

Recitation Guide - Week 12

Topics Covered: Digraphs, Graph Coloring

Problem 1

Prove that a graph with maximum degree at most k is $k + 1$ -colorable.

Solution:

We prove this claim by inducting on the number of vertices. Note that induction on k is possible, but is incredibly tedious.

Base Case: $n = 1$. The graph with one vertex has a maximum degree of 0, and is 1-colorable, so the base case holds.

Induction Step: Let $j \in \mathbb{Z}^+$ be arbitrary. Assume (IH) that a graph with j vertices and maximum degree at most k is $(k + 1)$ -colorable.

Consider an arbitrary graph G with $j + 1$ vertices. Create the graph G' by removing an arbitrary vertex v , as well as the edges connected to it. Because we have only removed a vertex, the maximum degree has not increased, so it is still at most k . Therefore, by the IH, G' is $k + 1$ -colorable.

When v is added to G' , it has at most k neighbors, so we can assign v a color that is different from those of its neighbors, as we have $k + 1$ colors to choose from. Thus, G is $(k + 1)$ -colorable, and the claim holds.

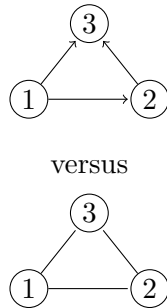
Problem 2 For any digraph $G = (V, E)$ without self-loops and without cycles of length 2 we define an undirected graph $G_u = (V, E_u)$ that has the same vertices as G and moreover in G_u we have an edge $u-v$ whenever we have the edge $u \rightarrow v$ or the edge $v \rightarrow u$ (or both) in G .

(Note: Going from G to G_u is sometimes called erasing direction. You can see that if G had self-loops or cycles of length 2 then erasing direction naively would produce features not allowed in undirected graphs.)

- (a) Prove that if G_u is acyclic then G is a DAG. Then give a counterexample that shows that the converse of this statement is false.
- (b) Prove that if G is strongly connected then G_u is connected. Then give a counterexample that shows that the converse of this statement is false.
- (c) Prove that if G is a DAG in which every sink is reachable from every source then G_u is connected.

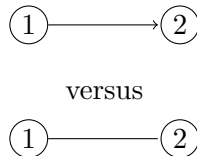
Solution:

- (a) Let us prove the contrapositive of the claim, that if G is not a DAG, then G_u is not acyclic. If G is a digraph but not a DAG, it contains a directed cycle $u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_k \rightarrow u_0$. Consider each edge (u_i, u_{i+1}) in the cycle. We know that the edge $\{u_i, u_{i+1}\}$ exists in G_u . Since $u_k \rightarrow u_0$ in G , we also know $\{u_k, u_0\}$ exists in G_u . Thus, the cycle $u_0 - u_1 - \dots - u_k - u_0$ exists in G_u , and it is not acyclic. As a counterexample to the converse of the statement, we have:



Here, the top graph is a DAG, but the bottom graph has a cycle.

- (b) Assume for contradiction that G is strongly connected but G_u is not connected. This means there exists vertices u and v such that there is no path between the two in G_u . Because G is strongly connected, we know that there exists a directed path $u \rightarrow \dots \rightarrow v$. Similar to the proof for (a), we know that every two consecutive vertices along this path also have an undirected edge between them in G_u , and thus a path from u to v exists in G_u . As a counterexample to the converse of the statement, we have:



The graph on the top is not strongly connected but the bottom graph is connected.

- (c) In order for us to show that G_u is connected, we need to first prove the following lemma:

Lemma 1: Every vertex is either a source, a sink or is along a path from a source to a sink.

Consider an arbitrary vertex, v which is neither a source nor a sink. We want to show that v is a vertex along a path from a source to a sink. The vertex must have at least one in-degree and one out-degree since the vertex is not a source nor a sink. Now we consider the maximal path with v , $u_0 \rightarrow u_1 \rightarrow \dots \rightarrow v \rightarrow \dots \rightarrow u_k$.

Consider u_0 and u_k .

u_0 must be a source. If u_0 is not a source, then it must have an in-degree of at least one. Since we considered the maximal path, this path cannot be extended, so some vertex within that path must be the predecessor to u_0 , resulting in a directed cycle. A directed cycle cannot exist in a DAG, which means u_0 must be a source.

By symmetry, we can argue that u_k must be a sink. Since u_0 and u_k is a source and sink respectively, v is a vertex along a path from a source to a sink. Therefore, Lemma 1 is proven.

Now we want to show that G_u is connected. If the edges' directionality are erased, every vertex along a path from a source to a sink is connected along that path. Erasing the direction of the edges converts these paths to a bi-directional path. Furthermore, by Lemma 1, we know that every vertex is along some path from source to vertex. Now we just need to show that it is possible for an arbitrary vertex r to reach another vertex u on another path. If we take the path r is on until we reach a sink, and take a path along which u is on, we have a path between v and u .

Problem 3 Consider a bipartite undirected graph G . Designate exactly one of its edges as “contrarian”. Put direction on all its edges as follows: on all non-contrarian edges, make the direction from red to blue. On the contrarian edge make the direction from blue to red.

- (i) Prove that the resulting digraph is in fact a DAG.
- (ii) Assume additionally that G is $K_{m,n}$. Count the number of distinct topological sorts of the resulting digraph.

Solution:

- (i) By contradiction, assume that the resulting digraph is not a DAG, i.e. there exists a directed cycle within the graph. Let this directed cycle be $u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_k \rightarrow u_0$. Since the graph is a bipartite graph, any two vertices with an edge between them must be colored a different color. WLOG, assume u_0 is red. Then u_1 must be blue and u_2 must be red. We have found the contrarian edge! Now consider u_k . Since u_0 is red, u_k must be blue. But this means we have found a second contrarian edge, which is impossible. Therefore, our contradiction statement is reached and the resulting digraph must be a DAG.
- (ii) G is now a complete bipartite graph with m red vertices and n blue vertices. Let (u, v) be the contrarian edge, where u is blue and v is red. In any topological sort, u must appear before v . We also observe that u must be immediately before v . If another red vertex a were between the two in the topological sort, there would be an edge back from a to u , and if another blue vertex b were between the two, there would be an edge back from v to b . The $m - 1$ other red vertices must appear in some order before u , since they each have an outgoing edge to u . Similarly, the $n - 1$ other blue vertices must appear in some order after v , since v has an outgoing edge to every blue vertex other than u . Since the $m - 1$ other red vertices have zero indegree and $n - 1$ other blue vertices have zero outdegree, we can choose the order of each group. This results in $(m - 1)!(n - 1)!$ possible topological sorts.

Problem 4 Prove that if G is a DAG with at least two distinct sinks such that there is a path from every source to every sink, then G must have at least one node of outdegree ≥ 2 .

Solution:

Since there is a path from every source to every sink, and we know from Lemma 22.15 that every DAG has at least one source, let's consider the paths from this specific source, u to two distinct sinks, v and w . Since both these paths start from the same source, u , and ends at different vertices, there must exist some vertex k such that the paths diverge. In other words, pick k such that k is the final similar vertex between paths $u - \dots - v$ and $u - \dots - w$. w must have outdegree of at least 2.