This homework is due electronically on Gradescope at 11:59PM EDT, December 4, 2023. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. **LATEX**: All solutions are required to be typeset in **LATEX**.

B. **Standard Deductions**:  
   - 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. **Solutions**: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. **Collaboration**: Please make sure to strictly follow our collaboration policy as clarified on Piazza.

E. **Citations**: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. **Outside Resources**: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. **Late Policy**: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two ‘T’ homeworks and two ‘H’ homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.
1. **[16 pts] Never Too Many Thank-you’s**

To show appreciation for the CIS 1600 Head TAs and the hardworking students, Rajiv wants to say thanks! But, since he does not feel like doing it himself, he decides that the two groups will say thank you to each other. He gathers \( n + m \) total people, made up of \( n \) Head TAs and \( m \) students, and decides to arrange some mutual "thank you's". Since he believes in spreading maximum gratitude across the two groups, Rajiv wants to ensure two requirements for the thank-you process: all thank-you’s must be between a Head TA and a student, and every student will say thank you to every Head TA (and vice versa, every Head TA will say thank you to every student). Since Rajiv loves graphs, he decides also to visualize the process. Below is an illustration of a valid thank-you process where \( n = 2 \) and \( m = 3 \), Head TAs are blue, students are red, and each line represents a mutual "thank-you" between a student and a Head TA.

(a) Worried that the Head TAs and students are getting more appreciation than they really deserve, Rajiv decides to remove some thank you’s from the process. He wants to do so such that the resulting graph of thank-you’s no longer has any cycles but still connects all \( n + m \) people, as he still believes everyone deserves at least one thanks. Let \( D \) be the number of thank-you’s that Rajiv removes. Prove that \( D \) is divisible by \( m - 1 \).

(b) Assume that \( m > n \). Prove that the length of a longest path in the original "thank you" graph is even.

2. **[14 pts] Piazza! or... Pizza?!**

We all know that Ishaan is an expert on Piazza, but it is lesser known that he is also an expert on pizza. To celebrate the end of the semester, Ishaan plans to prepare a pizza feast for the rest of the TAs. He plans to work with a set \( T \) of distinct toppings. He also has a set \( B \) of different sauces. Ishaan, being the pizza expert he is, knows that each sauce tastes good with some (possibly distinct) positive number of types of toppings in \( T \). He notices that for any given variety of sauce, the number of varieties of toppings that it tastes good with is always greater than or equal to the number of varieties of sauce that any one type of topping tastes good with. With this in mind, Ishaan wants to assemble \( |B| \) pizzas such that every variety of sauce goes with one type of topping and no type of topping goes with more than one variety of sauce and every pizza tastes good. Prove that Ishaan can realize his culinary dreams.

3. **[12 pts] Rubric’s Cube™**
Ishaan decides to hold a massive 1600 Rubric’s Cube™ contest between two teams, team \(A\) and team \(R\), led by Andrew and Rashmi, respectively. The contest has turns that last 1 minute, and each turn, a TA from each team comes forward to attempt solving a Rubric’s Cube mixed by Ishaan. If a TA cannot solve their Rubric’s Cube by the end of the turn, a new TA immediately comes forward, and Ishaan chooses a new Rubric’s Cube (so there is no time between turns). A team wins and the game ends when just one Rubric’s Cube is solved within a turn. Every turn, team \(A\) has a probability \(p\) of solving a Rubric’s Cube, and team \(R\) has a probability \(q\) of doing the same. Let \(X\) and \(Y\) be the random variables denoting how many turns it takes for a Rubric’s Cube to be solved within a turn by teams \(A\) and \(R\), respectively.

Ishaan has also invited Elisa over to observe the spectacle. Elisa, who has perfectly timed her arrival, tells Ishaan that she will arrive in \(k \in \mathbb{Z}^+\) minutes, and Ishaan immediately begins the first turn of the contest. For dramatic effect, Ishaan wants to know the probability that at least one team solves a Rubric’s Cube the exact minute Elisa arrives, and that neither team has won the game in a previous turn. Ishaan asks you for help: can you calculate \(\Pr\left[\min(X, Y) = k\right]\) for Ishaan?

4. **[14 pts]** Gift Giving Games

In the spirit of the holidays, the CIS 1600 TAs have decided to buy gifts for the wonderful head TAs. To ensure that each gift is perfect, the TAs have decided to work in pairs, with each pair choosing exactly one gift to give. Each TA has a set of eligible head TAs they could give gifts to, and two TAs can only be paired together if they share an eligible head TA in common. Luckily, each TA is guaranteed to have at least \(2k\) other TAs on staff with whom they share a common eligible head TA. Prove that the \(n > 2k > 0\) staff members can end up gifting the head TAs at least \(k\) gifts. In other words, prove that the staff members can create at least \(k\) disjoint pairs. Note that some TAs might not get paired up with anyone.

5. **[14 pts]** Rashimi’s Sashimi

Tired of being an amazing CIS 1600 Head TA, Rashmi decides to quit her job and pursue her true dream: opening a sushi restaurant! Using the money she saved up from her time on staff, she launches Rashimi’s Sashimi, the hottest new joint for Japanese cuisine the world had ever seen. In order to ensure that Rashimi’s Sashimi will always be able to serve hungry customers, Rashmi decides to source her ingredients from not just one factory, but from \(n\) different factories. Among these factories, there are \(m\) air routes that allow for travel between a pair of factories. Every pair of factories has at most one air route between it.

Rashmi wants to divide her factories into two disjoint groups, Main and Backup, such that each factory belongs to exactly one of the groups. This way, she can simplify the shipping logistics so that she’s only getting ingredients from the Main group of factories while still having
the Backup group just in case. She wants to ensure that there are at least \( \frac{m}{2} \) air routes between factories in the Main group and factories in the Backup group so that her factory system operates efficiently, but fears this might not be possible. Help assure Rashmi that she has nothing to worry about by proving it’ll always be possible to assign factories to Main and Backup groups such that this condition is satisfied.