CIS 160

# Recitation Guide - Week 9

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**Topics Covered:** Probability, Random Variables, Expectation

**Problem 1:**

AJ really loves his dice! He runs off on his own and rolls 6 fair, distinguishable, 6-sided die for fun. He comes back and tells you that the sum of his rolls equals to 32. Given this information, what is the probability that he only rolled 5s and 6s?

**Solution:**

The sample space $\Omega$ of AJ's original chance experiment can be represented by an ordered list of 6 numbers, $(d_1, d_2, \ldots, d_6)$ where each $d_i$ represents the value of the $i$th roll. We define the following events:

$A :=$ the event that AJ's rolls each take on only 5 or 6 as values.

$B :=$ the event that the sum of AJ's rolls takes the value of 32.

In terms of the events above, we are asked to find the probability $\Pr[A|B]$. From the definition of conditional probability, we have:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

We first find $\Pr[A \cap B]$. Because the dice are fair, we know each outcome is equally likely, and we have a uniform sample space. Therefore,

$$\Pr[A \cap B] = \frac{|A \cap B|}{|\Omega|}$$

$|\Omega| = 6^6$, as each roll has 6 possible outcomes, and there are 6 rolls.

To determine $|A \cap B|$, we wish to find the number of solutions to $d_1 + d_2 + \cdots + d_6 = 32$, where each $d_i$ can only be equal to 5 or 6. There are only 6 die, so we have two equations:

$$5x + 6y = 32$$
$$x + y = 6$$

Solving the system of equations gives us $x = 4$ and $y = 2$. Therefore, exactly 4 of the die have a value equal to 5, and the remaining 2 are rolled as a 6.

We can determine the number of solutions using the Multiplication Rule:

Step 1: Pick which die will take on the value 6. This can be done in $\binom{6}{2}$ ways.

Step 2: Assign the remaining die to have value 5. This can only be done in one way.

Therefore, we have

$$\begin{aligned}\Pr[A \cap B] &= \frac{|A \cap B|}{|\Omega|} \\ &= \frac{\binom{6}{2}}{6^6}\end{aligned}$$

We now determine $\Pr[B]$. The sample space is still uniform, so we need to find $|B|$. This represents the number of solutions to $d_1 + \cdots + d_6 = 32$ where each $d_i$ is an integer between 1 and 6.

We can count this using stars and bars, but see that, because each value is bounded above by 6, we can not blindly apply stars and bars to the original equation (We could get a solution where $d_1 = 32$ and $d_2 = \cdots = d_6 = 0$, for example.)

Instead, we see that the maximum possible value for the sum is 36, so we can view our equation as "taking away" 4 units from the highest possible sum. Algebraically, if we let $d_i = 6 - \bar{d}_i$, we know each $\bar{d}_i$ must be an integer between 0 and 5, and for each die, we get:

$$(6 - \bar{d}_1) + \cdots + (6 - \bar{d}_6) = 32 \qquad \Rightarrow \qquad \bar{d}_1 + \cdots + \bar{d}_6 = 4$$

Using stars and bars as in lecture 5 we see that there are 4 stars, and 6 categories (where each category can take between 0 and all four of the stars therefore we will use $6 - 1$ bars), so we have $\binom{4+6-1}{4} = \binom{9}{4}$ solutions to the equation $\bar{d}_1 + \cdots + \bar{d}_6 = 4$, and therefore $\binom{9}{4}$ solutions to the equation $d_1 + \cdots + d_6 = 32$. Therefore, we have

$$\begin{aligned}\Pr[B] &= \frac{|B|}{|\Omega|} \\ &= \frac{\binom{9}{4}}{6^6}\end{aligned}$$

Thus, for our final answer, we have:

$$\begin{aligned}\Pr[A|B] &= \frac{\binom{6}{2}/6^6}{\binom{9}{4}/6^6} \\ &= \frac{\binom{6}{2}}{\binom{9}{4}} \\ &= \boxed{\frac{5}{42}}\end{aligned}$$

**Problem 2:** Let's say we are playing the following game. I first roll a fair 6-sided die. If the number that shows is divisible by 3, I roll again and I pay you the dollar amount that shows up on the second roll and the game ends. If not, then I flip a fair coin. If it's tails, I take 10 dollars from you, and if it's heads, I pay you 5 dollars and again the game ends. What is your expected payoff?

**Solution:**

We will denote the payoff using a random variable $X$. $X$ can take on 7 values; 1, 2, 3, 4, 5, 6 and -10. Values 1,2,3,4,6 arise when the first dice roll is divisible by 3 and the second dice roll takes the values in the set $\{1, 2, 3, 4, 6\}$. $X$ takes on the value 5 in two ways. The first way is the through the method above with the dice rolls except the second roll is a 5. The second way is the first dice roll is not divisible by 3 and the resulting coin flip is a heads. Finally, $X$ can take on the value -10 when first dice roll is not divisible by 3 and the resulting coin flip is a tails.

Then, we will use the formula for the expected value of a random variable for each value $i$ that $X$ can take on.

$$E[X] = \sum_i i \Pr[X = i]$$

Let us partition the values $X$ can take based on the three payoff scenarios we described above.

***Case 1:***

For $X$ to take on the value of 1, the number on the first die must have been divisible by 3 and the number of the second die must have been 1.

