbb{N}\}$ and $B = \{n \mid n = 2j + 1 \text{ for some } j \in \mathbb{N}\}$. Is $A \subseteq B$?

Let $x$ be any arbitrary but particular element in $A$. Then,

\[
x = 2k + 5, \quad \text{for some integer } k.
\]

\[
= 2k + 4 + 1
\]

\[
= 2(k + 2) + 1
\]

Since $k \in \mathbb{N}$, $k + 2 \in \mathbb{N}$, and hence we have proved that any arbitrary element $x \in A$ also belongs to the set $B$. Thus $A \subseteq B$. 
**Problem 3:** Your favorite TA in the world, Andrew, is heading to the notorious E-formal. Andrew has 5 different spicy suits and 6 different ties bought from Amazon. In addition, he can add one of 20 different accessories, which are optional. On top of all of these choices, he can choose black shoes or tan shoes. How many different outfits can Andrew possibly wear?

We can apply the multiplication rule:

- **Step 1:** Choose the suit. (5 ways)
- **Step 2:** Choose the tie. (6 ways)
- **Step 3:** Choose the accessory. (20 accessories + 1 option for no accessories = 21 ways)
- **Step 4:** Choose the shoes. (2 ways)

By the Multiplication Rule, we get $5 \times 6 \times 21 \times 2 = 1260$ outfits.