Homework 10H

This homework is due electronically on Gradescope at 11:59PM EDT, November 6, 2023. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. \LaTeX: All solutions are required to be typeset in \LaTeX.

B. Standard Deductions:
   - 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.

C. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 1600.

D. Collaboration: Please make sure to strictly follow our collaboration policy as clarified on Piazza.

E. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

F. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

G. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two ‘T’ homeworks and two ‘H’ homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.
1. [10 pts] Travel of the Torte-Dilini
Karen has discovered a new type of pasta: Torte-Dilini. Karen wants to show all the CIS 1600 TAs her favorite Torte-Dilini dish. To find the TAs in the most efficient manner, Karen creates a map of all of the TAs dorm rooms. Each pair of dorm rooms on the map may be connected by at most one bidirectional zipline. After inspecting the map, Karen realize that she is able to reach any dorm room from any other dorm room by following a series of ziplines on the map and show them her favorite Torte-Dilini dish. Given that each dorm room has an even number of ziplines extending from it, prove that Karen’s claim still holds even if construction blocks off any one existing zipline between two dorm rooms.

We strongly encourage you to approach this problem without induction (our solution does not use induction). This is not to say it is not possible (correct proofs will receive credit), but induction proofs are not always so simple and it’s important to learn how to construct graph proofs using non-inductive methods as well.

2. [12 pts] All Roads Lead to Pasta
After venturing far and wide, you have finally arrived at the gates of Pastaland, a mythical kingdom from ancient legends. You find engravings on the gates in mysterious characters which you decipher to be the three problems given below. Solve these problems to unlock the gates and discover what lies beyond.

(a) Show that Markov’s inequality only applies to non-negative random variables. In other words, give an example of a random variable and its probability distribution for which Markov’s inequality gives an incorrect answer.

(b) Suppose $Z$ is a random variable that is always at least $-6$ and has expectation 0. Since $Z$ can take on negative values, Markov’s inequality does not apply directly. Still, show that the probability that $Z \geq 12$ is at most $\frac{4}{3}$. Use only Markov’s inequality, and known properties about expectations and random variables.

(c) Now suppose we additionally know that $Z$ has a variance of 12. With this information, show that $\Pr[Z \geq 12]$ is now at most $\frac{1}{12}$. Use only Markov’s inequality, and known properties about expectations, variances and random variables.

3. [10 pts] Dita-Li-ni Dinner, Farf-Lal-le Fiesta
Laura loves 1600 staff and decides to host a pasta dinner! The TA’s form a line to pick up their food. She knows that for the first 1600 dinner, 90% of the dishes are composed of Dita-Li-ni’s. Each dish of Dita-Li-ni’s has a $\frac{1}{20}$ chance of being a TA’s favorite dish once they try it for any given TA in line to grab their food, independent of whether or not the dish was a favorite for any previous TA in line. The remaining 10% of dishes are Farf-Lal-le’s, which are not fully
appreciated by the staff and have a $1/1000$ chance of being a TA’s favorite dish once they try it, again independent of their status from previous TA’s in line.

Laura chooses to unpack one dish for all the other TAs to try before the dinner, but having forgotten to label the dishes, is unable to differentiate between Dita-Li-ni’s and Farf-Lal-le’s. She randomly unpacks a dish brought to the dinner. Let $X$ be the random variable denoting the number of TA’s in line that must pass until this dish becomes some TA’s favorite dish.

A few months later, Laura decides to host a second pasta dinner as the first was a huge success! She decides to order the same set of dishes she did the first dinner. To get the TA’s started again, she unpacks the same exact dish early again as in the first party. Assume that this dish has the same probability of being a TA’s favorite as the corresponding dish from the first dinner. Let $Y$ be the random variable denoting the number of TA’s in line until this “new” dish is a TA’s favorite.

Are the random variables $X$ and $Y$ independent?

For this question, you do not need to define a sample space, but you must define any other events or random variables you may use.

4. [13 pts] Let’s Hope Aaron is Not Lactose Intolerant

Aaron has a debilitating cheese addiction and relies on his favorite pasta dish Mac-aaron-i & cheese to satiate his cheese cravings. He must periodically consume Mac-aaron-i & cheese, however, the number of Mac-aaron-i & cheese dishes he can eat in a day fluctuates every day for $n$ days, $n \geq 3$. Particularly, on the $i$th day for $1 \leq i \leq n$, he can eat $S_i$ Mac-aaron-i & cheese dishes in that day, where $S_i$ is an integer chosen uniformly at random between 1 and $V$ inclusively. Furthermore, the number of Mac-aaron-i & cheese dishes he can eat per day is never the same for any two days and $V > n$. Because he realizes that his cheese addiction is a slippery slope, he chooses to give in to his cravings and eat Mac-aaron-i & cheese dishes only on “Mac-aaron-i & cheese days”. (On all other days he will refrain from eating Mac-aaron-i & cheese.) The $i$th day is a Mac-aaron-i & cheese day if $S_{i-1} < S_i$ and $S_i > S_{i+1}$ for $1 < i < n$. The first day is a Mac-aaron-i & cheese day if $S_1 > S_2$ and the last day is a Mac-aaron-i & cheese day if $S_n > S_{n-1}$. Help Aaron calculate the expected number of days he will eat Mac-aaron-i & cheese.

5. [13 pts] Please Do Not Consume Raw Pasta!

In his application to a culinary school in Italy, Thomas constructs a pasta display aptly named: Thomas Fusil-Li. He arranges 18 pieces of Fusilli (his favorite), 10 of which are cooked and 8 of which are raw, uniformly at random to make the display. In expectation, how many pairs of consecutive pasta pieces are there, such that each is a different type (i.e. one fusilli is cooked
and the other is raw)?

For example, if the Thomas Fusil-Li display pasta is ordered CRRCCCRRRRCCCCR (where C is cooked fusilli and R is raw fusilli), we have 5 such pairs.

6. [12 pts] Gotta Eat 'Em All

Andrew and Ishaan are on a hunt to find the best aglio e olio in the city of Philadelphia. Andrew and Ishaan decide to visit every Italian restaurant, each of which has exactly one version of aglio e olio. Furthermore, each Italian Restaurant is reachable from every other Italian Restaurant via a series of roads.

Andrew and Ishaan want to make sure they can try every aglio e olio from every Italian Restaurant without wasting any time. They decide to start from one Italian Restaurant, walk through every road exactly once, and end at another restaurant other than the one from which they started at.

Prove that such a walk is possible if and only if there are exactly 2 restaurants that have an odd number of roads connecting them to other Italian Restaurants.