Mathematical Foundations of Computer Science
Lecture Outline
November 14, 2023

Matching in Bipartite Graphs

An independent set of a graph is a set of pair-wise non-adjacent vertices. A bipartite graph, \((U, V, E)\), is a graph whose vertex set is \(U \cup V\) and for each edge \(e = (u, v) \in E\), \(u \in U\) and \(v \in V\). In other words, \(U\) and \(V\) are independent sets and each edge in \(E\) connects a vertex in \(U\) to a vertex in \(V\).

Now consider the following scenario. There is a set of girls and a set of boys. Each girl likes some boys and dislikes others. What conditions would guarantee that each girl is paired-up with a boy that she likes and that no two girls are paired-up with the same boy.

We can model this situation using a bipartite graph, \((X, Y, E)\), where each vertex in \(X\) represents a girl, each vertex in \(Y\) represents a boy and an edge \((g, b) \in E\) means that girl \(g\) likes boy \(b\). We are interested in the conditions that would guarantee a matching that saturates every vertex in \(X\).

Hall’s theorem gives the necessary and sufficient conditions for the existence of such matchings in bipartite graphs.

Example. [Hall’s Theorem] Let \(G = (X, Y, E)\) be a bipartite graph. For any set \(S\) of vertices, let \(N_G(S)\) be the set of vertices adjacent to vertices in \(S\). Prove that \(G\) contains a matching that saturates every vertex in \(X\) iff \(|N_G(S)| \geq |S|, \forall S \subseteq X\). The condition “For all \(S \subseteq X, |N(S)| \geq |S|\)” is called Hall’s condition.

Solution. We prove that Hall’s condition is necessary as follows. Suppose \(G\) contains a matching \(M\) that saturates every vertex in \(X\). Let \(S\) be a subset of \(X\). Since each vertex in \(S\) is matched under \(M\) to a distinct vertex in \(N_G(S), |N_G(S)| \geq |S|\).

We will now prove the sufficiency of Hall’s condition, i.e., if \(|N_G(S)| \geq |S|, \forall S \subseteq X\) then \(G\) contains a matching that saturates every vertex in \(X\). We prove this by induction on the size of \(X\).

Base Case: \(|X| = 1\). If the only vertex in \(X\) is connected to at least one vertex in \(Y\) then clearly a matching exists.

Induction Hypothesis: Assume that Hall’s condition is sufficient when \(|X| = j\), for all \(j\) such that \(1 \leq j \leq k\).

Induction Step: We want to prove that the sufficiency of Hall’s condition when \(|X| = k + 1\). Let \(G = (X, Y, E)\) be a graph with \(k + 1\) vertices in \(X\) such that \(\forall S \subseteq X, |N_G(S)| \geq |S|\).
We consider the following two cases.

Case I: For every non-empty proper subset \( W \subseteq X, |N_G(W)| > |W| \). In this case, we pair-up an arbitrary vertex \( x \in X \) with one of its neighbors, say \( y \in Y \). Now consider the subgraph \( G' = (X', Y', E') \), where \( X' = X \setminus \{x\} \), \( Y' = Y \setminus \{y\} \), and \( E' = E \setminus \{(x, y)\} \). After the removal of \( y \), the neighborhood of any subset, \( S' \subseteq X' \) in \( G' \) is at most one less than its neighborhood in \( G \). But since \( |N_G(S')| > |S'| \), after removal of \( y \), it must be that \( |N_G'(S')| \geq |S'| \). Thus, Hall’s condition holds for \( G' \). By induction hypothesis, \( G' \) contains a matching \( M' \) that saturates every vertex in \( X' \). Hence, \( M' \cup \{(x, y)\} \) is a matching that saturates every vertex in \( X \).

Case II: For some non-empty proper subset \( W \subseteq X, |N(W)| = |W| \). For all \( S' \subseteq W \), we have \( N_G(S') \subseteq N_G(W) \). Hence, Hall’s condition holds for the subgraph induced by \( W \cup N(W) \). By induction hypothesis, there is a matching \( M_1 \) that matches every vertex in \( W \) to a vertex in \( N_G(W) \). Note that \( M_1 \) is a perfect matching. Consider the subgraph \( G' = (X', Y', E') \), where \( X' = X \setminus W \), \( Y' = Y \setminus N(W) \), and \( E' \) consists of all edges between \( X' \) and \( Y' \). If we can prove that Hall’s condition holds for \( G' \) then by induction hypothesis, \( G' \) has a matching \( M_2 \) that saturates every vertex in \( X' \). Then, \( M_1 \cup M_2 \) is clearly a matching in \( G \) that saturates every vertex in \( X \). It now remains to prove that \( \forall T \subseteq X', |N_G'(T)| \geq |T| \). Note that \( N_G(W \cup T) = N_G(W) \cup N_G'(T), |N_G(W)| = |W|, W \) and \( T \) are disjoint, and \( N_G(W) \) and \( N_G'(T) \) are disjoint. Then,

\[
|N_G(W \cup T)| \geq |W \cup T| \quad \text{(follows because } \forall S \subseteq X, |N_G(S)| \geq |S|) \\
|N_G(W)| + |N_G'(T)| \geq |W| + |T| \\
|W| + |N_G'(T)| \geq |W| + |T| \\
|N_G'(T)| \geq |T|
\]

This proves the sufficiency of Hall’s condition.

