This homework is due electronically on Gradescope at 11:59PM EDT, September 13, 2023. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. \LaTeX: Normally, we require all solutions to be typeset in \LaTeX. We have provided a \LaTeX primer video on Piazza and on the course website under the ‘resources’ tab, and have provided a template, should you choose to use \LaTeX. However, \LaTeX is not strictly required for this first assignment only.

B. Standard Deductions:
   - 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: You may not collaborate with anyone via any means.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two ‘T’ homeworks and two ‘H’ homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.
1. [9 pts] TAs Raise Praise to Sport Days

IT IS SPORTS SEASON!!!

The CIS 1600 staff has decided to host watch parties for different sports. Each TA picks their favorite out of tennis, football, and soccer, and goes to the respective watch party. The TAs that do not follow sports will go to grading as a punishment for not being an avid sports fan. We will call the set of TAs watching tennis \( T \), the set of TAs watching football \( F \), the set of TAs watching soccer \( S \), and the set of TAs grading \( G \).

Naturally, the CIS 1600 staff decided to prove or disprove the following statements before they began their watch party/grading session. (Note that since the TAs change their minds quite often about which set they want to be part of, the sets \( T, F, S \), and \( G \) are not the same between each subproblem).

(a) Suppose that \( T \), \( F \), and \( S \) are sets with \( T \cap F \cap S = \emptyset \). Prove or disprove:

\[
|T \cup F \cup S| = |T| + |F| + |S|
\]

(b) Let \( T, F, S \), and \( G \) be arbitrary sets. Prove or disprove:

\[
(T \cap S) \cup (F \cap G) \subseteq (T \cup F) \cap (S \cup G)
\]

(c) Let \( T \) and \( F \) be arbitrary sets. Prove or disprove:

\[
\mathcal{P}(T) \cup \mathcal{P}(F) = \mathcal{P}(T \cup F)
\]

2. [12 pts] Proof Protesters!

Arnya Sana-lenka is in the final of the US Open! But, Arthur Ashe Stadium has been invaded by a crowd of protesters demanding that their proofs get checked! Sana-lenka needs them to leave so that she can start warming up, but knows nothing about proofs, so she has asked you to answer several questions about the validity of their proofs.

For each of the “proofs” below, say whether the proof is valid or invalid. If it is invalid, indicate clearly as to where the logical error in the proof lies, and justify why this is a logical error. If the proof is valid, you may simply say so. Just stating that the claim is false will not be awarded credit.

(a) Claim: All natural numbers are divisible by 143.

Proof: Suppose, for the sake of contradiction, the statement were false. Let \( X \) be the set of counterexamples, i.e., \( X = \{ x \in \mathbb{N} \mid x \) is not divisible by 143\}. The supposition that
the statement is false means that \( X \neq \emptyset \). Since \( X \) is a nonempty set of natural numbers, it contains a least element \( z \).

Note that \( 0 \notin X \) because \( 0 \) is divisible by \( 143 \). So \( z \neq 0 \). Now consider \( z - 143 \). Since \( z - 143 < z \) (and \( z \) is the smallest counterexample) then \( z - 143 \) is not a counterexample to the original statement and is therefore not in \( X \). Therefore \( z - 143 \) is divisible by \( 143 \); that is, there is an integer \( a \) such that \( z - 143 = 143a \). So \( z = 143a + 143 = 143(a + 1) \) and \( z \) is divisible by \( 143 \), contradicting \( z \in X \).

(b) **Claim:** For all \( n \in \mathbb{N} \), \( 2n + 1 \) is a multiple of \( 3 \) \( \implies \) \( (n^2 + 1) \) is a multiple of \( 3 \).

**Proof:** We will prove the contrapositive. Assume \( (2n + 1) \) is not a multiple of \( 3 \).

- If \( n = 3k \), for \( k \in \mathbb{N} \), then \( n^2 + 1 = 9k^2 + 1 \) is not a multiple of \( 3 \).
- If \( n = 3k + 1 \) for \( k \in \mathbb{N} \), then \( (2n + 1) = 6k + 3 \) is a multiple of \( 3 \), so the original claim holds, as false implies everything.
- If \( n = 3k + 2 \) for \( k \in \mathbb{N} \), then \( n^2 + 1 = 9k^2 + 12k + 5 \) is not a multiple of \( 3 \).

In all cases, we have concluded \( n^2 + 1 \) is not a multiple of \( 3 \), so we have proved the claim.

(c) **Claim:** If \( x \) and \( y \) are integers then \( 4xy^3 \) has a different parity than \( x \).

**Proof:** Assume, without loss of generality, that \( x \) is odd. By definition of an odd integer, \( x = 2k + 1 \), for some integer \( k \). Thus:

\[
4xy^3 = (2k + 1)(4y^3) = 8ky^3 + 4y^3 = 2(4ky^3 + 2y^3)
\]

Since \( 4ky^3 + 2y^3 \) is an integer, \( 4xy^3 \) is even and hence has a different parity than \( x \).

3. **[9 pts] Game, Set, (Irrationality Proof), Match!**

Selina Qiu-lliams and Hasit-ael Nanda-l decide to have a friendly tennis match in preparation for the US Open. To decide who serves first, they race to prove the claim that for any positive integer \( a \) such that \( a = 19c^2 \), where \( c \in \mathbb{N} \), \( \sqrt{a} \) is always irrational. Selina is a master in proofs, but Hasit-ael needs your help. Help him complete the proof and earn the right to serve first! (You may assume that the square root of 19 is irrational without proof.)