This homework is due electronically on Gradescope at 11:59PM EDT, October 2, 2023. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. **LATEX**: All solutions are required to be typeset in LATEX.

B. **Standard Deductions**:  
   • 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. **Solutions**: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. **Collaboration**: You may not collaborate with anyone via any means.

E. **Citations**: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. **Outside Resources**: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. **Late Policy**: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two ‘T’ homeworks and two ‘H’ homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.
1. **[16 pts] Let’s Frolic on the Truffula Field**

   After reading the Lorax, the head TAs become super environmentally conscious and decide to plant a field of truffula trees on the high-rise fields. However, being the 1600 TAs that they are, they decide to make it a little more fun by creating walkways between all the different sections of trees.

   Rashmi and Elisa’s walkways connect \( n \geq 2 \) sections of trees. Of those sections, there exists a walkway between any two sections \( S, T \). However, to organize traffic throughout their field, they design each walkway to either go only from \( S \) to \( T \), only from \( T \) to \( S \), or in both directions.

   Unfortunately, while Rashmi and Elisa are great head TAs, their truffula field planting skills are less than competent, and they’ve discovered that their field has a problem: some sections might be dead-ends, meaning there are no walkways to let you leave, so someone could enter the beautiful truffula field but never be able to make it back out!

   Ishaan and Andrew decide to explore the field, but not before getting a better understanding of where they’re going. They are allowed to ask at most \( 2(n - 1) \) questions to Rashmi and Elisa, which take the form of “Can I get from section \( S \) to section \( T \) using a single walkway?” Rashmi and Elisa will truthfully answer either “Yes” or “No”. Prove, using induction, that if there are \( n \) sections, Ishaan and Andrew can find a dead-end section of the field, if it exists.

2. **[12 pts] Nothing Stops the Grinch**

   Sara, the Grinch, wants to break the indomitable Christmas spirit of Whoville. Mid-way through her heist of all things Christmas-related, she encounters Claire Lu Who struggling with an induction problem. She decides to help her out. Help her solve this problem so she can continue stealing Christmas.

   Prove using induction that for all positive integers \( n \), and for any integers \( a \) and \( b \) with \( a \neq b \), \( a^n - b^n \) is divisible by \( a - b \)

3. **[10 pts] You Are You < 3**

   “Today you are You, that is truer than true. There is no one alive who is Youer than You”. Dr. Suz recognizes that we are all unique human beings: this is exemplified by the fact that we all own a unique number of socks! Each of Dr. Suz’s friends has a different number of socks, ranging from 1 to \( 2n \), \( n \in \mathbb{Z}^+ \). As a thought experiment, Dr. Suz randomly chooses \( n + 1 \) of her friends. Prove that in any assortment of people that could be selected, there will always be two people whose numbers of socks are relatively prime.

4. **[14 pts] Who’s in Whoville?**

   Sam Ng-I-Am is the only Dr. Seuss Stan in the universe who can hear the people on the microscopic planet Whoville. He needs to convince everyone else that the tiny community
actually exists, before it gets destroyed! They all think he has lost his mind, so to prove that he is not crazy, help Sam Ng-I-Am solve the following problems about recurrence relations so that his fellow animals will believe and help him!

(a) Consider the following sequence,

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

Determine a recurrence relation $S(n)$ that captures this sequence, where $n \geq 0$. Make sure to specify the base cases.

(b) Prove using induction that

$$S(n) = \varphi^n + \psi^n, \quad \varphi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}$$

is another way of directly specifying $S(n)$, where $n$ is not in the base case(s). Hint: what are some algebraic relationships between $\varphi$ and $\psi$?

Please note that $\varphi$ is $\textbackslash \varphi$ and $\psi$ is $\textbackslash \psi$ in \LaTeX.

5. [18 pts] Fishy Business

One fish, two fish, red fish, blue fish! The CIS 1600 TAs love all kinds of fish! Avid fish lover Megan decides that she will buy $n$ fish for staff to admire. Initially, all fish are either blue or red. However, Megan’s goal is to transform all fish into gold fish! To do so, she has bought magical powder for these fish that will transform the fish’s color. Feeding the magical powder to fish that are already blue will turn them gold, and Megan can also only feed fish that are already blue. Additionally, when a fish turns gold, the adjacent fish will turn blue if they are red, red if they are blue, and stay gold if they are already gold. Any fish that turns gold will remain gold, and the only operation that Megan can perform is feed the magical powder to blue fish. For instance, with a starting configuration of

“R”, “B”, “B”, “R”

If Megan feeds the second fish from the left, the configuration becomes

“B”, “G”, “R”, “R”

If Megan feeds the only fish that is blue now, the configuration becomes

“G”, “G”, “R”, “R”

Prove that Megan can make all $n$ fish gold if and only if the number of fish that are initially blue is odd.