CIS 160

Recitation Guide - Week 3

Topics Covered: Proof Techniques: Contra* ; Truth Tables; Stars and Bars

Problem 1: Let $a, b \in \mathbb{Z}$ and n is a positive integer. If n does not divide ab (we use the notation $n \nmid ab$), then n does not divide a and n does not divide b ($n \nmid a$ and $n \nmid b$).

Solution:

We can prove the contrapositive of the above statement: if $n \mid a$ or $n \mid b$, then $n \mid ab$. To prove this, assume $n \mid a$ without loss of generality. Then $a = kn$ for some $k \in \mathbb{Z}$. Hence $ab = knb = n \times kb$. Since $k, b \in \mathbb{Z}$, we know that kb must be an integer, so $n \mid ab$.

Problem 2: Prove that $\sqrt{6}$ is irrational.

Solution:

We will use a proof by contradiction. Assume for the sake of contradiction that $\sqrt{6}$ is rational. By the definition of a rational number, write $\sqrt{6}$ as $\frac{a}{b}$ where a and b have no common divisors other than 1 (we call them **relatively prime** natural numbers) and $b \neq 0$. This means that

$$\begin{aligned} 6 &= \frac{a^2}{b^2} \\ 6b^2 &= a^2 \end{aligned}$$

If $6 \mid a^2$, then $2 \mid a^2$ which implies that a must be even (recall Lemma 5 proved in Lecture 4). Because a is even, let $a = 2c$ for some integer c.

$$\begin{aligned} 6b^2 &= a^2 \\ 2 \times 3 \times b^2 &= (2c)^2 \\ 2 \times 3 \times b^2 &= 2 \times 2 \times c^2 \\ 3b^2 &= 2c^2 \end{aligned}$$

If $2 \mid (3b^2)$, then $2 \mid b^2$ which implies that b must be even (see Lemma above). So, clearly, a and b are both even. However, this presents a contradiction: a and b must be relatively prime natural numbers, and thus cannot both be divisible by the same factor, 2.

Problem 3: Which of the following are logically equivalent to each other?

1. $p \wedge \neg(\neg p \wedge \neg q)$
2. $(\neg p \wedge \neg q) \Rightarrow q$
3. $p \vee C$

Solution:

1 and 3 are logically equivalent.

Consider the following truth tables:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$	$p \wedge \neg(\neg p \wedge \neg q)$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	F
F	F	T	T	T	F	F

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(\neg p \wedge \neg q) \Rightarrow q$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

p	q	C	$p \vee C$
T	T	F	T
T	F	F	T
F	T	F	F
F	F	F	F

Since $p \wedge \neg(\neg p \wedge \neg q)$ and $p \vee C$ have the same truth values for the same assignments to p and q, they are logically equivalent. In fact, note that both are logically equivalent to p.

Problem 4: The janitor needed to distribute soap bars and toilet paper to customers of the hotel. He starts his shift with 10 bars of soap and 10 rolls of toilet paper. After the 6th room, he discovers that he has run out of supplies. How many ways could he have distributed the toilet paper rolls and soap bars to the different rooms? He cannot tell the difference between any two toilet paper rolls and between any two soap bars. However, he can easily tell the difference between toilet paper and soap bars.

Solution:

We can break this problem down into separate stars and bars problems and combine them at the end.

There are 6 rooms in which we distribute 10 toilet paper rolls. Arrange 10 stars in a row. These stars represent the toilet paper rolls. Since the toilet paper rolls are indistinguishable, their ordering is irrelevant. We now also have $6 - 1 = 5$ bars to represent the rooms. Place the 5 bars between some of the stars (toilet paper rolls). The bars would then separate the stars into 6 parts, each of which represent one room. The number of arrangements of toilet paper rolls is then

$$\binom{10+6-1}{10} = \binom{15}{10}$$

We can do the same for the soap bars. 10 indistinguishable soap bars distributed along 6 rooms gives us

$$\binom{10+6-1}{10} = \binom{15}{10}$$

But this isn't the end! We still have to combine them. We can do this by using the multiplication rule, since the order in which the toilet paper rolls is distributed is independent of the order in which the soap bars are distributed.

> Step 1: Choose the order in which toilet paper rolls is distributed. From above, there are $\binom{15}{10}$ ways.
>
> Step 2: Choose the order in which soap bars are distributed. Again, there are $\binom{15}{10}$ ways.

which gives us

$$\binom{15}{10} \times \binom{15}{10} = \boxed{\binom{15}{10}^2}$$