This homework is due electronically on Gradescope at 11:59PM EDT, September 18, 2023. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. \LaTeX: All solutions are required to be typeset in \LaTeX.

B. Standard Deductions:
   • 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: You may not collaborate with anyone via any means.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two ‘T’ homeworks and two ‘H’ homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.
1. **[12 pts] Athletic Administrator Arranges Ace Athletes**

   Answer the following questions. No justification is necessary; however, if your answer is incorrect, you may receive some partial credit for nearly correct justification.

   (a) Aaron just recently became a high school sports coach with 99 players. For organizational purposes, he numbers his players 1, 2, . . . , 99, and wants to arrange them in a line such that the sum of the numbers of every pair of consecutive players in the line is odd. How many ways can he do this?

   (b) Regionals are coming up and he realizes that he needs to put together a team so that they can become regional champions. After a strenuous week of evaluating all 99 of his players, he realizes the ones numbered 1 through 24 (inclusive) are the only ones good enough to represent the school. Then, he splits the 24 players into two equal-sized teams A and B. Then, within both of these subteams, he assigns 2 players to each of the 6 roles in the team. Note that each of these 6 roles is distinct, and the order in which the players get assigned to their roles doesn’t matter. How many ways can Aaron organize his 24 finest players into the aforementioned subteams and the roles within them?

2. **[22 pts] Tennis Ticket Tactics**

   Luna is super desperate to watch the finals of the US Open and see her long-time idol compete. However, tickets sold out super fast and now she is standing outside the stadium with no way of getting in. The security guard decides to take pity on her and offers to let her in if she can help him formulate proofs for the following induction problems:

   (a) Prove that for all \( n \in \mathbb{Z}^+ \),
   \[
   \sum_{i=1}^{n} i^3 = \left( \sum_{i=1}^{n} i \right)^2
   \]

   (b) Prove by induction that for \( \forall n \in \mathbb{Z}^+ \),
   \[
   2^n + 3^n < 6^n.
   \]

3. **[12 pts] Alternate Ways to Feel Victorious**

   Nathan has always dreamed of winning the U.S. Open. However, after years of losing every tennis match that he played, he realizes that his chances of even qualifying are slim. To make himself feel like a winner, he decides to buy exactly \( r \geq 1 \) individual trophies to create his own custom trophy collection. He goes to the store and sees that there are \( k \geq 1 \) colors of trophies, with an infinite number of each color besides the legendary silver U.S. Open trophy replica, of which there is only 1. How many ways can Nathan create his trophy collection? Note that trophies of the same color are indistinguishable.
4. [12 pts] Be the Better Better

Laura and Luna are both avid sports betters. They each count their wins using mathematical operations and positive integers \( n \) and \( k \), where \( n \geq k \). Laura brags that she has correctly bet on \( k \binom{n}{k} \) matches, but Luna, who has bet correctly on \( n \binom{n-1}{k-1} \) matches, is convinced that she is the better sports better. As the judge, Rashmi points out that they actually both bet correctly on the same number of games. Help explain why they tied by giving a combinatorial proof for the following identity:

\[
k \binom{n}{k} = n \binom{n-1}{k-1}
\]

where \( k, n \in \mathbb{N}, 1 \leq k \leq n \).

5. [12 pts] Fall Football Fields

Yijia, an avid (American) football player, wants to spread her love for the sport to the CIS 1600 staff. As a result, she purchases 38 football fields - one for every member of staff except her. These football fields are numbered 1-38. However, gigantic worms have decided to invade each of the football fields. Yijia, who happens to be an expert horticulturalist, decides to build a plant potion to combat the worm infestation. There are three steps to creating this plant potion: crushing a red leaf, chopping a brown stick, and wrapping everything in a yellow leaf. Please note that these steps MUST be performed in chronological order (crush red leaf, chop brown stick, wrap in yellow leaf.) Yijia has numbered the red leaves 1-38, the brown sticks 1-38, and the yellow leaves 1-38.

(a) Yijia is trying to plan out her schedule to assemble the 38 distinguishable plant potions. She wants to make sure that each ingredient is used at its correspondingly numbered football field. An example schedule for Yijia is as follows: she can crush red leaf 1 for football field 1, chop brown stick 1 for football field 1, wrap in yellow leaf 1 for football field 1, then repeat this process for each of the remaining 37 football fields and their numbered red leaves, sticks, and yellow leaves. Note that Yijia doesn’t have to construct a single potion all at once, and can spread out the steps provided that the steps happen in the correct order for any single plant potion. For example, to create the plant potions for fields 1 and 2, Yijia could crush red leaf 1 for football field 1, crush red leaf 2 for football field 2, chop brown stick 1 for football field 1, chop brown stick 2 for football field 2, wrap in yellow leaf 1 for football field 1, and wrap in yellow leaf 2 for football field 2. How many ways are there for Yijia to schedule out how to assemble the 38 plant potions?

(b) Yijia is trying to plan out her schedule to assemble the 38 distinguishable plant potions except this time, she decides that any football field can get any numbered red leaf, brown stick, and yellow leaf. For example, Yijia could crush red leaf 38 for football field 1, chop brown stick 37 for football field 1, then wrap in yellow leaf 36 for football field 1 to assemble
the plant potion for football field 1. Note that once again, Yijia doesn’t have to construct a single plant potion all at once, and can spread out the steps provided that the steps happen in the correct order for any single plant potion. How many ways are there for Yijia to schedule out how to assemble the 38 plant potions?