Functions

Let $A$ and $B$ be sets. A function from $A$ to $B$ is a relation, $f$, from $A$ to $B$ such that for all $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. If $(a, b) \in f$, then we write $b = f(a)$. A function from $A$ to $B$ is also called a mapping from $A$ to $B$ and we write it as $f : A \rightarrow B$. The set $A$ is called the domain of $f$ and the set $B$ the codomain. If $a \in A$ then the element $b = f(a)$ is called the image of $a$ under $f$. The range of $f$, denoted by $\text{Ran}(f)$ is the set

$$\text{Ran}(f) = \{ b \in B \mid \exists a \in A \text{ such that } b = f(a) \}$$

Two functions are equal if they have the same domain, have the same codomain, and map each element of the domain to the same element in the codomain.

Example. Let $A$ and $B$ be finite sets of size $a$ and $b$, respectively. How many functions are there from $A$ to $B$?

Solution. The procedure of forming a function is as follows: in Step $i$ choose the image of the $i$th element in $A$. There are $a$ steps and there are $b$ ways to perform each step. Thus the total number of ways to create a function from $A$ to $B$ is $b^a$.

Let $f : A \rightarrow B$ be a function.

- $f$ is said to be one-to-one or injective, iff for every $x, y \in A$ such that $x \neq y, f(x) \neq f(y)$.
- $f$ is called onto or surjective, iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- $f$ is a one-to-one correspondence or bijection, if it is both one-to-one and onto.

Example. Classify the following functions.

- $f_1(x) = x^2$ from the set of integers to the set of integers.
- $f_2(x) = x^2$ from the set of non-negative real numbers to the set of non-negative real numbers.
- $f_3(x) = x + 1$ from the set of integers to the set of integers.
- $f_4(x) = x$ from a set $A$ to $A$. This function is called the identity function.
Solution.

injective: \( f_2, f_3, f_4 \)

surjective: \( f_2, f_3, f_4 \)

bijective: \( f_2, f_3, f_4 \)

Inverse and Composition

Let \( f \) be a one-to-one correspondence from the set \( A \) to the set \( B \). The inverse function of \( f \) is the function that maps an element \( b \in B \) to the unique element \( a \in A \) such that \( f(a) = b \). The inverse function of \( f \) is denoted by \( f^{-1} \). Hence \( f^{-1}(b) = a \) when \( f(a) = b \).

Note that if \( f \) is not bijective then its inverse does not exist.

Let \( f: A \to B \) and \( g: B \to C \) be functions. The composition of the function \( g \) with \( f \) is the function \( g \circ f: A \to C \), defined by 

\[
(g \circ f)(x) = g(f(x)), \forall x \in A 
\]

Example. Let \( g \) be the function from the set \( \{a, b, c\} \) to itself such that \( g(a) = b, g(b) = c, \) and \( g(c) = a \). Let \( f \) be the function from the set \( \{a, b, c\} \) to the set \( \{1, 2, 3\} \) such that \( f(a) = 3, f(b) = 2, \) and \( f(c) = 1 \). What is the composition of \( f \) with \( g \) and what is the composition of \( g \) with \( f \)?

Solution. The composition function \( f \circ g \) is as follows: \( (f \circ g)(a) = f(g(a)) = f(b) = 2 \), \( (f \circ g)(b) = f(g(b)) = f(c) = 1 \), and \( (f \circ g)(c) = f(g(c)) = f(a) = 3 \).

\( (g \circ f) \) is not defined as the range of \( f \) is not a subset of the domain of \( g \).

Example. Let \( f \) and \( g \) be the functions from the set of integers to the set of integers defined by \( f(x) = 2x + 3 \) and \( g(x) = 3x + 2 \). What is the composition of \( f \) and \( g \)? What is the composition of \( g \) and \( f \)?

Solution. \( (f \circ g)(x) = f(g(x)) = 2(3x + 2) + 3 = 6x + 7 \). Similarly, \( (g \circ f)(x) = g(f(x)) = 3(2x + 3) + 2 = 6x + 11 \). This example shows that commutative law does not apply to the composition of functions.