Let $A$ be the event that the number on the first die is divisible by 3. By total probability theorem, we can then write:

$$\begin{aligned}
\Pr[X = 1] &= \Pr[X = 1 \cap A] + \Pr[X = 1 \cap \bar{A}] \\
&= \Pr[X = 1|A] \times \Pr[A] \\
&= \frac{2}{6} \times \frac{1}{6}
\end{aligned}$$

Note that the second die roll is dependent on the first, so although we use multiplication, we can't justify it with "independence."

Similarly, $\Pr[X = 2], \Pr[X = 3], \Pr[X = 4]$, and $\Pr[X = 6]$ all equal $\frac{2}{6} \times \frac{1}{6}$.

***Case 2:***

For $X$ to take on the value of 5, either the number on the first die is divisible by 3 and the number on the second die is 5 or the number on the first die is not divisible by 3 and the coin toss lands as heads. Thus, using $A$ as defined above,

$$\begin{aligned}
\Pr[X = 5] &= \Pr[X = 5|A] \times \Pr[A] + \Pr[X = 5|\bar{A}] \times \Pr[\bar{A}] \\
&= \frac{2}{6} \times \frac{1}{6} + \frac{4}{6} \times \frac{1}{2}
\end{aligned}$$

***Case 3:***

Finally, for $X$ to take on the value of -10, the number on the first die must not be divisible by 3 and the coin must land as tails. Thus, $Pr[X = -10] = \frac{4}{6} \times \frac{1}{2}$. Thus, the expected value is:

$$\begin{aligned} E[X] &= \left(\frac{2}{6} \times \frac{1}{6}\right)(1 + 2 + 3 + 4 + 6) \\ &+ \left(\frac{2}{6} \times \frac{1}{6} + \frac{4}{6} \times \frac{1}{2}\right)(5) \\ &+ \left(\frac{4}{6} \times \frac{1}{2}\right)(-10) = -\frac{1}{2} \qquad \text{(dollars!)} \end{aligned}$$

Alternatively, we can sum over the outcomes of the sample space (rather than summing over all possible values of $X$) (by Proposition 16.9):

If the first roll is divisible by 3 (which happens with probability $\frac{2}{6} = \frac{1}{3}$), then the payoff for the outcome is just equal to the second roll. The probability that the second roll is equal to any value between 1 and 6 is $\frac{1}{6}$ since the die is fair. Thus, the probability that the first roll is divisible by 3 and the second roll is equal to any particular value is $\frac{1}{3} \cdot \frac{1}{6}$ by independence of the two rolls. Thus for all outcomes in which the first die's value was divisible by 3 and the second die had a value $k$ (for integer $k$ where $1 \leq k \leq 6$), the value of the outcome is $k$ with probability $\frac{1}{3} \cdot \frac{1}{6}$.

If the roll is not divisible by 3 (which happens with probability $\frac{4}{6} = \frac{2}{3}$, then the payoffs are equal to $-10$ or 5, depending on the flip (since the coin is fair, each has probability $\frac{1}{2}$). Thus, each outcome in which the first roll was not divisible by 3 occurs with probability $\frac{2}{3} \cdot \frac{1}{2}$ (since the roll and the flip are independent). Thus, the expected value is:

$$\begin{aligned} E[X] =& \frac{1}{3} \times \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) + \frac{2}{3} \times \frac{1}{2}(-10 + 5) \\ =& \frac{1}{18}(21) + \frac{1}{3}(-5) = -\frac{1}{2} \end{aligned}$$

**Problem 3:** There are $n$ people in a room. Each (unordered) pair of people has probability $p$ of being friends *mutually independently* among all the pairs of people. What is the expected number of friend groups of size $m$ in the room (in terms of $n$, $p$, and $m$)? Friend groups are groups of $m$ people in which everyone in the group is friends with everyone else in the group.

**Solution:**

Let the random variable $X$ denote the number of groups of size $m$ where everyone is friends with everyone else within that group. We are asked to find $E[X]$.

Note that there are $\binom{n}{m}$ distinct groups of $m$ people among $n$ people. We can then label these groups with natural numbers from $[1..\binom{n}{m}]$.

For each $i \in [1..\binom{n}{m}]$ define the indicator variables $X_i$ corresponding to the event "group $i$ is a group where everyone is friends with everyone else within that group." For each $i$ (where $m_x$ and $m_y$ are the $x^{th}$ and $y^{th}$ person in group $i$) we have

$$\Pr[X_i = 1] \;=\; \prod_{x,y|x<y,x,y\leq m} Pr[m_x \text{ and } m_y \text{ are friends}] = \; p^{\binom{m}{2}}$$

because every possible friendship between any two people in the group must exist, and these friendships are mutually independent. If mutual independence does not hold, then we cannot assume that $\Pr[X_i = 1] \;=\; p^{\binom{m}{2}}$ (i.e. we cannot multiply the probabilities together). If we only assume pairwise independence (i.e. every pair of friends are independent of any other pair), we could have subsets of friends who occur with a higher probability than others.

We can then compute the expectation:

$$\begin{aligned}
E[X] &= E[\sum_{i=1}^{\binom{n}{m}} X_i] \\
&= \sum_{i=1}^{\binom{n}{m}} E[X_i] && \text{(Linearity of Expectation)} \\
&= \sum_{i=1}^{\binom{n}{m}} Pr[X_i = 1] && \text{(by definition 16.15)} \\
&= \sum_{i=1}^{\binom{n}{m}} p^{\binom{m}{2}} \\
&= \binom{n}{m} \cdot p^{\binom{m}{2}}
\end{aligned}$$