Relations

A **binary relation** is a set of ordered pairs. For example, let \( R = \{(1, 2), (2, 3), (5, 4)\} \). Then since \( (1, 2) \in R \), we say that 1 is related to 2 by relation \( R \). We denote this by \( 1 \mathrel{R^2} 2 \).

Similarly, since \( (4, 7) \notin R \), 4 is not related to 7 by relation \( R \), denoted by \( 4 \not\mathrel{R} 7 \).

A binary relation \( R \) from set \( A \) to set \( B \) is a subset of the cartesian product \( A \times B \). When \( A = B \), we say that \( R \) is a relation on set \( A \).

Example. Let \( A = \{1, 2, 3, 4\} \) and \( B = \{a, b, c\} \). Consider the following relations.

\[
R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3)\} \\
R_2 = \{(1, 2), (2, 3), (3, 4), (4, 1), (4, 4)\} \\
R_3 = \{(1, a), (2, a), (3, b), (4, c)\} \\
R_4 = \{(a, 1), (a, 3), (a, 4), (c, 1)\} \\
R_5 = \{(a, a), (a, b), (1, c)\}
\]
$R_1$ and $R_2$ are relations on $A$. $R_3$ is a relation from $A$ to $B$. $R_4$ is a relation from $B$ to $A$. $R_5$ is not a relation on sets $A$ and $B$ and it is neither a relation from $A$ to $B$ nor a relation from $B$ to $A$. It is a relation on $A \cup B$.

Below are some more examples of relations.

- If $S$ is a set then “is a subset of”, $\subseteq$, is a relation on $\mathcal{P}(S)$, the power set of $S$.
- “is a student in” is a relation from the set of students to the set of courses.
- “=” is a relation on $\mathbb{Z}$.
- “has a path in $G$ to” is a relation on $V(G)$, the set of vertices in $G$.

**Example.** How many relations are there on a set $A$ of $n$ elements?

**Solution.** Note that any relation on $A$ is a subset of $A \times A$ and since the power set of $A \times A$ contains all subsets of $A \times A$, the number of possible relations on $A$ is the cardinality of the power set of $A \times A$. Since $|A \times A| = n^2$, the cardinality of the power set of $A \times A$ is $2^{n^2}$. Thus our answer is $2^{n^2}$.

**Properties of Relations**

Let $R$ be a relation defined on set $A$. We say that $R$ is

- reflexive, if for all $x \in A$, $(x, x) \in R$.
- irreflexive, if for all $x \in A$, $(x, x) \notin R$.
- symmetric, if for all $x, y \in A$, $(x, y) \in R \implies (y, x) \in R$.
- antisymmetric, if for all $x, y \in A$, $x R y$ and $y R x \implies x = y$.
- transitive, if for all $x, y, z \in A$, $x R y$ and $y R z \implies x R z$.

Note that the terms symmetric and antisymmetric are not opposites. A relation may be both symmetric and antisymmetric or can neither be symmetric nor be antisymmetric.

**Example.** What are the properties of the following relations?

\begin{align*}
R_1 &: \text{equality relation on } \mathbb{Z}.
R_2 &: \text{“is a sibling of” relation on the set of all people.}
R_3 &: \text{“$\leq$” relation on } \mathbb{Z}.
R_4 &: \text{“$<$” relation on } \mathbb{Z}.
R_5 &: \text{“$|$” relation on } \mathbb{Z}^+.
R_6 &: \text{“$|$” relation on } \mathbb{Z}.
R_7 &: \text{“$\subseteq$” relation on the power set of a set } S.
R_8 &: \{(x, y) \in \mathbb{R}^2 : |x - y| < \epsilon\}, \text{ where } \epsilon = 0.001
\end{align*}
Solution.

Reflexive : $R_1, R_3, R_5, R_7, R_8$
Irreflexive : $R_2, R_4$
Symmetric : $R_1, R_2, R_8$
Antisymmetric : $R_1, R_3, R_4, R_5, R_7$
Transitive : $R_1, R_3, R_4, R_5, R_6, R_7$

Note that $R_6$ is not reflexive because $(0, 0) \not\in R_6$; it is not antisymmetric because for any integer $a$, $a \mid -a$ and $-a \mid a$, but $a \neq -a$. $R_2$ is not transitive because $x$ and $z$ could be the same person. Observe that $R_6$ is an example of a relation that is neither symmetric nor antisymmetric. $R_1$ is an example of a relation that is symmetric and antisymmetric.