Example. Let \( f: A \to B \) and \( g: B \to C \) be two functions. Then

i. if \( f \) and \( g \) are surjective then so is \( g \circ f \).

ii. if \( f \) and \( g \) are injective then so is \( g \circ f \).

iii. if \( f \) and \( g \) are bijective then so is \( g \circ f \).
Solution. Let \( c \in C \). Since \( g \) is surjective there must be a \( b \in B \) such that \( g(b) = c \). Since \( f \) is surjective there must be a \( a \in A \) such that \( f(a) = b \). Thus \( (g \circ f)(a) = g(f(a)) = g(b) = c \). This proves that \( g \circ f \) is surjective.

Let \( a, a' \in A \) such that \( (g \circ f)(a) = (g \circ f)(a') \). This means that \( g(f(a)) = g(f(a')) \). Since \( g \) is injective we have \( f(a) = f(a') \). Then since \( f \) is injective, we have \( a = a' \).

The bijectivity of \( (g \circ f) \) follows from the injectivity and surjectivity of \( (g \circ f) \).

The Ramsey number \( R(k, l) \) is the smallest number \( n \) such that any graph with \( n \) vertices has clique of size \( k \) or an independent set of size \( l \). Another way to formulate this is: in any two-coloring (say, red and blue) on edges of the complete graph on \( n \) vertices, there is a monochromatic red clique of size \( k \) or a monochromatic blue clique of size \( l \). Diagonal Ramsey Number asks for the value of \( R(k, k) \) for any integer \( k \). Finding a diagonal Ramsey number even for \( k = 6 \) is very difficult. We want to find a lower bound on \( R(k, k) \).

Example. If \( \binom{n}{k} 2^{1-(\frac{k}{2})} < 1 \), then \( R(k, k) > n \). In particular, \( R(k, k) > \lceil 2^{\frac{k}{2}} \rceil \), for \( k \geq 3 \).

Solution. Consider a complete graph \( G \) in which each edge is colored red or blue with a probability of \( 1/2 \). Let \( S \) be a any subset of \( k \) vertices and \( E(S) \) be the set of edges with both endpoints in \( S \).

\[
\Pr[\text{edges in } E(S) \text{ are monochromatic}] = 2 \cdot 2^{−\binom{k}{2}}
\]

Since there are \( \binom{n}{k} \) subsets of size \( k \), the probability that some subset of size \( k \) is monochromatic is at most

\[
2 \binom{n}{k} 2^{−\binom{k}{2}} = \binom{n}{k} 2^{1−\binom{k}{2}} \tag{1}
\]

Since the last expression is less than 1 (given as a condition in the problem statement), there is a 2-coloring of edges of a complete graph on \( n \) vertices in which there is no monochromatic clique of size \( k \). Thus \( R(k, k) > n \).

If \( n = \lceil 2^{k/2} \rceil \) then

\[
\binom{n}{k} 2^{1−\binom{k}{2}} \leq \frac{n^k}{k!} \cdot 2^{1−\frac{k(k−1)}{2}} \leq \left(\frac{2^{k^2/2}}{k!}\right) 2^{1−\frac{k^2}{2}+\frac{k}{2}} = \frac{2^{1+\frac{k}{2}}}{k!}
\]

Note that the last expression is less than 1, if \( k \geq 3 \).

It can be shown that \( R(k, k) < 4^k \). These are the best known bounds on the size of \( R(k, k) \), so a lot of progress is yet to be made. What is known is that \( R(2, 2) = 2, R(3, 3) = 6 \), and \( R(4, 4) = 10 \). The values of \( R(k, k) \) are not known for \( k \geq 5 \).

Recall that a tournament is a directed graph with exactly one directed edge between any pair of vertices. A tournament \( G = (V, E) \) is called \( k \)-dominated if for every set of \( k \) vertices \( v_1, v_2, \ldots, v_k \), there exists another vertex \( u \in V \) such that \( (u, v_i) \in E \), for \( i = 1, 2, \ldots, k \).